RAY PRESERVING METRICS AND APPLICATIONS

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1. Introduction

Let Σ be an m-dimensional minimal submanifold of $\mathbf{R}^n$ and p an interior point of Σ. Define the *cone* $p \times\!\!\!\!\times \partial\Sigma$ *over* $\partial\Sigma$ *with vertex* p as the union of the line segments from p to q, over all $q \in \partial\Sigma$. And for a k-dimensional submanifold $N \subset \mathbf{R}^n$ and a point $q \in \mathbf{R}^n$, we define the *density of* N *at* q to be the limit

$$\Theta_N(q) := \lim_{\varepsilon \to 0} \frac{\text{Volume}(N \cap B_\varepsilon(q))}{\omega_k \varepsilon^k}, \tag{1}$$

where ω_k is the volume of a k-dimensional unit ball. Then we have the following interesting relationships between Σ and $p \times\!\!\!\!\times \partial\Sigma$ [C1]:

$$\Theta_\Sigma(p) \le \Theta_{p \times\!\!\!\!\times \partial\Sigma}(p) \tag{2}$$

and

$$\text{Volume}(\Sigma) \le \text{Volume}(p \times\!\!\!\!\times \partial\Sigma). \tag{3}$$

In (2) equality holds if and only if Σ is part of an m-plane. (2) and (3) also hold for a minimal submanifold Σ of n-dimansional hyperbolic space $\mathbf{H}^n$ [CG1]. However, they do not hold for a minimal submanifold of a general Riemannian manifold.

(2) and (3) played key roles in the proof of the isoperimetric inequality for minimal surfaces in $\mathbf{R}^n$ and $\mathbf{H}^n$ [C1, CG1]. And (2) was an important estimate when Ekholm-White-Wienholtz proved the embeddedness of a minimal surface in $\mathbf{R}^n$. In this survey article we introduce a constant curvature metric on the cone $p \times\!\!\!\!\times \partial\Sigma$ such that the above two estimates still hold for a minimal surface Σ in a Riemannian manifold M with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. With these estimates we can obtain the isoperimetric inequality for the minimal surface $\Sigma \subset M$ [C2], and with more estimates on the cone with constant curvature metric, we can prove the embeddedness of some minimal surface $\Sigma \subset M$ [CG3].

In order to construct a constant curvature metric on $p \times\!\!\!\!\times \partial\Sigma := C$, write $\partial\Sigma = \Gamma$ and start with an arc-length parameter s along Γ. Let $r(s)$ be the distance in $p \times\!\!\!\!\times \partial\Sigma$ from the corresponding point of Γ to p. Then choose a point $\widehat{p} \in \mathbf{H}^2(-\kappa^2)$, and let a curve $\widehat{\Gamma}$ locally isometric to Γ be traced out in $\mathbf{H}^2(-\kappa^2)$ so that the distance from $\widehat{p}$ equals $r(s)$. Let $\widehat{C} = \widehat{p} \times\!\!\!\!\times \widehat{\Gamma}$, which may be in a covering of $\mathbf{H}^2(-\kappa^2)$ branched over $\widehat{p}$, and finally glue

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$\widehat{C}$ along the geodesic segments from $\widehat{p}$ to the initial and final points (cf. [C2], p. 211.) Note that the angle between two geodesics at p becomes larger under $\widehat{g}$, as we shall see in Proposition 3 below that

$$\Theta_C(p) \leq \Theta_{\widehat{C}}(p).$$

One can think of $\widehat{C}$ as C equipped with the constant curvature metric $\widehat{g}$.

More precisely, Let $\widehat{g}$ be a new metric on $C = p \mathbin{⋇} \partial\Sigma$ with constant Gauss curvature $-\kappa^2$ such that the distance from p remains the same as in the original metric g, and so does the arclength element of Γ. In other words, every geodesic from p under g remains a geodesic of equal length under $\widehat{g}$, the length of any arc of Γ remains the same, and the angles between the tangent vector to Γ and the geodesic from p remain unchanged.

2. Comparisons of Density, Area and Geodesic Curvature

In this section we derive various estimates similar to (2) and (3) for Σ, C and $\widehat{C}$. Assume that M is an n-dimensional complete, simply connected Riemannian manifold with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. Let Γ be a C^2 immersed curve in M.

Write $G(r) := \log\tanh(\kappa r/2)$ for the Green's function of the two-dimensional hyperbolic plane $\mathbf{H}^2(-\kappa^2)$ with Gauss curvature $\equiv -\kappa^2 < 0$, and $G(r) := \log r$ for $\mathbf{R}^2$, if $\kappa = 0$. We compute $dG/dr = \kappa/\sinh\kappa r$ or $dG/dr = 1/r$, respectively. Choose a point $p \in M$, and define $\rho(x) := d(x,p)$, using the distance function $d(\cdot,\cdot)$ of M.

Lemma 1. *Let N^2 be a two-dimensional manifold immersed in a complete, simply connected Riemannian manifold M whose sectional curvature is bounded above by $-\kappa^2,\ \kappa \geq 0$. Then*

(a) *except at p,*

$$\triangle_N G(\rho) \geq 2\kappa^2 \frac{\cosh\kappa\rho}{\sinh^2\kappa\rho}\left(1 - |\nabla_N\rho|^2\right) + \kappa\frac{d\rho(\vec{H})}{\sinh\kappa\rho}\ in\ case\ \kappa > 0,$$

and

$$\triangle_N G(\rho) \geq \frac{2}{\rho^2}\left(1 - |\nabla_N\rho|^2\right) + \frac{d\rho(\vec{H})}{\rho}\ in\ case\ \kappa = 0,$$

where $\vec{H}$ is the mean curvature vector of N.

(b)

$$\triangle_N \log(1+\cosh\kappa\rho) \geq \kappa^2 + \kappa\tanh(\kappa\rho/2)\,d\rho(\vec{H})\ in\ case\ \kappa > 0,$$

and

$$\triangle_N \rho^2 \geq 4 + 2\rho\,d\rho(\vec{H})\ in\ case\ \kappa = 0.$$

Proof. By the Hessian comparison theorem, the Hessian of the distance function ρ of M satisfies

$$\overline{\nabla}^2\rho \geq \kappa\coth\kappa\rho(g - \overline{\nabla}\rho\otimes\overline{\nabla}\rho) \text{ for } \kappa > 0, \text{ and } \overline{\nabla}^2\rho^2 \geq 2g \text{ for } \kappa = 0, \tag{4}$$

where g is the metric tensor of M (see [SY], p. 4). The trace formula states that

$$\triangle_N G = \sum_{\alpha=1}^{2} \overline{\nabla}^2 G(e_\alpha, e_\alpha) + dG(\vec{H}),$$

where $\{e_1, e_2\}$ is an orthonormal basis for the tangent plane to N. These formulas are well known (see e. g. [CG2], pp. 172, 174.) Choosing $\{e_1, e_2\}$ with $d\rho(e_2) = 0$ and $d\rho(e_1) = |\nabla_N \rho|$, we have

$$\overline{\nabla}^2 G(e_1, e_1) \geq \kappa^2 \frac{\cosh \kappa\rho}{\sinh^2 \kappa\rho}(1 - 2\, d\rho(e_1)^2)$$

and

$$\overline{\nabla}^2 G(e_2, e_2) \geq \kappa^2 \frac{\cosh \kappa\rho}{\sinh^2 \kappa\rho}.$$

The conclusion of part **(a)** follows.

For the proof of part **(b)**, we again use the trace formula and note that

$$\overline{\nabla}^2 \log(1+\cosh \kappa\rho) \geq \frac{\kappa^2}{1 + \cosh \kappa\rho} \left[\cosh \kappa\rho \cdot g + (1 - \cosh \kappa\rho)\overline{\nabla}\rho \otimes \overline{\nabla}\rho\right] \text{ for } \kappa > 0. \qquad \square$$

Corollary 1. **(a)** *If Σ^2 is a branched minimal surface in M, then $G(\rho)$ is subharmonic on Σ.*
(b) *If $\widehat{C}$ is the cone $p \mathbin{⋇} \partial\Sigma$ over the pole p of the distance function ρ in M with the metric $\widehat{g}$ of Gauss curvature $\equiv -\kappa^2$, then $G(\rho)$ is harmonic on $\widehat{C}$, except at p.*
(c) *Further, on $\widehat{C}$*

$$\triangle_{\widehat{C}} \log(1 + \cosh \kappa\rho) = \kappa^2 \text{ for } \kappa > 0, \text{ and}$$

$$\triangle_{\widehat{C}} \rho^2 = 4 \text{ for } \kappa = 0.$$

Proof. **(a)** On Σ, the mean curvature vector of Σ vanishes and $|\nabla_\Sigma \rho| \leq 1$, hence $\triangle_\Sigma G(\rho) \geq 0$, except at p, according to Lemma 1**(a)**. If $p \in \Sigma$, then the outward normal derivative of $G(\rho)$ on $\partial B_\varepsilon(p) \cap \Sigma$ approaches $+\infty$ as $\varepsilon \to 0$, which implies that G is subharmonic everywhere on Σ. **(b)** On the cone $\widehat{C}$, however, we apply Lemma 1**(a)** with $M = N = \widehat{C}$, so that $\vec{H} \equiv 0$ and $|\nabla_{\widehat{C}} \rho| \equiv 1$. Moreover constancy of the Gauss curvature on $\widehat{C}$ forces all the inequalities in the proof of Lemma 1**(a)** to become equality and consequently $\triangle_{\widehat{C}} G(\rho) \equiv 0$. Similarly for part **(c)**. $\square$

Proposition 1. *(Density Comparison) Let Σ^2 be a branched minimal surface in an n-dimensional simply connected Riemannian manifold M with sectional curvature $\leq -\kappa^2$. Then*
(a) $\Theta_\Sigma(p) < \Theta_{\widehat{C}}(p)$ *unless Σ is totally geodesic with constant Gauss curvature $-\kappa^2$;*
(b) $\mathrm{Area}(\Sigma) \leq \mathrm{Area}(\widehat{C})$.

Proof. **(a)** By Corollary 1, we have $\triangle_\Sigma G(\rho) \geq 0$ and $\triangle_{\widehat{C}} G(\rho) \equiv 0$, where, as above, $G(\rho(x)) := \log \tanh(\kappa\rho(x)/2)$ and $\rho(x) := d_M(x, p)$ or $d_{\widehat{C}}(x, p)$ respectively. For small $\varepsilon > 0$, write $\widehat{C}_\varepsilon := \widehat{C} \backslash B_\varepsilon(p)$ and $\Sigma_\varepsilon := \Sigma \backslash B_\varepsilon(p)$, where $B_\varepsilon(p)$ denotes the geodesic ball in M of radius ε and center p. Then the boundary of Σ_ε is $\Gamma \cup (\Sigma \cap \partial B_\varepsilon(p))$. (The component $\Sigma \cap \partial B_\varepsilon(p)$ may be empty.) Let ν_Σ be the outward unit normal vector tangent to Σ_ε at $\partial\Sigma_\varepsilon$. Then

$$0 \leq \int_{\Sigma_\varepsilon} \triangle_\Sigma G(\rho)\,dA = \int_{\partial\Sigma_\varepsilon} \nu_\Sigma \cdot \overline{\nabla} G\,ds = \int_{\Sigma\cap\partial B_\varepsilon(p)} \kappa\frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sinh\kappa\varepsilon}\,ds + \int_\Gamma \kappa\frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sinh\kappa\rho}\,ds.$$

Along the small boundary component $\Sigma \cap \partial B_\varepsilon(p)$, as $\varepsilon \to 0$, $\nu_\Sigma \cdot \overline{\nabla}\rho \to -1$ uniformly, and

$$\kappa\frac{L(\Sigma\cap\partial B_\varepsilon(p))}{2\pi\sinh\kappa\varepsilon} \to \Theta_\Sigma(p).$$

Let ν_C be the outward unit normal vector tangent to C along its boundary. Then it should be noted that

$$\nu_\Sigma \cdot \overline{\nabla}\rho \leq \nu_C \cdot \overline{\nabla}\rho \text{ along } \Gamma.$$

Thus we find that the inequality above implies

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \kappa\frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh\kappa\rho}\,ds. \tag{5}$$

Note here that ν_C, considered as a tangent vector to C, is also the outward unit normal vector in the metric $\widehat{g}$. Along the intrinsic distance sphere $\partial\widehat{B}_\varepsilon(p) \subset \widehat{C}$, $-\nabla\rho$ is the outward unit normal vector tangent to $\widehat{C}_\varepsilon$. Hence by Corollary 1(b), assuming $C\backslash\{p\}$ is immersed, as $\varepsilon \to 0$,

$$0 = \int_{\widehat{C}_\varepsilon} \triangle_{\widehat{C}} G(\rho)\,dA \to -2\pi\Theta_{\widehat{C}}(p) + \int_\Gamma \kappa\frac{\nu_C \cdot \nabla\rho}{\sinh\kappa\rho}\,ds.$$

Therefore, by equation (5),

$$2\pi\Theta_{\widehat{C}}(p) = \int_\Gamma \kappa\frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh\kappa\rho}\,ds \geq 2\pi\Theta_\Sigma(p),$$

which is the desired estimate.

If equality holds, then $\triangle_\Sigma G \equiv 0$, which requires $|\nabla_\Sigma\rho| \equiv 1$ according to Lemma 1. But this means that Σ is a cone over p, as well as being minimal, which can only occur when Σ is totally geodesic. Moreover, $\triangle_\Sigma G \equiv 0$ now implies that $\triangle_\Sigma\rho \equiv \kappa\coth\kappa\rho$, which, along with $K_\Sigma \leq -\kappa^2$, implies that Σ has constant Gauss curvature $K_\Sigma \equiv -\kappa^2$.
(b) Integrate Lemma 1(b) over Σ to get

$$\begin{aligned}
\kappa^2\text{Area}(\Sigma) &\leq \int_\Sigma \triangle_\Sigma \log(1+\cosh\kappa\rho) = \int_\Gamma \frac{\kappa\sinh\kappa\rho}{1+\cosh\kappa\rho}\frac{\partial\rho}{\partial\nu} \\
&\leq \int_\Gamma \frac{\kappa\sinh\kappa\rho}{1+\cosh\kappa\rho}\frac{\partial\rho}{\partial\eta} = \int_{\widehat{C}} \triangle_{\widehat{C}} \log(1+\cosh\kappa\rho) \text{ (by Corollary 1(c))} \\
&= \kappa^2\text{Area}(\widehat{C}). \qquad \Box
\end{aligned}$$

Proposition 2. *(Geodesic Curvature Comparison) Let Γ be a C^2 curve in M^n, a manifold with sectional curvatures $\leq -\kappa^2$, and let C be the cone $p ⋇ \Gamma$. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as in Introduction above, then $k(q) \geq \widehat{k}(q)$ for almost all $q \in \Gamma$, where k and $\widehat{k}$ denote the inward geodesic curvatures of Γ in C and $\widehat{C}$, respectively.*

Proof. We first assume that $C\backslash\{p\}$ is immersed. For $\rho_0 > 0$, let $\Gamma_0 = C \cap \partial B_{\rho_0}(p)$, and let k_0 be the geodesic curvature of Γ_0 in C. Also, let $\widehat{k}_0$ be the geodesic curvature of Γ_0 in $\widehat{C}$. To estimate k_0 and $\widehat{k}_0$ let us define V ($\widehat{V}$, respectively) to be a Jacobi field in C ($\widehat{C}$, respectively) along the unit-speed geodesic γ from p to $q \in \Gamma$, satisfying

$$V(p) = \widehat{V}(p) = 0 \text{ and } V \perp \dot{\gamma},\ \widehat{V} \perp \dot{\gamma}. \tag{6}$$

For each $q \in \Gamma$, since $g = \widehat{g}$ along Γ, we may also impose the boundary conditions

$$V(q) = \widehat{V}(q), |V(q)| = |\widehat{V}(q)| = 1, \tag{7}$$

thereby determining V and $\widehat{V}$ uniquely, since K and $\widehat{K}$, the Gauss curvatures of C and $\widehat{C}$ respectively, are nonpositive. In fact, $V = \widehat{V}$ as vector fields on $C\backslash\{p\}$. V and $\widehat{V}$ satisfy the Jacobi equations

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}V = R(\dot{\gamma}, V)\dot{\gamma} \ \text{ and } \ \widehat{\nabla}_{\dot{\gamma}}\widehat{\nabla}_{\dot{\gamma}}\widehat{V} = \widehat{R}(\dot{\gamma}, \widehat{V})\dot{\gamma}, \tag{8}$$

where $\nabla, \widehat{\nabla}$ denote the connections for the metrics $g, \widehat{g}$ respectively, while $R, \widehat{R}$ denote the Riemann curvature tensors of g and $\widehat{g}$, respectively. Write $f(t) = ||V(\gamma(t))||$, and similarly $\widehat{f}(t) = ||\widehat{V}(\gamma(t))||$, where the norms are measured using g and $\widehat{g}$, respectively. Since C and $\widehat{C}$ have dimension 2, equations (8) are equivalent to the scalar Jacobi equations

$$f''(t) + K(\gamma(t))f(t) = 0, \ \ \widehat{f}''(t) + \widehat{K}(\gamma(t))\widehat{f}(t) = 0. \tag{9}$$

By the Gauss equation we have

$$K = R_M(\dot{\gamma}, V, V, \dot{\gamma})/||V||^2 + \det(B),$$

where R_M is the Riemann curvature tensor of M and B is the second fundamental form of C in M. Since C is a cone, we have $\det(B) = 0$, and it follows that C has Gauss curvature

$$K \leq -\kappa^2.$$

We next compute k_0 and $\widehat{k}_0$. Extend V and $\widehat{V}$ as normal Jacobi fields along all radial geodesics from p. Also, let W be the unit vector field which is tangent to the radial geodesics. Then $[V, W] \equiv 0$ and $\langle V, W\rangle \equiv 0$. Similarly, $[\widehat{V}, W] \equiv 0$ and $\langle \widehat{V}, W\rangle \equiv 0$. Then

$$||V||^2 k_0 = -\langle \overline{\nabla}_V V, W\rangle = \langle V, \overline{\nabla}_V W\rangle = \langle V, \overline{\nabla}_{\dot{\gamma}} V\rangle = \dot{\gamma}(||V||^2)/2 = f'(t)f(t).$$

Thus $k_0(\gamma(t)) = f'(t)/f(t)$. Similarly, we compute $\widehat{k}_0(\gamma(t)) = \widehat{f}'(t)/\widehat{f}(t)$. As is well known, the scalar Jacobi equations (9) are equivalent to the Riccati equations

$$k_0'(\gamma(t)) + k_0(\gamma(t))^2 = -K(\gamma(t)) \geq \kappa^2,$$

and

$$\widehat{k}_0'(\gamma(t)) + \widehat{k}_0(\gamma(t))^2 = -\widehat{K}(\gamma(t)) = \kappa^2.$$

It follows that the difference satisfies a homogeneous linear differential inequality

$$(k_0 - \widehat{k}_0)' + (k_0 + \widehat{k}_0)(k_0 - \widehat{k}_0) = -K + \widehat{K} \geq 0.$$

Meanwhile, $k_0 - \widehat{k}_0 = (f'\widehat{f} - \widehat{f}'f)/(\widehat{f}f) \to 0$ as $t \to 0$, as follows from L'Hospital's rule using the equations (9). Therefore

$$f'/f - \widehat{f}'/\widehat{f} = k_0 - \widehat{k}_0 \geq 0. \tag{10}$$

We are now in a position to compare the respective inward geodesic curvatures k and $\widehat{k}$ of Γ. Write $T = (V/f)\cos\varphi - W\sin\varphi$ for the unit tangent vector to Γ: T has unit length with respect to either metric g or $\widehat{g}$. Then $\nabla_T T = -k\,\nu_C$ and $\widehat{\nabla}_T T = -\widehat{k}\,\nu_C$, where $\nu_C = (V/f)\sin\varphi + W\cos\varphi$ is the outward unit normal vector to Γ, with respect to either metric, and $\cos\varphi \geq 0$. We compute $\nabla_W W = \nabla_W(V/f) = 0,\ \nabla_{V/f}(V/f) = -k_0 W$ and $\nabla_V W = k_0 V$. It follows in a straightforward fashion that $-k\,\nu_C = \nabla_T T = -k_0\,\nu_C\cos\varphi - \nu_C T(\varphi)$. Thus $k = k_0\cos\varphi + T(\varphi)$, and similarly $\widehat{k} = \widehat{k}_0\cos\varphi + T(\varphi)$. Hence

$$k - \widehat{k} = (k_0 - \widehat{k}_0)\cos\varphi \geq 0. \qquad \square$$

Remark 1. The proof of Proposition 2 holds more generally, for any two metrics $g,\ \widehat{g}$ on a cone which have the same unit-speed geodesics from the vertex, agree at the boundary, and whose Gaussian curvatures satisfy $K \leq \widehat{K}$.

Remark 2. The metric $\widehat{g}$ can be called *ray preserving* in that the geodesics from p are preserved under $\widehat{g}$. As a matter of fact, $\widehat{g}$ can be made ray preserving without having constant Gaussian curvature. Such a metric $\widehat{g}$ can be obtained through a *ray preserving map* h between two surfaces S and $\widehat{S}$ as follows. Let p be a point of S and $\Gamma \subset S$ an open curve parametrized by arclength s. Let $\phi(s)$ be the angle between the tangent vector to Γ at $\Gamma(s)$ and the geodesic from p to $\Gamma(s)$ and let $r(s)$ be the distance from p to $\Gamma(s)$. For simplicity let us assume $0 < \phi(s) < \pi$, i.e., $\Gamma(s)$ is moving in one direction when viewed from p. We want to find a ray preserving map $h : p⋇\Gamma \subset S \to \widehat{S}$ which preserves the distance $r(s)$ from p to $\Gamma(s)$ and the arclength element of Γ. h can be found by constructing a curve $\widehat{\Gamma}$ which we want to be the image of Γ under h. Suppose $\widehat{\Gamma}$ is parametrized by arclength s, with distance $\widehat{r}(s)$ from $\widehat{p} = h(p)$, and making an angle of $\widehat{\phi}(s)$ with the geodesic from $\widehat{p}$. In order for h to be ray preserving, $\widehat{\Gamma}$ should satisfy $r(s) = \widehat{r}(s)$. However, it is easier to require $\widehat{\Gamma}$ to satisfy

$$\phi(s) = \widehat{\phi}(s) \tag{11}$$

because (11) implies $dr/ds = d\widehat{r}/ds$ and hence $r(s) = \widehat{r}(s)$. (11) gives an ODE on $\widehat{S}$ and $\widehat{\Gamma}$ is its unique solution. Therefore h is ray preserving, and the pull-back of $\widehat{S}$ under h gives a ray preserving metric on S.

Proposition 3. *(Density and Area Comparison) Let Γ be a C^2 curve in M^n, and let $C = p⋇\Gamma$, as in Proposition 2. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as in Introduction above, then the densities $\Theta_C(p) \leq \Theta_{\widehat{C}}(p)$ and the areas* $\mathrm{Area}(C) \leq \mathrm{Area}(\widehat{C})$.

Proof. The inequality (10) above implies that $f(t)/\widehat{f}(t)$ is increasing. Recalling the normalization $f = \widehat{f}$ at each $q \in \Gamma$ and $f = \widehat{f} = 0$ at p, we see that $f(t) \leq \widehat{f}(t)$ along $\gamma,\ f' \geq$

$\widehat{f}'$ at q, and $f' \leq \widehat{f}'$ at p. Note that $\mathrm{Area}(C)$ and $\mathrm{Area}(\widehat{C})$ may be written as the same double integral with respective integrands f and $\widehat{f}$. $\square$

Proposition 4. *(Gauss-Bonnet)* **(a)** *For any geodesic cone $\widehat{C} = p \times\!\!\!\!\times \Gamma, p \notin \Gamma$, with constant curvature $-\kappa^2$ over an immersed C^2 curve Γ in M^n, $n \geq 2$,*

$$2\pi\Theta_{\widehat{C}}(p) + \kappa^2\,\mathrm{Area}(\widehat{C}) = \int_\Gamma \widehat{k}\; ds,$$

where $\widehat{k}$ is the geodesic curvature of Γ in $\widehat{C}$.
(b) *If $p \in \Gamma$, then*

$$2\pi\Theta_{\widehat{C}}(p) + \kappa^2\,\mathrm{Area}(\widehat{C}) = \int_\Gamma \widehat{k}\; ds - \pi.$$

Proof. **(a)** Consider $p \notin \Gamma$. By the Gauss-Bonnet formula on $\widehat{C}_\varepsilon := \widehat{C} \backslash B_\varepsilon(p)$,

$$\int_{\widehat{C}_\varepsilon} \widehat{K}\, dA + \int_\Gamma \widehat{k}\, ds + \int_{\widehat{C}\cap\partial B_\varepsilon(p)} \widehat{k}\, ds = 2\pi\chi(\widehat{C}_\varepsilon) = 0, \tag{12}$$

where $\widehat{K} \equiv -\kappa^2$ is the intrinsic Gauss curvature of $\widehat{C}_\varepsilon$. Since $\widehat{C}_\varepsilon$ is an immersed annulus, the Euler number $\chi(\widehat{C}_\varepsilon) = 0$.

The geodesic curvature of $\widehat{C} \cap \partial B_\varepsilon(p)$ is the negative of the curvature of $\partial B_\varepsilon(p)$ as a curve in $\mathbf{H}^2(-\kappa^2)$, namely, $-\kappa \coth \kappa\varepsilon$. Thus,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\widehat{C}\cap\partial B_\varepsilon(p)} \widehat{k}\, ds &= -\lim_{\varepsilon\to 0} (\kappa \coth \kappa\varepsilon) L(\widehat{C} \cap \partial B_\varepsilon(p)) \\
&= -\lim_{\varepsilon\to 0} (\cosh \kappa\varepsilon) 2\pi\Theta_{\widehat{C}}(p) = -2\pi\Theta_{\widehat{C}}(p).
\end{aligned}$$

Since $\mathrm{Area}(\widehat{C}_\varepsilon) \to \mathrm{Area}(\widehat{C})$, the Gauss-Bonnet formula (12) now implies

$$-\kappa^2\,\mathrm{Area}(\widehat{C}) + \int_\Gamma \widehat{k}\, ds - 2\pi\Theta_{\widehat{C}}(p) = 0, \tag{13}$$

which proves part **(a)**.

The proof of part **(b)** is analogous. However, when $p \in \Gamma$, for small ε, $\widehat{C}_\varepsilon$ is a topological disk, so that $\chi(\widehat{C}_\varepsilon) = 1$. Also, the boundary of $\widehat{C}_\varepsilon$ consists of the arc $\widehat{C} \cap \partial B_\varepsilon(p)$ and the arc $\Gamma_\varepsilon := \Gamma \backslash B_\varepsilon(p)$. For small $\varepsilon > 0$, these arcs meet at two points forming exterior angles $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. Equation (12) becomes

$$-\int_{\widehat{C}_\varepsilon} \kappa^2\, dA + \int_{\Gamma_\varepsilon} \widehat{k}\, ds - \int_{\widehat{C}\cap\partial B_\varepsilon(p)} \vec{k} \cdot \nu_C\, ds + \alpha(\varepsilon) + \beta(\varepsilon) = 2\pi.$$

Since Γ is smooth, $\alpha(\varepsilon) \to \pi/2$ and $\beta(\varepsilon) \to \pi/2$ as $\varepsilon \to 0$, which yields

$$-\kappa^2\,\mathrm{Area}(\widehat{C}) + \int_\Gamma \widehat{k}\, ds - 2\pi\Theta_{\widehat{C}}(p) = \pi. \qquad \square \tag{14}$$

3. Embeddedness of Minimal Surfaces in Negatively Curved Spaces

After the formidable problem of Plateau in Euclidean $\mathbf{R}^n$ was settled by Douglas and Radó in 1930, mathematicians' attention was drawn to the uniqueness and embeddedness of their solutions (see [D] and [R1].). The first uniqueness result was obtained by Radó ([R2], p. 100). He proved that if a simple closed curve $\Gamma \subset \mathbf{R}^3$ has a one-to-one projection onto the boundary of a convex region $R \subset \mathbf{R}^2$, then Γ bounds a unique minimal disk. In fact any minimal surface bounded by Γ is a graph over R, and hence is simply connected and embedded. Later Nitsche [N2] showed that if Γ is analytic with total curvature $\leq 4\pi$, then Γ bounds exactly one minimal disk.

The embeddedness of the minimal disk bounded by a Jordan curve Γ was first obtained by Gulliver and Spruck [GS] under the assumption that Γ has total curvature $\leq 4\pi$ and is extreme (that is, it lies on the boundary of a convex set). In the same paper, they conjectured that either condition alone would be sufficient for the embeddedness of an area-minimizing disk. Indeed Tomi-Tromba [TT], Almgren-Simon [AS], and Meeks-Yau [MY] derived the embeddedness of a minimal disk bounded by an extreme Γ. But the sufficiency of the total curvature condition alone, when Γ is not assumed to be extreme, remained open for 25 years.

However, in a very recent paper, Ekholm, White, and Wienholtz [EWW] ingeniously proved the embeddedness of any minimal surface bounded by a curve Γ in $\mathbf{R}^n$ with total curvature $\leq 4\pi$.

In this section we introduce a result of [CG3] which extends the Ekholm-White-Wienholtz result to minimal surfaces in an n-dimensional Riemannian manifold M with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. It is proved that if Γ is a Jordan curve in M^n with total curvature

$$\mathcal{C}_{\rm tot}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2 \inf_{p \in M} \mathrm{Area}(p \times \Gamma),$$

then every branched minimal surface bounded by Γ is embedded (Theorem 1.)

Definition 1. *Define the* minimum cone area *of* Γ *as*

$$\mathcal{A}(\Gamma) := \inf_{p \in \mathcal{H}_{\rm cvx}(\Gamma)} \mathrm{Area}(p \times \Gamma).$$

Theorem 1. *Let Σ^2 be a branched minimal surface (of arbitrary topological type) in an n-dimensional complete, simply connected Riemannian manifold M whose sectional curvature is bounded above by a nonpositive constant $-\kappa^2$. Write $\Gamma = \partial\Sigma$, which we assume to be a C^2 Jordan curve, i. e. a C^2 embedding of the circle S^1. If the total curvature of Γ satisfies*

$$\mathcal{C}_{\rm tot}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2 \mathcal{A}(\Gamma), \tag{15}$$

then $\overline{\Sigma}$ is an embedding.

Proof. Let Σ^2 be a branched minimal surface in M whose boundary $\partial\Sigma = \Gamma$ is a C^2 Jordan curve satisfying the hypothesis (15):

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2 \mathcal{A}(\Gamma),$$

where $-\kappa^2$ is an upper bound on sectional curvatures of the ambient manifold M. We need to show that $\overline{\Sigma}$ has no branch points and is embedded. Thus, it will suffice to show that $\Theta_\Sigma(p) < 2$ at all $p \in M \backslash \Gamma$ and that $\Theta_\Sigma(p) < 3/2$ at $p \in \Gamma$.

Consider any $p \in \Sigma \backslash \Gamma$, and let $C = p \divideontimes \Gamma$ be the geodesic cone over Γ with vertex p. If Σ is totally geodesic, then Σ is embedded, since there are no compact totally geodesic surfaces and no geodesic loops in M. Otherwise, by Proposition 1 and Proposition 4**(a)**, we have

$$2\pi\Theta_\Sigma(p) < 2\pi\Theta_{\widehat{C}}(p) = \int_\Gamma \widehat{k}\, ds - \kappa^2 \operatorname{Area}(\widehat{C}).$$

Recall that $\Sigma \subset \mathcal{H}_{\mathrm{cvx}}(\Gamma)$. Hence Proposition 3 implies that $\operatorname{Area}(\widehat{C})$ is at least equal to the minimum cone area $\mathcal{A}(\Gamma)$, and since $\widehat{k} \leq k \leq |\vec{k}|$ almost everywhere along Γ by Proposition 2, we find

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\mathrm{tot}}(\Gamma) - \kappa^2 \mathcal{A}(\Gamma).$$

Therefore, hypothesis (15) implies $\Theta_\Sigma(p) < 2$. If $p \in \Gamma$, apply Proposition 4**(b)** to show $\Theta_\Sigma(p) < 3/2$. Then the embedded character of $\overline{\Sigma}$ follows. $\square$

Theorem 1 implies a substantial extension of the Fáry-Milnor Theorem, which was proved for $\kappa = 0$ in [AB] and [S].

Theorem 2. *Let Γ be a C^2 Jordan curve in a complete, simply connected Riemannian 3-manifold M with sectional curvature $\leq -\kappa^2$. If the total curvature of Γ satisfies*

$$\int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2\, \mathcal{A}(\Gamma),$$

then Γ is unknotted.

The readers are referred to [CG3] for the proof of Theorem 2.

Example 1. *This example shows that the hypothesis*

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) \leq 4\pi + \kappa^2 \mathcal{A}(\Gamma)$$

of Theorems 1 and 2 is sharp.

Let Γ_0 be the double cover of the circle of radius R in a totally geodesic $\mathbf{H}^2 \subset \mathbf{H}^3$. Here $\mathbf{H}^n$ is the n-dimensional hyperbolic space of constant sectional curvature $-\kappa^2 = -1$. Given any choice of positive integer m, the example is a one-parameter family of $(2, 2m+1)$-torus knots Γ_η in $\mathbf{H}^3,\ \eta > 0$, with $\Gamma_\eta \to \Gamma_0$ and with

$$\mathcal{C}_{\mathrm{tot}}(\Gamma_\eta) < 4\pi + \mathcal{A}(\Gamma_\eta) + \eta.$$

In fact, Γ_0 has length $4\pi \sinh R$, curvature $|\vec{k}| \equiv \coth R$, $\mathcal{C}_{\mathrm{tot}}(\Gamma_0) = 4\pi \cosh R$, and $\mathcal{A}(\Gamma_0) = 4\pi(\cosh R - 1)$. $\square$

4. Isoperimetric Inequality of Minimal Surfaces in a Negatively Curved Riemannian Manifold

Let D be a domain in a simply connected surface of constant Gauss curvature K. The area A of D and the perimeter L satisfy the isoperimetric inequality

$$4\pi A \ \leq \ L^2 + KA^2, \tag{16}$$

where equality holds if and only if D is a geodesic disk. The case $K = 0$ was proved by Steiner in 1842 [S], $K > 0$ by Bernstein in 1905 [B], and $K < 0$ by Schmidt in 1940 [Sc].

Let M be a simply connected Riemannian manifold of constant sectional curvature K. The isoperimetric inequality (16) holds for any domain on a totally geodesic surface in M. Since a totally geodesic surface is minimal in M, it has been naturally conjectured that (16) should hold for every minimal surface in M.

The first result of this nature is due to Carleman [C], who showed in 1921 that (16) holds for a simply connected domain on a minimal surface in $\mathbf{R}^n$. So far (16) has been proved only for minimal surfaces with one or two boundary components in $\mathbf{R}^n$ [LSY, C1] and in H^n [CG1].

Consider minimal surfaces in a simply connected Riemannian manifold M of varying sectional curvature. Suppose the sectional curvature of M is bounded above by a constant $K = -\kappa^2$. In this section we introduce the result of [C2] which proves that (16) holds also for a minimal surface Σ with one or two boundary components in M when $K \leq 0$.

Proposition 1(a) implies that if $p \in \Sigma$, then

$$2\pi \leq \Theta_{\widehat{C}}(p). \tag{17}$$

Hence from the cutting and pasting arguments and the approximation argument as in Lemma 4 of [CG1] it follows that

$$4\pi \mathrm{Area}(\widehat{C}) \ \leq \ \mathrm{Length}(\partial\Sigma)^2 + K\mathrm{Area}(\widehat{C})^2.$$

Therefore using Proposition 1(b) and the monotonicity of the quadratic function $4\pi A - KA^2$ of $A > 0$, we obtain the desired isoperimetric inequality for Σ in case $K < 0$.

If equality holds in the isoperimetric inequality, then

$$\mathrm{Area}(\Sigma) \ = \ \mathrm{Area}(\widehat{C})$$

and therefore equality should hold in Lemma 1(b). Consequently equality holds in (4) and $|\nabla r| \equiv 1$ on Σ as we easily see in the proof of Lemma 1(b). It follows that $\Sigma = p ⋇ \partial\Sigma$ and, by Index Lemma, Σ is constantly curved and hence totally geodesic. Thus Schmidt's theorem [Sm] completes the proof in case $K < 0$.

The theorem for $K = 0$ follows from (17), Proposition 1(b) and the arguments of [C1].

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