CAPILLARY SURFACES IN A CONVEX CONE

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Spheres and the Delaunay surfaces have long been known as surfaces of constant mean curvature(CMC) in $\mathbb{R}^3$. The former is compact and round and the latter is noncompact and rotational. In 1950's Alexandrov [Al] showed that a compact embedded CMC hypersurface in $\mathbb{R}^{n+1}$ is a round hypersphere, and Hopf [Ho] proved that an immersed CMC sphere is a round sphere. However, contrary to Hopf's conjecture, Wente [We] constructed an immersed CMC torus. Since then many compact immersed CMC surfaces have been found [Ka], [Ab].

The simplest compact CMC surface with nonempty boundary is a spherical cap. In fact Nitsche [Ni] showed that an immersed disk-type CMC surface in a ball which makes a constant contact angle with the boundary sphere is a spherical cap. Moreover the first-named author [Ch] proved that in a domain $D \subset \mathbb{R}^3$ bounded by planes or spheres every immersed disk-type CMC surface in D which makes a constant contact angle with ∂D and has less than four vertices is part of a round sphere. The upper bound of three on the number of vertices in this result is critical in applying the Poincaré-Hopf theorem.

Now one can ask a natural question: When are the capillary surfaces with more than three vertices in a domain with piecewise umbilic boundary necessarily part of a sphere? A capillary surface in a domain D is a CMC surface making a constant contact angle with ∂D, assuming no influence of gravity. In this paper we show that if $S \subset \mathbb{R}^{n+1}$ is a compact embedded hypersurface with constant higher order mean curvature in a convex piecewise smooth cone C which is perpendicular to ∂C, then S is part of a round hypersphere (Theorem 1). Furthermore, if an embedded capillary surface in a convex polyhedral cone, i.e., a domain bounded by planes containing a point has nonnegative Gaussian curvature near ∂S and if the number of faces of C is at most 7, then S is part of a round sphere (Theorem 2).

In the proof of Theorem 1, the Minkowski formula and the Reilly formula are used as in [Ro]. For Theorem 2 we use the Poincaré-Hopf theorem [Ch] and the Bonnet transform [Pa].

1. Minkowski formula

In order to prove Theorem 1 for hypersurfaces in $\mathbb{R}^{n+1}$, we need to extend the Minkowski formula in this section and introduce the Reilly formula in the next section.

Throughout this paper the volume forms in the integrals will be dropped for notational convenience.

Let S be a compact immersed hypersurface in $\mathbb{R}^{n+1}$ and A its volume. When S is embedded, S encloses a domain whose volume is denoted by V. Even when S has self intersection and nonempty boundary one can naturally define the enclosed volume V with respect to $p \in S$ to be the volume of the cone $p \divideontimes S$ by counting multiplicity. If η denotes the outward unit normal to S and $X(p)$ the position vector of S at p, then we have

$$(n+1)V = \int_S \langle X, \eta \rangle. \tag{1}$$

One can easily get (1) by integrating $\bar{\Delta}|X|^2 = 2(n+1)$ on S, where $\bar{\Delta}$ is the Laplacian on $\mathbb{R}^{n+1}$. On the other hand, the first variation of S under the homothetic expansion in $\mathbb{R}^{n+1}$ gives

$$A = \int_S H \langle X, \eta \rangle, \tag{2}$$

where H is the mean curvature of S. (2) can also be obtained by integrating

$$\frac{1}{2n} \Delta |X|^2 = 1 - H \langle X, \eta \rangle \tag{3}$$

on S, where Δ is the Laplacian on S.

Minkowski generalized (1) and (2) as follows. Let S_t be a parallel surface of S, i.e., the set of all points with distance t from S in η direction. If dS and κ_1,...,κ_n denote the volume form and the principal curvatures of S, respectively, then the volume form of S_t is

$$dS_t = (1+\kappa_1 t) \cdots (1+\kappa_n t) dS = P_n(t) dS, \tag{4}$$

where

$$P_n(t) := (1+\kappa_1 t) \cdots (1+\kappa_n t) = 1 + \begin{pmatrix} n \\ 1 \end{pmatrix} H_1\, t + \cdots + \begin{pmatrix} n \\ n \end{pmatrix} H_n\, t^n.$$

Being the elementary symmetric polynomial of degree k in κ_1,...,κ_n, H_k is called the k-th *order mean curvature* of S ($H_1 = H$). Furthermore the mean curvature of S_t is

$$H(t) = \frac{1}{n} \sum_i \frac{\kappa_i}{1+\kappa_i t} = \frac{{P_n}'(t)}{n P_n(t)}.$$

Hence, integrating (3) on S_t with $\partial S_t = \phi$, we get for all sufficiently small t

$$0 = \int_{S_t} \{1 - H(t) \langle X + t\eta, \eta \rangle\} = \int_S \{P_n(t) - \frac{t}{n} {P_n}'(t) - \frac{1}{n} {P_n}'(t) \langle X, \eta \rangle\}. \quad \text{(by (4))} \tag{5}$$

Equating the like terms in (5) yeilds the Minkowski formula:

$$\int_S (H_{k-1} - H_k \langle X, \eta \rangle) = 0, \quad k = 1, ..., n, \quad \text{with } H_0 = 1. \tag{6}$$

We now obtain the Minkowski formula for immersed hypersurfaces with nonempty boundary in the following.

Proposition 1. *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary and with vertex at the origin. Let S be an immersed hypersurface in $\mathbb{R}^{n+1}$ with boundary in ∂C such that near ∂S S is inside C and perpendicular to ∂C. Then we have*

$$\int_S (H_{k-1} - H_k\langle X, \eta\rangle) = 0, \quad k = 1, ..., n. \tag{7}$$

2. Reilly formula

A basic tool in tensor analysis is the Ricci identity: If X, Y, Z are vector fields on a Riemannian manifold M with curvature tensor R and α is a 1-form on M, then

$$((\nabla_X\nabla_Y - \nabla_Y\nabla_X)\alpha)\,(Z) = \alpha(R(X, Y)Z).$$

Given a smooth function f on M, one can obtain Bochner's formula by applying the Ricci identity to df and taking trace:

$$\langle \Delta df, df\rangle = |\nabla df|^2 + \frac{1}{2}\Delta|df|^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

If we integrate Bochner's formula on a domain D in M^{n+1} and use the Stokes theorem, we can get the Reilly formula [Re]:

$$\int_D \{(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \mathrm{Ric}(\bar{\nabla} f, \bar{\nabla} f)\} = \int_{\partial D} \{(2\Delta f + nH\frac{\partial f}{\partial\eta})\frac{\partial f}{\partial\eta} + \mathrm{II}(\nabla f, \nabla f)\}.$$

where $\bar{\Delta} f, \bar{\nabla}^2 f, \bar{\nabla} f$ are the Laplacian, the Hessian, the gradient of f in M and $\Delta f, \nabla f, H, \eta, \mathrm{II}$ are the Laplacian of f, the gradient of f, the mean curvature, the outward unit normal, and the second fundamental form of ∂D, respectively.

Ros [Ro] used the Reilly formula to prove $\int_S 1/H \geq (n+1)V$ for a compact embedded hypersurface $S \subset \mathbb{R}^{n+1}$. We extend his result as follows.

Proposition 2. *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary and with vertex at the origin O. Let $S \subset C$ be an embedded hypersurface with boundary in ∂C such that S is perpendicular to ∂C along ∂S. Let H be the mean curvature of S and V the volume of the domain D enclosed by S and ∂C. If $H > 0$ on S, then*

$$\int_S \frac{1}{H} \geq (n+1)V \tag{8}$$

and equality holds if and only if S is a spherical cap.

3. Hypersurfaces with constant H_ℓ

With Propositions 1 and 2 we are now ready to prove the second theorem.

Theorem 1. *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary and with vertex at the origin. Let $S \subset C$ be an embedded hypersurface of constant ℓ-th order mean curvature with boundary in ∂C such that S is perpendicular to ∂C along ∂S. Then S is a spherical cap.*

4. Capillary surfaces in $\mathbb{R}^3$

In this section, we give a sufficient condition for a disk type capillary surface S (with nonzero constant mean curvature) in a convex polyhedral cone $C \subset \mathbb{R}^3$ to be spherical. Taylor's boundary regularity theorem for capillary surfaces [T] implies that S is analytic up to ∂S. In the following, we assume that each analytic component of ∂S is of $C^{2,\alpha}$ up to the singular points, which are called the vertices. Let O be the vertex of C. Let us label the faces of C by W_i, $i = 1, \ldots, n$, and denote the vertex $S \cap \overline{W}_i \cap \overline{W}_{i+1}$ by v_i. Since the curve $S_i = S \cap W_i$ is a curvature line of W_i, the Joachimstahl's theorem [Sp] implies that S_i is a curvature line of S. Let κ_i be the curvature of S_i. Then the principal curvature of S in the tangent direction of S_i is $\kappa_i \sin\gamma$.

Theorem 2. *Let S be a disk type capillary surface in a convex polyhedral cone C with contact angle $\gamma < \pi/2$. Suppose that S has only one vertex on each edge and ∂S is $C^{2,\alpha}$ up to the vertices. If the number of faces of C is at most 7, then S is spherical.*

If the contact angle is bigger than $\pi/2$, then $|X|$ increases as X moves from the boundary in the inward conormal direction. Therefore $\max_{X\in S}|X|$ is attained at an interior point, which implies that the mean curvature vector $\vec{H}$ points inside. Similarly, if the contact angle is $< \pi/2$, then $\vec{H}$ points outside. Let $\eta_H = \vec{H}/H$ and let B be the second fundamental form of X. Though the case $\gamma = \pi/2$ was treated under more general condition in Theorem 1, we include $\gamma = \pi/2$ in the following to make the arguments simpler and clearer.

First we recall the parallel H-surface, denoted by $\tilde{S}$, of S. The position vector $\tilde{X}$ for a point of $\tilde{S}$ (corresponding to $p \in S$) is given by

$$\tilde{X} = HX(p) + \eta_H(p).$$

Let us fix a a conformal coordinate $w = u + iv$ on S and let K be the Gaussian curvature of S. It is straighforward to see that

$$|\tilde{X}_u|^2 = |\tilde{X}_v|^2 = (H^2 - K)|X_u|^2, \ \langle \tilde{X}_u, \tilde{X}_v\rangle = 0, \tag{9}$$

$$\tilde{X}_u \wedge \tilde{X}_v = -(H^2 - K)X_u \wedge X_v, \tag{10}$$

For the intrinsic Laplacian Δ_S of S, we have

$$\Delta_S X = 2H\eta_H,$$
$$\Delta_S\ \eta_H = -|B|^2\eta_H.$$

From the above equations, it follows that

$$\Delta_{\tilde{S}}\tilde{X} \quad = \quad 2\frac{\tilde{X}_u \wedge \tilde{X}_v}{|\tilde{X}_u \wedge \tilde{X}_v|}. \tag{11}$$

Equation (9) says that $\tilde{S}$ is singular at the image of umbilic points of S. In fact, $\tilde{S}$ is branched at the singular points. From (10) and (11), we see that $\tilde{S}$ has mean curvature 1 and the mean curvature vector of $\tilde{S}$ is $-\eta_H$. Since S meets W_i at constant angle γ, $\tilde{S}_i$ is contatined in a plane P_i, which is parallel to W_i with distance $\cos\gamma$. The regularity assumption on S guarantees that $\tilde{S}$ is well-defined at the vertices of S and $\tilde{v}_i$ lies on the line $P_i \cap P_{i+1}$. The P_i's define two special cones C_1 and C_2: C_1 is a parallel translation of C and C_2 is the reflection of C_1 about the vertex of C_1.

Now we introduce the rotation index for a family of curvature lines on S [Ho]. Let

$$II = Ldu^2 + 2Mdudv + Ndv^2$$

be the second fundamental form of S. The Hopf differential Φdw^2 is a quadratic differential defined by

$$\Phi(w, \bar{w}) = L - N - 2iM.$$

Since the Hopf differential is holomorphic on cmc surfaces in $\mathbb{R}^3$ [Ho], the zeros of Φ are isolated unless it is identically zero. Let us fix one family $\mathcal{F}$ of curvature lines on S. The rotation index I of $\mathcal{F}$ is given by

$$I = -\frac{1}{4\pi}\delta(\arg\Phi),$$

where δ denotes the variation along a small curve around p in the positive sense.

The first-named author generalized the ratation index to cmc surfaces with boundary [Ch]. Lemma 2 of [Ch] says that

i) The rotation index of a nonvertex boundary umbilic point is $\leq -\frac{1}{4}$.

ii) The rotation index of a vertex with angle $< \pi$ *is not bigger than* $\frac{1}{4}$. *If the vertex is umbilic, then the rotation index is nonpositive. If the angle of a vertex is* $> \pi$, *then the rotation index is* $\leq -\frac{1}{4}$.

In the following, we assume that the mean curvature of S is 1.

Lemma 1. *Assume that the conditions of Theorem 2 hold. For two consecutive faces of* C, *say* W_1 *and* W_2, *it is impossible for the curvatures of* κ_1 *and* κ_2 *of* S_1 *and* S_2 *to satisfy* $\kappa_1\sin\gamma < 1$ *and* $\kappa_2\sin\gamma > 1$ *or* $\kappa_1\sin\gamma > 1$ *and* $\kappa_2\sin\gamma < 1$. *(The curvatures are computed with respect to* η_H*.) Therefore there is at least one umbilic point on* S_1 *or* S_2.

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