THE PERIODIC PLATEAU PROBLEM AND ITS APPLICATION

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Abstract. Given a noncompact disconnected periodic curve Γ of infinite length with two components and no self-intersection in $\mathbb{R}^3$, it is proved that there exists a noncompact simply connected periodic minimal surface spanning Γ. As an application, it is shown that for any tetrahedron T with dihedral angles $\leq 90°$, there exist four embedded minimal annuli in T, which are perpendicular to ∂T along their boundary. It is also proved that every Platonic solid of $\mathbb{R}^3$ contains a free boundary embedded minimal surface of genus zero.

Keywords: Plateau problem, helically periodic, minimal surface, free boundary, Platonic solid
MSC: 53A10, 49Q05

1. introduction

The famous problem of finding a surface of the least area spanning a given Jordan curve called the Plateau problem, was settled by Douglas and Radó independently in 1931. Since then, many questions have been raised about the Douglas-Radó solution: the uniqueness, the embeddedness, the topology of the solution, and the number of solutions.

In this paper, we are concerned with the Plateau problem for a noncompact disconnected complete curve Γ with two components in $\mathbb{R}^3$, which is periodic. Γ is said to be periodic if Γ has a fundamental piece $\bar{\gamma}$ with two components such that Γ is the infinite union of the congruent copies of $\bar{\gamma}$ obtained in a periodic way. In particular, Γ is *helically periodic* if it is the union of images of $\bar{\gamma}$ under the cyclic group $\langle\sigma\rangle$ generated by a screw motion σ. Γ is *translationally periodic* if it is

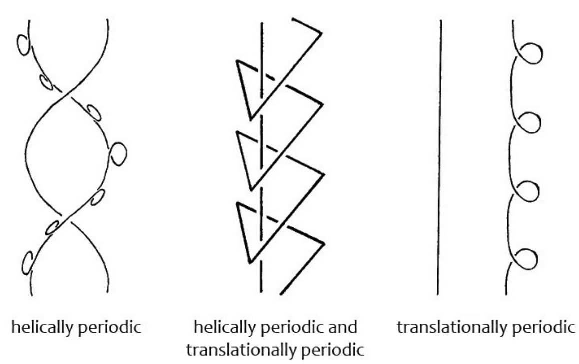

Figure 1.

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invariant under the cyclic group $\langle\tau\rangle$ generated by a translation τ(see Figure 1). The extensions by screw motions and translations are to be performed infinitely until Γ becomes complete.

We prove that for every complete noncompact disconnected periodic curve Γ in $\mathbb{R}^3$ there exists a noncompact simply connected minimal surface $\Sigma \subset \mathbb{R}^3$ spanning Γ such that Σ inherits the periodicity of Γ (Theorem 2.2). Furthermore, in case Γ consists of the x_3-axis and a complete connected translationally periodic curve γ_1 winding around the x_3-axis such that a fundamental piece of γ_1 admits a one-to-one orthogonal projection onto a convex closed curve in the x_1x_2-plane, we can show that Σ is unique and embedded (Theorem 3.1). These two theorems have an interesting application. Smyth [8] showed that given a tetrahedron T, there exist three embedded minimal disks in T which meet ∂T orthogonally along their boundary. From T, Smyth considered a quadrilateral Γ whose edges are perpendicular to the faces of T. Γ bounds a unique minimal graph Σ. He then showed that the conjugate minimal surface of Σ is the desired minimal surface in T.

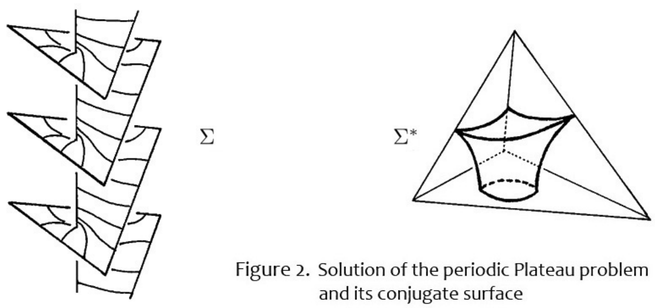

Figure 2. Solution of the periodic Plateau problem and its conjugate surface

In this paper, we will first see that the tetrahedron T gives rise to a noncompact, disconnected, translationally periodic, piecewise linear curve Γ such that the edges (=line segments) of a fundamental piece $\bar{\gamma}$ of Γ are perpendicular to the faces of T. In fact, $\bar{\gamma}$ has two components $\bar{\gamma}_0$, $\bar{\gamma}_1$, where $\bar{\gamma}_0$ has only one edge and $\bar{\gamma}_1$ has 3 edges. So one of the two components of Γ is a straight line ℓ. By Theorem 2.2 Γ bounds a noncompact simply connected translationally periodic minimal surface Σ. Let Σ^* be its conjugate minimal surface. In Theorem 5.1, we will prove that if ℓ is properly chosen relative to $\bar{\gamma}_1$, then Σ^* is a minimal annulus in T which is perpendicular to ∂T (see Figure 2). One boundary component of Σ^* is a convex closed curve lying in one face of T and the other component traces along the remaining three faces. Since there are four lines perpendicular to a face of T we conclude that there exist four free boundary minimal annuli in T if the dihedral angles of T are $\leq 90°$. If at least one dihedral angle of T is $> 90°$, there exist four minimal annuli that are not necessarily inside T but still perpendicular to the planes containing the faces of T along their boundary.

In general, one cannot generalize Theorem 5.1 to construct a free boundary minimal annulus in a polyhedron other than a tetrahedron. However, in case P_y is a right pyramid with a regular polygonal base B and apex p (i.e., $P_y = p \times B$, the cone), we can show the existence of a free boundary minimal annulus Σ^* in P_y(Theorem 6.1). Consequently, it is proved that every Platonic solid P_s bounded by regular n-gons contains a free boundary embedded minimal surface of genus 0.

2. Periodic Plateau problem

A Jordan curve is simple and closed. So, it has no self-intersection and is homeomorphic to a circle. If a simple curve $\Gamma \subset \mathbb{R}^3$ is not closed but homeomorphic to

$\mathbb{R}^1$ and has infinite length, one cannot, in general, find a minimal surface spanning Γ. However, if there exists a surface of finite area spanning Γ, one can easily show the existence of a minimal surface spanning Γ. The same is true if Γ is the union of simple open curves of infinite length bounding a surface of finite area. If Γ cannot bound a surface of finite area, one must impose extra conditions on Γ to get a minimal surface spanning Γ. This section shows that the periodicity of Γ is a sufficient condition for this purpose. Our proof for this periodic case follows the standard proof for a Jordan curve Γ, modifying the arguments in each periodic situation.

Definition 2.1. a) Let $\Gamma \subset \mathbb{R}^3$ be the union of two complete rectifiable curves γ_1, γ_2 and let U be a convex polyhedral domain in $\mathbb{R}^3$. Γ is said to be *periodic* if Γ is the infinite union of the congruent copies of $\bar{\gamma} := \Gamma \cap U$. $\bar{\gamma}$ is called a *fundamental piece* of Γ.

b) Γ is said to be *translationally periodic* if it is the union of translated fundamental pieces $\tau^n(\bar{\gamma})$ for the cyclic group $\langle\tau\rangle$ generated by a parallel translation τ. Γ is invariant under $\langle\tau\rangle$. Moreover, Γ is said to be *helically periodic* if it is the union of $\sigma^n(\bar{\gamma})$ for the cyclic group $\langle\sigma\rangle$ generated by a screw motion σ. Assume that the screw motion σ is the rotation about the x_3-axis by angle β composed with the translation by e, that is,

$$\sigma(r\cos\theta, r\sin\theta, x_3) = (r\cos(\theta+\beta), r\sin(\theta+\beta), x_3+e). \tag{2.1}$$

Every translationally periodic Γ can be helically periodic as well with respect to σ for $\beta = 0$.

c) Given a surface Σ spanning Γ, the periodicity of Σ can be defined in the same way as Γ.

Γ is complete because translations and screw motions are performed infinitely.

Theorem 2.2. *Let $\Gamma \subset \mathbb{R}^3$ be the union of two complete curves γ_1, γ_2, which is helically periodic. Then there exists a simply connected helically periodic minimal surface Σ spanning Γ. The fundamental region of Σ has the least area among the fundamental regions of all the simply connected helically periodic surfaces spanning Γ. The same conclusion holds for the translationally periodic Γ as well.*

Proof. Let's first prove the theorem when Γ is helically periodic. We assume that Γ is invariant under the σ defined by (2.1). We may further assume that σ maps the fundamental piece $\bar{\gamma}$ of Γ to its adjoining piece, that is, $\bar{\gamma}$ is connected to $\sigma(\bar{\gamma})$ through their common endpoints. Γ uniquely determines the angle $\beta > 0$ of (2.1), which we call the *period* of Γ. $\hat{\Sigma}$ is a fundamental region of Σ if and only if

$$\Sigma = \bigcup_{k\in\mathbb{Z}} \sigma^k(\hat{\Sigma}) \text{ and } \hat{\Sigma} \cap \sigma(\hat{\Sigma}) = \emptyset.$$

Definition 2.3. To each complete helically periodic curve Γ we associate the class $\mathcal{C}_{a,\Gamma}$ of admissible maps φ from the infinite strip $I_a := [0,a] \times \mathbb{R}$ to $\mathbb{R}^3$ with the following properties:

(1) φ is a piecewise C^1 immersion in the interior of I_a and is continuous in I_a;
(2) $\varphi(x, y+k\beta) = \sigma^k(\varphi(x,y))$, $(x,y) \in I_a$, k : integer, β : fixed > 0;
(3) $\varphi\big|_{\partial I_a}$ is a *monotone* map onto Γ, i.e., the set $\varphi^{-1}(p)$ is connected for each $p \in \Gamma$.

To normalize $\mathcal{C}_{a,\Gamma}$ let's assume that $\varphi(0,0) = p$ for a fixed point p of Γ. φ is said to be *invariant under the screw motion* σ with *period* β if φ satisfies property (2).

Define the *area functional* A on $\mathcal{C}_{a,\Gamma}$ by

$$A(\varphi) = \int\int_{[0,a]\times[0,\beta]} |\varphi_x \wedge \varphi_y| dx\, dy$$

and the *Dirichlet integral* $D(\varphi)$ of $\varphi \in \mathcal{C}_{a,\Gamma}$ by

$$D(\varphi) = \int\int_{[0,a]\times[0,\beta]} |\nabla\varphi|^2 dx\, dy.$$

Since

$$|\varphi_x \wedge \varphi_y| \leq \frac{1}{2}\left(|\varphi_x|^2 + |\varphi_y|^2\right)$$

we have

$$A(\varphi) \leq \frac{1}{2} D(\varphi),\ \ \varphi \in \mathcal{C}_{a,\Gamma} \tag{2.2}$$

where equality holds if and only if φ is almost conformal. To obtain the equality case, we must prove the existence of periodic isothermal coordinates invariant under σ on the surface $\varphi(I_a)$.

Proposition 2.4. *For any $\varphi \in \mathcal{C}_{a,\Gamma}$ there exist $\bar{b} > 0$ and a periodic homeomorphism $H : I_a \to I_{\bar{b}} := [0,\bar{b}] \times \mathbb{R}$ such that H^{-1} has period β and the reparametrized map $\varphi \circ H^{-1} : I_{\bar{b}} \to \varphi(I_a)$ is a conformal map in $\mathcal{C}_{\bar{b},\Gamma}$.*

Proof. Let N be the annulus obtained from $[0,a] \times [0,\beta]$ by identifying the two line segments $[0,a] \times \{0,\beta\}$. Let g be the metric on N which is pulled back by φ from the metric of $\varphi(I_a)$. g is well-defined since φ is invariant under the screw motion σ determined by Γ. Let's consider the Dirichlet problem on (N, g) for a constant $b > 0$:

$$\Delta u = 0,\ \ u = 0 \text{ on } \{0\} \times [0,\beta],\ \ u = b \text{ on } \{a\} \times [0,\beta].$$

A unique solution $u = h_b$ exists for this problem. The harmonic function h_b has a conjugate harmonic function h_b^* which is multi-valued on (N, g). But h_b^* is well-defined on its universal cover $\widetilde{N} = I_a$. Let $\tau(b) > 0$ be the period of h_b^* on N. $\tau(b)$ is an increasing function which varies from 0 to ∞ as b does so. Hence there exists $\bar{b} > 0$ such that $\tau(\bar{b}) = \beta$. Note that $h_{\bar{b}}$ can also be lifted to $h_{\bar{b}}$ on I_a. Then the map $H : I_a \to I_{\bar{b}}$ defined by $H(q) = (h_{\bar{b}}(q), h^*_{\bar{b}}(q))$ is a periodic homeomorphism and yields a conformal map $\varphi \circ H^{-1} : I_{\bar{b}} \to \varphi(I_a)$. Note that H^{-1} has period β and $\varphi \circ H^{-1}$ is invariant under the screw motion σ. This completes the proof of the proposition. □

To prove the existence of an area-minimizing surface spanning Γ, let's define

$$a_\Gamma = \inf_{\varphi \in \mathcal{C}_{a,\Gamma},\, a>0} A(\varphi) \ \text{ and } \ d_\Gamma = \inf_{\varphi \in \mathcal{C}_{a,\Gamma},\, a>0} D(\varphi).$$

Then by (2.2) and the existence of the isothermal coordinates, we have

$$a_\Gamma = \frac{1}{2} d_\Gamma.$$

Therefore

$$D(\psi) = d_\Gamma \text{ for some } \psi \in \mathcal{C}_{a,\Gamma} \iff A(\psi) = a_\Gamma \text{ and } \psi \text{ is almost conformal.}$$

Thus, to solve the periodic Plateau problem it suffices to find $\bar{a} > 0$ and a map $\psi \in \mathcal{C}_{\bar{a},\Gamma}$ which minimizes the Dirichlet integral $D(\varphi)$ on $[0,a] \times [0,\beta]$ among all φ in $\mathcal{C}_{a,\Gamma}$ and all $a > 0$. First we shall fix $a > 0$ and apply the periodic Dirichlet principle on $\mathcal{C}_{a,\Gamma}$ as follows.

Lemma 2.5. *For each admissible map φ in $\mathcal{C}_{a,\Gamma}$ there exists a unique harmonic admissible map $\psi \in \mathcal{C}_{a,\Gamma}$ with $\psi|_{\partial I_a} = \varphi|_{\partial I_a}$. Moreover, $D(\psi) \leq D(\varphi)$.*

Proof. Let x, y be the Euclidean coordinates of $\mathbb{R}^2$ and set $t = x + iy$. Define

$$f_1(t) = e^{\pi it/a} \text{ and } f_2(z) = \frac{iz+1}{z+i}.$$

Then $z = f_1(t)$ maps the infinite vertical strip I_a one-to-one onto the upper half plane $\{\operatorname{Im} z \geq 0\} \setminus \{0\}$ and $w = f_2(z)$ maps $\{\operatorname{Im} z \geq 0\} \setminus \{0\}$ one-to-one onto the unit disk $\{|w| \leq 1\} \setminus \{i, -i\}$. Furthermore, we see that $f_2(f_1(\partial I_a)) = \{|w| = 1\} \setminus \{i, -i\}$. Let's consider the vector-valued Dirichlet problem for $u = (u_1, u_2, u_3)$ in $D := \{w : |w| < 1\}$:

$$\Delta u = 0 \text{ in } D, \quad u = \varphi \circ {f_1}^{-1} \circ f_2^{-1} \text{ on } \partial D, \quad \varphi = (\varphi_1, \varphi_2, \varphi_3). \tag{2.3}$$

Since φ satisfies $\varphi(x, y + k\beta) = \sigma^k(\varphi(x, y))$ for the screw motion σ defined by (2.1), we see that φ_1, φ_2 are bounded and

$$\varphi_3(x, y + k\beta) = \varphi_3(x, y) + ke. \tag{2.4}$$

The Dirichlet problem (2.3) has a unique bounded solution for u_1, u_2 because of the boundedness of φ_1, φ_2. Even though φ_3 is unbounded, by (2.4) $\varphi_3 - \frac{e}{\beta}y$ is bounded and periodic in I_a. So, if the Dirichlet problem

$$\Delta v = 0 \text{ in } I_a, \quad v = \varphi_3 - \frac{e}{\beta}y \text{ on } \partial I_a \tag{2.5}$$

has a bounded solution, it must be unique and periodic with period β. To find its bounded solution, we convert it to a new Dirichlet problem on D:

$$\Delta w = 0 \text{ in } D, \quad w = (\varphi_3 - \frac{e}{\beta}y) \circ {f_1}^{-1} \circ f_2^{-1} \text{ on } \partial D. \tag{2.6}$$

The boundedness of $(\varphi_3 - \frac{e}{\beta}y) \circ {f_1}^{-1} \circ f_2^{-1}$ gives the existence of a unique bounded solution $w = \tilde{h}_3$ to (2.6). As $\frac{e}{\beta}y \circ f_1^{-1} \circ f_2^{-1}$ is harmonic in D, it is easy to see that $u_3 := \tilde{h}_3 + \frac{e}{\beta}y \circ f_1^{-1} \circ f_2^{-1}$ is the third component of a desired solution to (2.3).

Pulling back (u_1, u_2, u_3) by $f_2 \circ f_1$ to I_a, one can obtain a harmonic map $\psi : I_a \to \mathbb{R}^3$ having the same boundary value as φ on ∂I_a. We now show that ψ is invariant under the screw motion σ, in other words,

$$\psi(x, y + \beta) = \sigma(\psi(x, y)).$$

Let $h_1, h_2, h_3 : I_a \to \mathbb{R}$ be the harmonic components of ψ, that is,

$$\psi(x, y) = (h_1(x, y), h_2(x, y), h_3(x, y)).$$

(One easily sees that $h_3 = \tilde{h}_3 \circ f_2 \circ f_1 + \frac{e}{\beta}y$.) Define

$$\psi_A(x, y) = \psi(x, y + \beta) \text{ and } \psi_B(x, y) = \sigma(\psi(x, y)).$$

Since $\tilde{h}_3 \circ f_2 \circ f_1$ is periodic with period β, we have

$$h_3(x, y + \beta) = h_3(x, y) + e.$$

So the third component of $\psi_A(x, y)$ equals that of $\psi_B(x, y)$. On the other hand,

$$\psi_B(x, y) = (\cos\beta\, h_1(x, y) - \sin\beta\, h_2(x, y), \sin\beta\, h_1(x, y) + \cos\beta\, h_2(x, y), h_3(x, y) + e).$$

Hence ψ_A, ψ_B are harmonic maps. As h_1, h_2 are bounded, so is $\psi_A - \psi_B$. Since $\sigma(\Gamma) = \Gamma$, $\psi_A - \psi_B$ vanishes on ∂I_a. Then $(\psi_A - \psi_B) \circ f_1^{-1} \circ f_2^{-1}$ is a bounded harmonic map vanishing on ∂D and so $\psi_A - \psi_B \equiv 0$. Therefore ψ is invariant under σ. ψ is a unique admissible harmonic map in $\mathcal{C}_{a,\Gamma}$ having the same boundary values as φ.

Set $\Phi = \varphi - \psi$. Then Φ is also invariant under σ and hence

$$D(\varphi) = D(\Phi) + D(\psi) + 2D(\Phi, \psi)$$

where

$$D(\Phi,\psi)=\int\int_{[0,a]\times[0,\beta]}\left(\langle\frac{\partial\Phi}{\partial x},\frac{\partial\psi}{\partial x}\rangle+\langle\frac{\partial\Phi}{\partial y},\frac{\partial\psi}{\partial y}\rangle\right)dxdy.$$

Green's identity implies that

$$D(\Phi,\psi)=\int_{\partial([0,a]\times[0,\beta])}\langle\Phi,\frac{\partial\psi}{\partial\nu}\rangle ds-\int\int_{[0,a]\times[0,\beta]}\langle\Phi,\Delta\psi\rangle dxdy,$$

where ν is the outward unit normal to $\partial([0,a]\times[0,\beta])$. But

$$\Phi=0 \text{ on } \{0,a\}\times[0,\beta] \quad\text{and}\quad \frac{\partial\psi}{\partial\nu}\Big|_{[0,a]\times\{\beta\}}=-\frac{\partial\psi}{\partial\nu}\Big|_{[0,a]\times\{0\}}$$

because of the invariance of ψ under σ. Hence $D(\Phi,\psi)=0$. It then follows that

$$D(\psi)\le D(\varphi),$$

which completes the proof of the lemma. □

Define

$$d_{a,\Gamma}=\inf_{\varphi\in\mathcal{C}_{a,\Gamma}}D(\varphi).$$

We claim here that $d_{a,\Gamma}$ goes to infinity as $a\to\infty$ and as $a\to 0$.

$$\begin{aligned}D(\varphi)&\ge\int_0^a\int_0^\beta|\varphi_y|^2dydx=\int_0^a\int_0^\beta\sum_{i=1}^3\left(\frac{\partial\varphi_i}{\partial y}\right)^2dydx\\&\ge\frac{1}{\beta}\int_0^a\left(\int_0^\beta\frac{\partial\varphi_3}{\partial y}dy\right)^2dx=\frac{1}{\beta}\int_0^a(\varphi_3(x,\beta)-\varphi_3(x,0))^2dx\\&=\frac{ae^2}{\beta}.\end{aligned}$$

So $\lim_{a\to\infty}d_{a,\Gamma}=\infty$. On the other hand,

$$\begin{aligned}D(\varphi)&\ge\int_0^\beta\int_0^a|\varphi_x|^2dxdy=\int_0^\beta\int_0^a\sum_{i=1}^3\left(\frac{\partial\varphi_i}{\partial x}\right)^2dxdy\\&\ge\frac{1}{a}\int_0^\beta\sum_{i=1}^3\left(\int_0^a\frac{\partial\varphi_i}{\partial x}dx\right)^2dy=\frac{1}{a}\int_0^\beta\sum_{i=1}^3(\varphi_i(a,y)-\varphi_i(0,y))^2dy\\&\ge\frac{\beta d^2}{a},\end{aligned}$$

where d is the distance between the two components γ_0, γ_1 of Γ which are written as $\gamma_0=\varphi(\{0\}\times\mathbb{R}),\gamma_1=\varphi(\{a\}\times\mathbb{R})$. Hence $\lim_{a\to 0}d_{a,\Gamma}=\infty$ as well.

Therefore, we can conclude that there exists a positive constant $\bar{a}$ such that

$$d_\Gamma=d_{\bar{a},\Gamma}.$$

To finish the proof of Theorem 2.2 we need the following.

Lemma 2.6. *Let M be a constant $>d_\Gamma$. Then, for any $a>0$, the family of functions*

$$\mathcal{F}_a=\{\varphi|_{\partial I_a}:\varphi\in\mathcal{C}_{a,\Gamma},\ D(\varphi)\le M\}$$

is compact in the topology of uniform convergence.

Proof. For each $z \in \partial I_a$ and each $r > 0$, define C_r to be the intersection of I_a with the circle of radius r centered at z, and denote by s the arc length parameter of C_r. Choose any $\varphi \in \mathcal{C}_{a,\Gamma}$ with $D(\varphi) \leq M$. For $0 < \delta < \min(1, a^2)$, consider the integral

$$K := \int_\delta^{\sqrt{\delta}} \int_{C_r} |\varphi_s|^2 ds\, dr \leq D(\varphi) \leq M.$$

One can see that

$$K = \int_\delta^{\sqrt{\delta}} f(r)\, d(\log r), \quad f(r) := r \int_{C_r} |\varphi_s|^2 ds.$$

By the mean value theorem there exists ρ with $\delta \leq \rho \leq \sqrt{\delta}$ such that

$$K = f(\rho) \int_\delta^{\sqrt{\delta}} d(\log r) = \frac{1}{2}\, f(\rho) \log(\frac{1}{\delta}).$$

Hence

$$\int_{C_\rho} |\varphi_s|^2 ds \leq \frac{2M}{\rho \log(\frac{1}{\delta})}.$$

Denote the length of the curve $\varphi(C_r)$ by $L(\varphi(C_r))$. Then $L(\varphi(C_\rho)) = \int_{C_\rho} |\varphi_s| ds$ and from the Cauchy-Schwarz inequality it follows that

$$L(\varphi(C_\rho))^2 \leq \frac{2\pi M}{\log(\frac{1}{\delta})}. \tag{2.7}$$

Given a number $\varepsilon > 0$, by the compactness of $\Gamma/\langle\sigma\rangle$ we see that there exists $d > 0$ such that for any p, p' in Γ with $0 < |pp'| < d$, the diameter of the bounded component of $\Gamma \setminus \{p, p'\}$ is smaller than ε. Choose $\delta < \min(1, a^2)$ such that $\frac{2\pi M}{\log(\frac{1}{\delta})} < d^2$. Then for any $z \in \partial I_a$, there exists a number ρ with $\delta < \rho < \sqrt{\delta}$ such that by (2.7), $L(\varphi(C_\rho)) < d$. Let E_z be the interval in ∂I_a between z_1 and z_2, the two endpoints of C_ρ. Then $|\varphi(z_1)\varphi(z_2)| < d$ and hence the diameter of $\varphi(E_z)$ is smaller than ε. Therefore for any $z, z' \in \partial I_a$ with $|z - z'| < \delta$ and z being the center of C_ρ, we have $\varphi(z), \varphi(z') \in \varphi(E_z)$ and thus

$$|\varphi(z) - \varphi(z')| < \varepsilon.$$

Since δ was chosen independently of z, z' and φ, we obtain the equicontinuity of $\mathcal{F}_a$.

In Douglas's solution for the existence of a conformal harmonic map $\varphi : D \to \mathbb{R}^n$ spanning a Jordan curve Γ, it was essential to prescribe $\varphi(z_i) = p_i$ for arbitrarily chosen points $z_1, z_2, z_3 \in \partial D$ and $p_1, p_2, p_3 \in \Gamma$. This was to derive the equicontinuity in a minimizing sequence. Fortunately, we do not need this prescription for our compact set $\Gamma/\langle\sigma\rangle$ as $\varphi(I_a)/\langle\sigma\rangle$ is not a disk. Yet we need to avoid an unwanted situation resulting from the disconnectedness of Γ: we have to show that $\varphi(\{a\} \times \mathbb{R})$ does not drift away from $\varphi(\{0\} \times \mathbb{R})$ (recall that $\varphi(0, 0)$ is fixed). This can be done by deriving a length bound from a bound on $D(\varphi)$ as above.

For each $y \in [0, \beta]$ let ℓ_y denote the line segment $[0, a] \times \{y\}$. Choose $\varphi \in \mathcal{C}_{a,\Gamma}$ and suppose $D(\varphi) \leq M$. Consider the integral

$$K := \int_0^\beta \int_{\ell_y} |\varphi_x|^2 dx\, dy \leq D(\varphi) \leq M.$$

Then

$$K = \int_0^\beta \tilde{f}(y)\, dy, \quad \tilde{f}(y) := \int_{\ell_y} |\varphi_x|^2 dx.$$

The mean value theorem implies that there exists $0 < \bar{y} < \beta$ such that

$$K = \beta\, \tilde{f}(\bar{y}) \leq M.$$

Hence

$$L(\varphi(\ell_{\bar{y}}))^2 = \left(\int_{\ell_{\bar{y}}} |\varphi_x| dx\right)^2 \le a\,\tilde{f}(\bar{y}) \le \frac{aM}{\beta}. \tag{2.8}$$

We say that $\varphi(\{a\} \times \mathbb{R})$ drifts away from $\varphi(\{0\} \times \mathbb{R})$ if $\lim_{x\to a} |\varphi_3(x,y)| = \infty$ for some $0 \le y \le \beta$. Therefore (2.8) means that no drift occurs under φ if $D(\varphi)$ is bounded, as claimed. Thus, by Arzela's theorem, the equicontinuity yields the compactness of $\mathcal{F}_a$. This completes the proof of Lemma 2.6. □

Finally, let $\{\varphi_n\}$ be a minimizing sequence in $\mathcal{C}_{\bar{a},\Gamma}$, that is, $\lim_{n\to\infty} D(\varphi_n) = d_\Gamma$. From Lemma 2.6 it follows that there exists a subsequence $\{\varphi_{n_i}\}$ such that $\{\varphi_{n_i}|_{\partial I_{\bar{a}}}\}$ converges uniformly to $\bar{\varphi}|_{\partial I_{\bar{a}}}$ for some $\bar{\varphi} \in \mathcal{C}_{\bar{a},\Gamma}$. By Lemma 2.5 there exist harmonic maps $\psi_i, \psi \in \mathcal{C}_{\bar{a},\Gamma}$ such that

$$\psi_i|_{\partial I_{\bar{a}}} = \varphi_{n_i}|_{\partial I_{\bar{a}}}, \quad D(\psi_i) \le D(\varphi_{n_i}), \quad \psi|_{\partial I_{\bar{a}}} = \bar{\varphi}|_{\partial I_{\bar{a}}}, \quad \psi = \lim_{i\to\infty} \psi_i.$$

Then, Harnack's principle gives

$$D(\psi) \le \liminf_i D(\psi_i) \le d_\Gamma.$$

Consequently, $D(\psi) = d_\Gamma$ and so ψ is almost conformal and harmonic. This completes the proof of Theorem 2.2 when Γ is helically periodic and, therefore, when it is also translationally periodic. Since ψ is periodically area minimizing in $\mathbb{R}^3$ it has no interior branch point (see [3]). □

3. Uniqueness and Embeddedness

Under what condition can Γ guarantee the uniqueness and embeddedness of the periodic Plateau solution Σ? For the Douglas solution with Jordan curve Γ, Nitsche [4] and Ekholm-White-Wienholtz [2] proved the uniqueness and the embeddedness, respectively, if the total curvature of $\Gamma \le 4\pi$. But even before Douglas, Radó [6] showed that the Dirichlet solution of the minimal surface equation for any continuous boundary data over the boundary of a convex domain in $\mathbb{R}^2$ exists as a graph, which is unique and embedded. In the same spirit, we have a partial answer for our periodic Plateau problem.

Theorem 3.1. *Let γ_0 be the x_3-axis and γ_1 a complete connected curve winding around γ_0. Define $\Gamma = \gamma_0 \cup \gamma_1$ and let τ be a vertical translation by e. If Γ is translationally periodic with respect to τ and a fundamental piece of γ_1 admits a one-to-one orthogonal projection onto a convex closed curve in the x_1x_2-plane, then the translationally periodic minimal surface Σ spanning Γ has the following properties:*

- (a) *The Gaussian curvature of Σ is negative at any point $p \in \gamma_0$;*
- (b) *Σ is embedded and its fundamental region (not including γ_0) is a graph over its projection onto the x_1x_2-plane;*
- (c) *Σ is unique.*

Proof. (a) γ_0 is parametrized by x_3. At any point $p(x_3)$ of γ_0, Σ has a tangent *half plane* $Q_{p(x_3)}$. In a neighborhood of $p(x_3)$, Σ is divided by $Q_{p(x_3)}$, like a half pie, into $m(\ge 2)$ regions (see Figure 3). Define $\theta(x_3)$ to be the angle between $Q_{p(x_3)}$ and the positive x_1-axis. $\theta(x_3)$ is a well-defined analytic function satisfying $\theta(x_3 + e) = \theta(x_3) + 2\pi$. It is known (to be proved shortly) that

$$m = 2 \text{ at } p(x_3) \ \Leftrightarrow\ K(x_3) < 0 \ \Leftrightarrow\ \theta'(x_3) \ne 0, \tag{3.1}$$

where $K(x_3)$ is the Gaussian curvature of Σ at $p(x_3)$.

We claim that $m \equiv 2$ on γ_0. Suppose $m \geq 3$ at $p(x_3)$ so that $Q_{p(x_3)} \cap \Sigma \setminus \gamma_0$ is the union of at least two analytic curves $C_1, C_2, ..., C_k$ emanating from $p(x_3)$. Since $Q_{p(x_3)}$ intersects γ_1, at least one of $C_1, C_2, ..., C_k$ should reach γ_1. So we have two possibilities: either **(i)** only one of them, say C_1, reaches γ_1, or **(ii)** two of them, say C_1, C_2, reach γ_1 (see Figure 3). In the first case, since C_2 is disjoint from γ_1

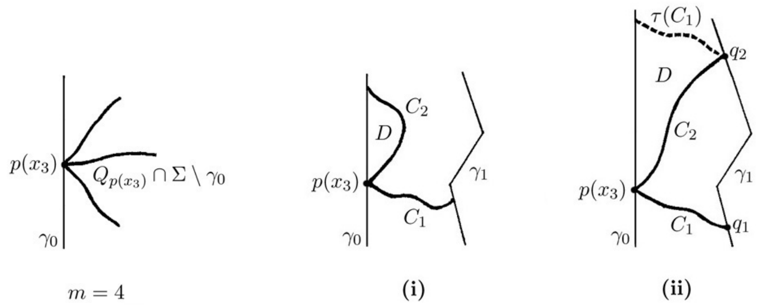

Figure 3. Intersection with the tangent half plane

and translationally periodic, it cannot be unbounded and should be in a fundamental region of Σ. Hence C_2 comes back to γ_0. C_2 and γ_0 should then bound a domain $D \subset \Sigma$ with $\partial D \subset Q_{p(x_3)}$ as Σ is simply connected. But this contradicts the maximum principle because D has a point that attains the maximum distance from $Q_{p(x_3)}$. In case of **(ii)**, set $C_1 \cap \gamma_1 = \{q_1\}$ and $C_2 \cap \gamma_1 = \{q_2\}$. Denote by π the projection onto the x_1x_2-plane. Due to the convexity of $\pi(\gamma_1)$, $Q_{p(x_3)}$ intersects any fundamental piece of γ_1 only at one point. Therefore $\{q_1, q_2\}$ should be the boundary of a fundamental piece of γ_1. Hence $\tau(q_1) = q_2$, interchanging q_1 and q_2 if necessary. So the two curves $\tau(C_1)$ and C_2 meet at q_2. Then $\tau(C_1)$, C_2 and γ_0 bound a domain $D \subset \Sigma$. Again ∂D is a subset of $Q_{p(x_3)}$, which contradicts the maximum principle. Therefore $m \equiv 2$ on γ_0, as claimed.

To give a proof of the equivalences (3.1), let's view Σ in a neighborhood of $p \in \gamma_0$ as a graph over Q_p, the tangent half plane of Σ at p. Introduce x, y, z as the coordinates of $\mathbb{R}^3$ such that $z \equiv 0$ on Q_p, $x \equiv 0$ on γ_0 and $p = (0, 0, 0)$. Then Σ is the graph of an analytic function $z = f(x, y)$ and the lowest order term of its Taylor series is $f_m(x, y) = c_m \operatorname{Im}(x + iy)^m, m \geq 2$, when m is an even integer and $f_m(x, y) = c_m \operatorname{Re}(x + iy)^m$ when m is odd. It follows that Σ is divided by Q_p into m regions in a neighborhood of p and that $K(p) = 0$ if $m \geq 3$ and $K(p) < 0$ if $m = 2$, which is the first equivalence in (3.1). Hence $K < 0$ on γ_0 by the claim above and this proves (a). The second equivalence follows from the expression for the Gaussian curvature in terms of the Weierstrass data on Σ, a 1-form fdz and the Gauss map g:

$$K = -\frac{16|g'|^2}{|f|^2(1 + |g|^2)^4}. \tag{3.2}$$

(b) First we show that $\Sigma \setminus \gamma_0$ has no vertical tangent plane. Suppose not; let q be an interior point of Σ at which the tangent plane P is vertical. Remember that $\pi(\gamma_1)$ is convex. Hence, P intersects γ_1 only at two points in its fundamental piece. $P \cap \Sigma$ is locally the union of at least four curves $C_1, \ldots, C_k, k \geq 4$, emanating from q, and two of them should reach γ_1. If we assume only four curves emanate from q in $P \cap \Sigma$, two of them will reach γ_1, and then either the remaining two will reach γ_0, or they will be connected to each other by the translation τ as in Figure 4: **(i)** C_1, C_2 will intersect γ_1 and C_3, C_4 will intersect γ_0; **(ii)** C_1, C_2 will intersect γ_1 and C_3, C_4 will be disjoint from $\gamma_0 \cup \gamma_1$ so that C_4 will be connected to $\tau(C_3)$.

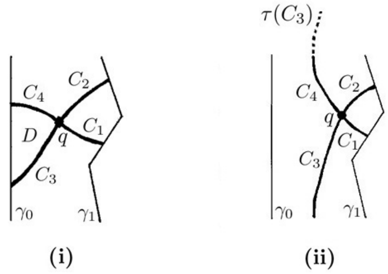

Figure 4. Intersection of Σ with its tangent plane

In case of **(i)**, $C_3 \cup C_4 \cup \gamma_0$ will bound a domain $D \subset \Sigma$. But this contradicts the maximum principle since $\partial D \subset P$. In case of **(ii)**, γ_0 is disjoint from P. Then γ_0 and $P \cap \Sigma$ bound an infinite strip $S \subset \Sigma$ lying on one side of P. Since $S/\langle\tau\rangle$ is compact, there exists a point $p_S \in S$ which has the maximum distance from P among all points of S. γ_0 is a constant distance away from P and the inward unit conormals to γ_0 on Σ wind around it once in its fundamental piece. So there is a point in γ_0 at which the inward unit conormal to γ_0 points away from P. Then, in that direction, the distance from P increases. Hence, p_S is not a point of γ_0 but an interior point of S. However, this contradicts the maximum principle. Consequently, no tangent plane to Σ can be vertical at any point of Σ_0. Even if $P \cap \Sigma$ consists of six curves or more, the same argument works.

We now show that the interior of Σ does not intersect γ_0. Let $\psi : [0, \bar{a}] \times \mathbb{R} \to \mathbb{R}^3$ be the periodically area minimizing conformal harmonic map such that $\psi([0, \bar{a}] \times \mathbb{R}) = \Sigma$, $\psi(\{0\} \times \mathbb{R}) = \gamma_0$ and $\psi(\{\bar{a}\} \times \mathbb{R}) = \gamma_1$. Suppose there exists an interior point $p \in (0, \bar{a}) \times \mathbb{R}$ such that Σ intersects γ_0 at $\psi(p)$. Define $f(q) = x_1(q)^2 + x_2(q)^2$ for $q \in \Sigma$. Let $\mathcal{F}$ be the family of all arcs on Σ connecting γ_0 to $\psi(p)$. Find a saddle point in Σ for the function f. Define

$$A = \min_{\alpha\in\mathcal{F}} \max_{q\in\alpha} f(q).$$

Clearly there exists a saddle point q_0 in Σ such that $f(q_0) = A$. Suppose $A = 0$. Then there is an arc $\tilde{\alpha} \subset [0, \bar{a}] \times \mathbb{R}$ connecting $\{0\} \times \mathbb{R}$ to p such that $f \equiv 0$ on $\psi(\tilde{\alpha})$. Since Σ periodically minimizes area, it has no interior branch point. Neither does Σ have a boundary branch point on γ_0. Hence ψ is an immersion on $[0, \bar{a}) \times \mathbb{R}$. But ψ maps $(\{0\} \times \mathbb{R}) \cup \tilde{\alpha}$ onto γ_0 if $f \equiv 0$ on $\psi(\tilde{\alpha})$. This is not possible for the immersion ψ. Hence, A cannot be equal to 0. Since $\nabla f = 0$ at q_0, the tangent plane to Σ at q_0 is parallel to γ_0, so it must be vertical. This is a contradiction. Therefore, the interior of Σ does not intersect γ_0.

Henceforth we show that $\hat{\Sigma} \setminus \gamma_0$ is a graph over the x_1x_2-plane, where $\hat{\Sigma}$ is a fundamental region of Σ. By (a), we know that $m \equiv 2$ on γ_0. Hence, given a vertical half plane Q emanating from γ_0 and a suitably chosen fundamental region $\hat{\Sigma}$ of Σ, $Q \cap \hat{\Sigma} \setminus \gamma_0$ is a single smooth curve joining γ_0 to γ_1. Since the interior of Σ does not intersect γ_0, the projection map $\pi|_{Q\cap\hat{\Sigma}\setminus\gamma_0}$ is one-to-one near γ_0. As $\pi(\gamma_1)$ is convex and $\pi|_{\hat{\Sigma}\cap\gamma_1}$ is one-to-one, $\pi(\Sigma)$ lies inside $\pi(\gamma_1)$ and $\pi|_{\hat{\Sigma}}$ is one-to-one near γ_1. Suppose the curve $Q \cap \hat{\Sigma} \setminus \gamma_0$ contains a point p at which its tangent line is vertical. Then, the tangent plane to Σ at p is also vertical, which is a contradiction. Hence $Q \cap \hat{\Sigma} \setminus \gamma_0$ admits a one-to-one projection into $\pi(Q)$ for all Q. It follows that $\hat{\Sigma} \setminus \gamma_0$ is a 2-dimensional graph over $\pi(\hat{\Sigma} \setminus \gamma_0)$. Hence, Σ is embedded.

(c) Suppose there exist two periodic Plateau solutions Σ_1, Σ_2 spanning Γ. Assume that their fundamental regions $\hat{\Sigma}_1, \hat{\Sigma}_2$ are the graphs of analytic functions f_1, f_2 :

$D \subset x_1x_2$-plane $\to \mathbb{R}$, $D := \pi(\Sigma_1 \setminus \gamma_0) = \pi(\Sigma_2 \setminus \gamma_0)$. Assume also that $f_1 \geq f_2$. If there exists an interior point $p \in D$ such that $(f_1 - f_2)(p) = \max_{q\in D}(f_1 - f_2)(q)$, we have a contradiction to the maximum principle. Hence, $f_1 - f_2$ has no interior maximum in D. Since $f_1 - f_2 \equiv 0$ on $\pi(\gamma_1)$, it can have a maximum only at $\pi(\gamma_0) = (0, 0)$. However, the maximum is attained *anglewise* as follows. Let $M = \sup_{q\in D}(f_1 - f_2)(q)$. Given a half plane Q emanating from γ_0, let $M_Q = \sup_{q\in Q\cap D}(f_1 - f_2)(q)$. Then $M = \max_Q M_Q$. Hence there exists a half-plane Q_1 emanating from γ_0 such that

$$M = \lim_{q\in\ell,\, q\to(0,0)} (f_1 - f_2)(q), \text{ where } \ell = Q_1 \cap D.$$

Then the parallel translate of Σ_2 by M, denoted as $\Sigma_2 + M$, still contains γ_0 as Σ_1 does, lies on one side of Σ_1 (above Σ_1) and is tangent to Σ_1 at $x_3 = q_1 := \lim_{q\in\ell,\, q\to(0,0)} f_1(q)$. Hence, by the boundary maximum principle(boundary point lemma), $f_2 + M \equiv f_1$, that is, $\Sigma_2 + M = \Sigma_1$. Since $\Sigma_2 + M$ spans $\Gamma + M$ and Σ_1 does Γ, M must equal 0 and thus follows the uniqueness of Σ. □

4. Smyth's Theorem

It was H.A. Schwarz [7] who first constructed a triply periodic minimal surface in $\mathbb{R}^3$. He started from a regular tetrahedron, four edges forming a Jordan curve, generating a unique minimal disk. Schwarz found this surface using specific Weierstrass data. By applying his reflection principle, he was able to extend the minimal disk across its linear boundary to obtain the D-surface. Then Schwarz introduced its conjugate surface, which he called the P-surface. This surface is embedded and triply periodic, just like the D-surface. Moreover, part of it is a free boundary minimal surface in a cube.

It is interesting to notice that both D-surface and P-surface have fundamental regions which are free boundary minimal disks in two specific tetrahedra, respectively. However, this is not an accident; B. Smyth [8] showed surprisingly that *any* tetrahedron contains as many as *three* free boundary minimal disks. In the remainder of the paper, we want to apply Smyth's method to the periodic Plateau solutions. To do so, we shall first review Smyth's theorem in this section.

Given a tetrahedron T in $\mathbb{R}^3$, let F_1, F_2, F_3, F_4 be its faces and $\nu_1, \nu_2, \nu_3, \nu_4$ the outward unit normals to the faces, respectively. Then, any three of $\nu_1, \nu_2, \nu_3, \nu_4$ are linearly independent, but all four are not. Hence there should exist positive numbers c_1, c_2, c_3, c_4 such that

$$c_1\nu_1 + c_2\nu_2 + c_3\nu_3 + c_4\nu_4 = 0. \tag{4.1}$$

We may assume

$$c_i = \text{Area}(F_i), \ \ i = 1, 2, 3, 4.$$

This is due to the divergence theorem applied on the domain T to the gradients of the harmonic functions x_1, x_2, x_3, the Euclidean coordinates of $\mathbb{R}^3$.

By (4.1) we see that there exists an oriented skew quadrilateral Γ whose edges (as vectors) are $c_1\nu_1, c_2\nu_2, c_3\nu_3, c_4\nu_4$. The Jordan curve Γ bounds a unique minimal disk Σ, which is the image $X(D)$ of a conformal harmonic map $X := (x_1, x_2, x_3)$. It is well known that x_1, x_2, x_3 are also harmonic on Σ. Hence, there exist their conjugate harmonic functions x_1^*, x_2^*, x_3^* on Σ. Then $X^* := (x_1^*, x_2^*, x_3^*)$ defines a conformal harmonic map from D onto Σ^* in $\mathbb{R}^3$. $X^* \circ X^{-1} : \Sigma \to \Sigma^*$ is a local isometry because of the Cauchy-Riemann equations. Therefore, Σ^* is a minimal surface locally isometric to Σ.

Let $y_i = b_i^1x_1 + b_i^2x_2 + b_i^3x_3$ be a linear function in $\mathbb{R}^3$ such that $\nabla y_i = c_i\nu_i$, $i = 1, 2, 3, 4$. Then y_i is constant$(= d_i)$ on the face F_i. Suppose u, v are isothermal coordinates on Σ such that v is constant along the edge $c_i\nu_i$. Then $dX(\frac{\partial}{\partial v})$ is perpendicular to the vector $c_i\nu_i$ on the edge $c_i\nu_i$. Hence $\frac{\partial y_i}{\partial v} = 0$, and by Cauchy-Riemann $\frac{\partial y_i^*}{\partial u} = 0$ on $c_i\nu_i$ as well, where $y_i^* := b_i^1 x_1^* + b_i^2 x_2^* + b_i^3 x_3^*$. Therefore y_i^* is constant along the edge $c_i\nu_i$, meaning that the image $X^*(c_i\nu_i)$ lies on the plane $\{y_i^* = d_i^*\}$ for some constant d_i^*.

$dX(\frac{\partial}{\partial u})$ is parallel to ∇y_i along the edge $c_i\nu_i$. By Cauchy-Riemann, there exists a number $c(p)$ at $p \in c_i\nu_i$ such that

$$c(p)(b_i^1, b_i^2, b_i^3) = dX(\frac{\partial}{\partial u}) = dX^*(\frac{\partial}{\partial v}). \tag{4.2}$$

Hence $dX^*(\frac{\partial}{\partial v})$ is parallel to (b_i^1, b_i^2, b_i^3). Therefore Σ^* is perpendicular to the plane $\{y_i^* = d_i^*\}$ along $X^*(c_i\nu_i)$. In conclusion, Σ^* is locally isometric to Σ and is a free boundary minimal surface in a tetrahedron T' similar to T. Thus T contains a free boundary minimal surface which is a homothetic expansion of Σ^*.

The skew quadrilateral Γ depends on the order of $c_1\nu_1, c_2\nu_2, c_3\nu_3, c_4\nu_4$. Any edge of the four can be chosen to be the first in a quadrilateral. Hence, there are $6 = 3!$ orderings of the four edges. But they can be paired off into three quadrilaterals with two opposite orientations. To be precise, for example, if the quadrilateral Γ_1 determined by four ordered vectors (u, v, w, x) is reversely traversed, we get the quadrilateral $-\Gamma_1$ for the ordering $(-u, -x, -w, -v)$. Define an orthogonal map $\xi(p) = -p$, $p \in \mathbb{R}^3$, then $\xi(-\Gamma_1)$ is the quadrilateral determined by (u, x, w, v). $\xi(-\Gamma_1)$ cannot be obtained from Γ_1 by a Euclidean motion. Even so, the two minimal disks spanning Γ_1 and $\xi(-\Gamma_1)$ are intrinsically isometric. Moreover, their conjugate surfaces are extrinsically isometric, i.e., they are identical modulo a Euclidean motion. Therefore, the six orderings of the four edges yield three geometrically distinct conjugate minimal disks, which, if properly expanded, will be free boundary minimal surfaces in T.

5. Free boundary minimal annulus

By generalizing Smyth's method to a translationally periodic solution of the periodic Plateau problem, we will construct four free boundary minimal annuli in a tetrahedron.

Theorem 5.1. *Let T be a tetrahedron with faces F_1, F_2, F_3, F_4 in $\mathbb{R}^3$ and let π_i be the orthogonal projection onto the plane P_i containing $F_i, i = 1, 2, 3, 4$.*

- (a) *If every dihedral angle of T is $\leq 90°$, there exist four free boundary minimal annuli A_1, A_2, A_3, A_4 in T.*
- (b) *If at least one dihedral angle of T is $> 90°$, there exist four minimal annuli $A_1, A_2, A_3,$ A_4 which are perpendicular to $\cup_{j=1}^4 P_j$ along ∂A_i. Part of A_i may lie outside T if a dihedral angle is nearer to $180°$.* (*See* Figure 5, right.) *Near ∂A_i, however, A_i lies in the same side of P_j as T does. Moreover, ∂A_i equals $\Gamma_i^1 \cup \Gamma_i^2$, where Γ_i^1 is a closed convex curve in P_i and Γ_i^2 is a closed, piecewise planar curve in $P_j \cup P_k \cup P_l$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*
- (c) *If the three dihedral angles along ∂F_i are $\leq 90°$, then A_i lies inside T. Γ_i^1 is a closed convex curve in F_i and Γ_i^2 is a closed, piecewise planar curve in $\partial T \setminus F_i$.* (*See* Figure 5, left.)
- (d) *Each planar curve in Γ_i^2 is convex and is perpendicular to the lines containing the edges of T at its endpoints.*
- (e) *A_i is an embedded graph over $\pi_i(A_i)$.*

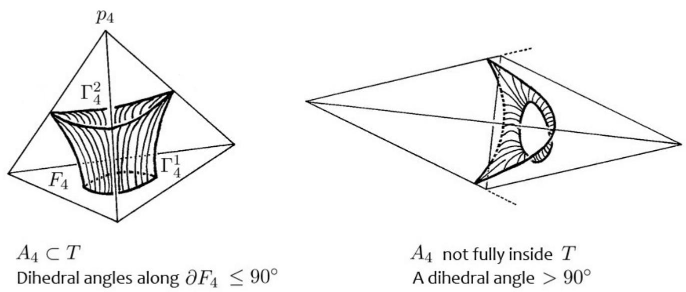

Figure 5.

Proof. As in the preceding section, ν_i denotes the outward unit normal to F_i. Again, there are positive constants $c_i = \text{Area}(F_i)$ such that $c_1\nu_1 + c_2\nu_2 + c_3\nu_3 + c_4\nu_4 = 0$. Assume that ν_4 is parallel to the x_3-axis so that F_4 is contained in the x_1x_2-plane. Denote the x_1x_2-plane by P_4 and recall that π_4 denotes the orthogonal projection onto P_4. Since

$$\pi_4(c_1\nu_1) + \pi_4(c_2\nu_2) + \pi_4(c_3\nu_3) = 0,$$

$\pi_4(c_1\nu_1), \pi_4(c_2\nu_2), \pi_4(c_3\nu_3)$ determine the boundary of a triangle $\Delta_4 \subset P_4$, that is, $\pi_4(c_i\nu_i)$ is the ith oriented edge of Δ_4, $i = 1, 2, 3$. $\pi_4(c_i\nu_i)$ is perpendicular to the boundary edge $F_i \cap F_4$ of F_4. Also $\pi_4(c_i\nu_i)$ is perpendicular to the corresponding edge of $J(\Delta_4)$, where J denotes the counterclockwise 90° rotation on P_4. Therefore Δ_4 is similar to F_4.

Choose a point q from the interior $\check{\Delta}_4$ of Δ_4 and let $\bar{\gamma}_q$ be the vertical line segment starting from q and corresponding to (i.e., having the same length and direction as) $-c_4\nu_4$. Let $\bar{\gamma}_1$ be a connected piecewise linear open curve starting from a vertex of Δ_4 that is the starting point of the oriented edge $\pi_4(c_1\nu_1)$ such that $\bar{\gamma}_1$ is the union of the three oriented line segments corresponding to the ordered vectors $c_1\nu_1, c_2\nu_2, c_3\nu_3$. Then $\pi_4(\bar{\gamma}_1) = \partial\Delta_4$. Also the endpoints of $\bar{\gamma}_1$ and $\bar{\gamma}_q$ are in Δ_4 and in its parallel translate. One can extend $\bar{\gamma}_q \cup \bar{\gamma}_1$ into a complete translationally periodic curve $\Gamma_q := \gamma_q \cup \gamma_1$ such that $\bar{\gamma}_q \cup \bar{\gamma}_1, \bar{\gamma}_q, \bar{\gamma}_1$ become fundamental pieces of $\Gamma_q, \gamma_q, \gamma_1$, respectively. By Theorem 2.2 and Theorem 3.1, there uniquely exists a simply connected minimal surface Σ_q spanning Γ_q. Σ_q has the same translational periodicity as Γ_q does. (See Figure 6.)

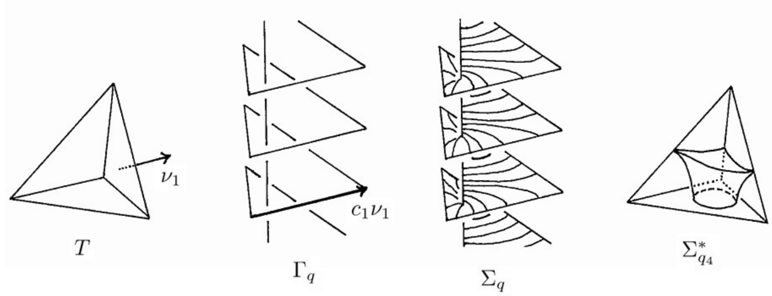

Figure 6. Construction procedure

Let Σ_q^* be the conjugate minimal surface of Σ_q and denote by $Y_q^* = X_q^* \circ X_q^{-1}$ the local isometry from Σ_q to Σ_q^*. By Smyth's arguments in the preceding section, we see that the image $Y_q^*(c_i\nu_i)$ of the edge $c_i\nu_i$ is in a plane parallel to the face F_i. More precisely, $Y_q^*(c_i\nu_i)$ lies in the plane $\{y_i^* = d_i^*\}$, where $\nabla y_i^* = c_i\nu_i$. However, $Y_q^*(\bar{\gamma}_q)$ is not closed in general because Y_q^* may have nonzero period along $\bar{\gamma}_q$. But note that by Cauchy-Riemann the period of Y_q^* along $Y_q^*(\bar\gamma_q)$ equals the flux of Σ_q along $\bar\gamma_q$. Therefore, to make Σ_q^* a well-defined compact minimal annulus, we need to find a suitable point q in $\check\Delta_4$ for which the flux of Σ_q along $\bar\gamma_q$ becomes the zero vector. Note here that the flux of Σ_q along $\bar\gamma_1$ vanishes if and only if the flux of Σ_q along $\bar\gamma_q$ does.

Let $n(p)$ be the inward unit conormal to $\bar\gamma_q$ on Σ_q at $p \in \bar\gamma_q$ and define

$$f(q) = \int_{p\in\bar\gamma_q} n(p).$$

Then $f(q)$ is the flux of Σ_q along $\bar\gamma_q$ and f is a map from the interior $\check\Delta_4$ to the set N of vectors parallel to the plane P_4. f is a smooth map and can be extended continuously to the closed triangle Δ_4. Let $\Delta_4 \times \mathbb{R}$ be the vertical prism over Δ_4. Obviously Σ_q lies inside $\Delta_4 \times \mathbb{R}$. Since $\bar\gamma_1$ winds around $\bar\gamma_q$ once, so does $n(p)$ as p moves along $\bar\gamma_q$. But as q approaches a point $\tilde q \in \partial\Delta_4$, Γ_q converges to a complete translationally periodic curve $\Gamma_{\tilde q} := \gamma_{\tilde q} \cup \gamma_1$ of which $\bar\gamma_{\tilde q} \cup \bar\gamma_1$ is a fundamental piece. Let τ be the translation defined by $\tau(\bar p) = \bar p - c_4\nu_4$, $\bar p \in \mathbb{R}^3$. Since $\bar\gamma_{\tilde q}$ intersects $\bar\gamma_1$, $\Gamma_{\tilde q}$ is a periodic union of Jordan curves, or more precisely, $\Gamma_{\tilde q} = \cup_n \tau^n(\gamma_{1\tilde q})$, where $\gamma_{1\tilde q}$ is a Jordan curve which is a subset of $(\bar\gamma_{\tilde q} \cup \bar\gamma_1) \cup \tau(\bar\gamma_{\tilde q} \cup \bar\gamma_1)$. $\gamma_{1\tilde q}$ consists of five (or four if $\bar\gamma_{\tilde q}$ passes through a vertex of $\bar\gamma_1$) line segments. It is known that the total curvature of $\gamma_{1\tilde q}$ equals the length of its tangent indicatrix $T_{1\tilde q}$. $T_{1\tilde q}$ is comprised of (i) a geodesic triangle and a geodesic with multiplicity 2 in case $\gamma_{1\tilde q}$ consists of five line segments or (ii) four geodesics connecting the four points in $\mathbb{S}^2$ that correspond to $\nu_1, \nu_2, \nu_3, \nu_4$. Since the length of a geodesic triangle is less than 2π and the length of a geodesic is less than π, the total length of $T_{1\tilde q}$ is smaller than 4π in either case. Thus by [4] there exists a unique minimal disk spanning $\gamma_{1\tilde q}$. We can easily extend the proof of Theorem 3.1 (c) to the limiting case where $\bar\gamma_1$ intersects $\bar\gamma_0$. So we can see that $\gamma_{1\tilde q}$ bounds a unique minimal surface $\hat\Sigma_{\tilde q} \subset \Delta_4 \times \mathbb{R}$ regardless of its topology. As $q \to \tilde q \in \partial\Delta_4$, a fundamental region of Σ_q converges to $\hat\Sigma_{\tilde q}$. Hence, by continuity of the extended map $f : \Delta_4 \to N$, $f(q)$ converges to $f(\tilde q) = \int_{p\in\bar\gamma_{\tilde q}} n(p)$ which is the flux of $\hat\Sigma_{\tilde q}$ along $\bar\gamma_{\tilde q} \subset \partial\Delta_4 \times \mathbb{R}$. Therefore, as $n(p)$ points into the interior of Δ_4 at any $p \in \bar\gamma_{\tilde q}$, $f(\tilde q)$ is a nonzero horizontal vector pointing toward the interior of Δ_4.

Now, we are ready to show a point q in the interior $\check\Delta_4$ at which the flux $f(q)$ vanishes. Suppose $f(q) \neq 0$ for all $q \in \check\Delta_4$ and define a map $\tilde f : \Delta_4 \to \mathbb{S}^1$ by

$$\tilde f(q) = \frac{f(q)}{|f(q)|}.$$

Then $\tilde f$ is continuous and $\tilde f|_{\partial\Delta_4}$ has winding number 1 because the nonzero horizontal vector $f(\tilde q)$ points toward the interior $\check\Delta_4$ at any $\tilde q \in \partial\Delta_4$. But this is a contradiction since the induced homomorphism $\tilde f_* : \pi_1(\Delta_4) \to \pi_1(\mathbb{S}^1)$ must then be surjective. Therefore there should exist $q_4 \in \check\Delta_4$, and a minimal surface Σ_{q_4} which has zero flux $f(q_4) = 0$ along $\bar\gamma_{q_4}$. Thus the conjugate surface $\Sigma_{q_4}^*$ is a well-defined minimal annulus. (See Figure 6.)

It remains to show that a homothetic expansion of $\Sigma_{q_4}^*$ is in T and perpendicular to ∂T along its boundary. According to the arguments of Smyth's theorem, there exist constants $d_1^*, d_2^*, d_3^*, d_4^*$ such that the curve $Y_{q_4}^*(c_i\nu_i)$ is in the plane $\{y_i^* = d_i^*\}$ and $\Sigma_{q_4}^*$ is perpendicular to that plane along $Y_{q_4}^*(c_i\nu_i)$. Moreover, the outward unit conormal to $Y_{q_4}^*(c_i\nu_i)$ on $\Sigma_{q_4}^*$ is ν_i and hence near $Y_{q_4}^*(c_i\nu_i)$, $\Sigma_{q_4}^*$ lies in the same side of the plane $\{y_i^* = d_i^*\}$ as T' does. Remember that the four planes $\cup_{i=1}^4\{y_i = d_i\}$ enclose the tetrahedron T and $\cup_{i=1}^4\{y_i^* = d_i^*\}$ enclose the tetrahedron T'. Since

$y_i = b_i^1 x_1 + b_i^2 x_2 + b_i^3 x_3$ and $y_i^* = b_i^1 x_1^* + b_i^2 x_2^* + b_i^3 x_3^*$, T' is similar to T. As ν_4 is assumed to be parallel to the x_3 axis, $y_4^* = b_4^* x_3^*$.

Obviously a homothetic expansion of $\Sigma_{q_4}^*$ will give a minimal annulus A_4 which is perpendicular to $\cup_{i=1}^4 \{y_i = d_i\}$ along ∂A_4. Working with a new plane P_j containing $F_j, j = 1, 2, 3$, instead of F_4 and using the triangles $\Delta_j \subset P_j$, obtained from the relation for the projection π_j into P_j:

$$\left(\sum_{i=1}^{4} \pi(c_i\nu_i)\right) - \pi_j(c_j\nu_j) = 0, \ \ j = 1, 2 \text{ or } 3,$$

one can similarly find minimal annuli A_1, A_2, A_3 which are homothetic expansions of $\Sigma_{q_j}^*$ for some $q_j \in \Delta_j, j = 1, 2, 3$. This proves (b) except for the convexity of the closed curve.

Let's denote by F_j' the face of T', similar to the face F_j of T, $j = 1, 2, 3, 4$. Is it true that $\partial\Sigma_{q_j}^* \subset \partial T'$? Here we have to be careful because $Y_{q_j}^*(\bar\gamma_{q_j})$ and $Y_{q_j}^*(\bar\gamma_1)$ are *disconnected*. (Notice that $\partial\Sigma^*$ is connected in Smyth's case.) Consequently, for $j = 4$, $Y_{q_4}^*(\bar\gamma_1)$ is not necessarily a subset of $\partial T' \setminus \{y_4^* = d_4^*\}$ and it may intersect the plane $\{y_4^* = d_4^*\}(= \{x_3^* = 0\})$ as in Figure 5, right. To get some information about the location of $\partial\Sigma_{q_4}^*$, let's first assume that (d) and (e) are true. Since near $Y_{q_4}^*(c_i\nu_i), i = 1, 2, 3$, $\Sigma_{q_4}^*$ lies in the same side of the plane $\{y_i^* = d_i^*\}$ as T' does and since $Y_{q_4}^*(c_i\nu_i)$ are convex and are perpendicular on their endpoints to the three lines containing the edges $F_1' \cap F_2'$, $F_2' \cap F_3'$, $F_3' \cap F_1'$, respectively, one can conclude that *(i)* $Y_{q_4}^*(\bar\gamma_1)$ lies in the tangent cone $TC_{p_4'}(\partial T')$ of $\partial T'$ at p_4', the vertex of T' opposite F_4'. As $\Sigma_{q_4}^*$ is a graph over $\pi_4(\Sigma_{q_4}^*)$, *(ii)* $Y_{q_4}^*(\bar\gamma_{q_4})$ is surrounded by $\pi_4(Y_{q_4}^*(\bar\gamma_1))$ in the plane $\{y_4^* = d_4^*\}$.

Now let's prove a lemma, which is more general than (c). If the dihedral angles along ∂F_4 are $\le 90°$, the unit normals ν_1, ν_2, ν_3 are pointing upward and $\bar\gamma_1$ goes upward. So one can consider the following generalization.

Lemma 5.2. *Let $\Gamma = \gamma_0 \cup \gamma_1$ be a translationally periodic curve and γ_0 the x_3-axis. Assume that Σ_Γ is a translationally periodic Plateau solution spanning Γ. If x_3 is a nondecreasing function on γ_1, then the boundary component of Σ_Γ^* corresponding to γ_0 is in the $x_1^*x_2^*$-plane and Σ_Γ^* is on and above the $x_1^*x_2^*$-plane.*

Proof. Σ_Γ has no horizontal tangent plane $T_p\Sigma_\Gamma$ at any interior point $p \in \Sigma_\Gamma$. This can be verified as follows. Every horizontal plane $\{x_3 = h\}$ intersects Γ either at two points only or at infinitely many points (the second case occurs when $\{x_3 = h\} \cap \gamma_1$ is a curve of positive length). If $T_p\Sigma_\Gamma = \{x_3 = h\}$, then $\{x_3 = h\} \cap \Sigma_\Gamma$ is the set of at least four curves emanating from p. But then three of them intersect γ_1, and hence there exists a domain $D \subset \Sigma_\Gamma$ with $\partial D \subset \{x_3 = h\}$, which contradicts the maximum principle. Hence $\{x_3 = h\}$ is transversal to Σ_Γ for every h and therefore x_3^* is an increasing function on every horizontal section $\{x_3 = h\} \cap \Sigma_\Gamma$. Since $x_3^* = 0$ on γ_0, x_3^* must be nonnegative on Σ_Γ^*. □

If the dihedral angles along ∂F_4 are $\le 90°$, then by the above lemma $Y_{q_4}^*(\bar\gamma_1) \subset \partial T' \setminus F_4'$. By (e), which will be proved independently, $Y_{q_4}^*(\bar\gamma_{q_4})$ is surrounded by $\pi_4(Y_{q_4}^*(\bar\gamma_1))$ and hence $Y_{q_4}^*(\bar\gamma_{q_4})$ lies inside F_4'. This proves (c) (except for convexity) and (a) as well.

We now derive the convexity of $\partial\Sigma_{q_4}^*$ as follows. Henceforth, our proof will be independent of (a), (b), and (c). It should be mentioned that $\Sigma_{q_4}^*$ has been constructed independently of (d) and (e). Let Q be a vertical half plane emanating from $\bar\gamma_{q_4}$, that is, $\partial Q \supset \bar\gamma_{q_4}$. Then $Q \cap \bar\gamma_1$ is a single point unless Q contains the two boundary points of $\bar\gamma_1$. Let q be a point of $\bar\gamma_{q_4}$ which is the end point of $Q \cap (\Sigma_{q_4} \setminus \bar\gamma_{q_4})$.

Here we claim that in a neighborhood U of q, $C := U \cap Q \cap (\Sigma_{q_4} \setminus \bar{\gamma}_{q_4})$ is a single curve emanating from q. If not, $U \cap Q \cap (\Sigma_{q_4} \setminus \bar{\gamma}_{q_4})$ is the union of at least two curves $C_1, C_2, \ldots$ emanating from q. These curves can be extended up to $\bar{\gamma}_{q_4} \cup \bar{\gamma}_1$. In case $Q \cap \partial\bar{\gamma}_1 = \emptyset$, $Q \cap \bar{\gamma}_1$ is a single point, then only one of $C_1, C_2, \ldots$, say C_1, can reach the point $Q \cap \bar{\gamma}_1$ and C_2 can only reach $\bar{\gamma}_{q_4}$. Since Σ_{q_4} is simply connected, C_2 and $\bar{\gamma}_{q_4}$ bound a domain $D \subset \Sigma_{q_4}$ with $\partial D \subset Q$. This contradicts the maximum principle. In case Q intersects $\bar{\gamma}_1$ at its boundary points p_1, p_2, there exist two curves, say $C_1, C_2 \subset Q \cap \Sigma_{q_4}$ emanating from q, such that $p_1 \in C_1$ and $p_2 \in C_2$. Remember that $\bar{\gamma}_{q_4} \cup \bar{\gamma}_1$ is a fundamental piece of Γ_{q_4} which is translationally periodic under the vertical translation τ by $-c_4\nu_4$. Hence $\tau(p_1) = p_2$ and therefore the two distinct curves $\tau(C_1), C_2 \subset Q \cap \Sigma_{q_4}$ emanate from p_2. But this is not possible since in a neighborhood of p_2, $Q \cap \Sigma_{q_4}$ is a single curve emanating from p_2. Hence, the claim follows.

Note that $\log g = i \arg g$ on the straight line γ_{q_4} containing $\bar{\gamma}_{q_4}$ because $|g| \equiv 1$ there. If $(d/dx_3)\arg g = 0$ at a point $q \in \gamma_{q_4}$ (x_3: the parameter of γ_{q_4}), then for the vertical half plane Q tangent to Σ_{q_4} at q, $Q \cap (\Sigma_{q_4} \setminus \gamma_{q_4})$ will be the union of at least two curves emanating from q, contradicting the claim. Hence $g' \neq 0$ on γ_{q_4}. Therefore $g' \neq 0$ on $\Sigma^*_{q_4} \cap \{y^*_4 = d^*_4\} = Y^*_{q_4}(\gamma_{q_4})$ as well and so $\Sigma^*_{q_4} \cap \{y^*_4 = d^*_4\}$ is convex. Similarly, let Q_j be a half plane emanating from the line segment L in $\bar{\gamma}_1$ corresponding to $c_j\nu_j$, $j = 1, 2, 3$. Being nonvertical, Q_j intersects γ_{q_4} only at one point. Hence $Q_j \cap (\Sigma_{q_4} \setminus L)$ is a single curve joining a point $p \in L$ to $Q_j \cap \gamma_{q_4}$ and p is a tangent point of Q_j and Σ_{q_4}. If we rotate Σ_{q_4} in such a way that $|g| \equiv 1$ on L, we can conclude $g'(p) \neq 0$ in the same way as above, as long as p is an interior point of L. On the other hand, $g' = 0$ at the boundary of L because the interior angle at the boundary of L is $< \pi$. Note that any interior point of L can be a tangent point of Q_j and Σ_{q_4} for some Q_j emanating from L and that Q_j intersects γ_{q_4} at one point only. Therefore $g' \neq 0$ in the interior of $L \subset \Sigma_{q_4}$ and hence $g' \neq 0$ in the interior of $\Sigma^*_{q_4} \cap \{y^*_j = d^*_j\} = Y^*(L)$. Thus $\Sigma^*_{q_4} \cap \{y^*_j = d^*_j\}$ is convex, $j = 1, 2, 3$. Since $\Sigma^*_{q_4}$ is perpendicular to $\{y^*_i = d^*_i\}$ and to $\{y^*_j = d^*_j\}$ at $p = \Sigma^*_{q_4} \cap \{y^*_i = d^*_i\} \cap \{y^*_j = d^*_j\}, 1 \leq i \neq j \leq 3$, so is $\partial\Sigma^*_{q_4}$ to the edge $\{y^*_i = d^*_i\} \cap \{y^*_j = d^*_j\}$ at p. This proves (d).

Remark that $Q \cap \bar{\gamma}_1$ being a single point is the key to the convexity of $\Sigma^*_{q_4} \cap \{y^*_4 = d^*_4\}$. Therefore, one can easily prove the following generalization, which is dual to Lemma 5.2.

Lemma 5.3. *Let $\Gamma = \gamma_0 \cup \gamma_1$ be a translationally periodic curve and γ_0 the x_3-axis. Assume that Σ_Γ is a translationally periodic Plateau solution spanning Γ and that its conjugate surface Σ^*_Γ is a well-defined minimal annulus. If a fundamental piece $\bar{\gamma}_1$ of γ_1 has a one-to-one projection into the x_1x_2-plane $\{x_3 = 0\}$, then the closed curve $\Sigma^*_\Gamma \cap \{x^*_3 = 0\}$ is convex.*

Finally, let's prove (e). Theorem 3.1 (b) implies that $\hat{\Sigma}_{q_4} \setminus \gamma_{q_4}$ is a graph over $\pi_4(\Sigma_{q_4} \setminus \gamma_{q_4})$. The two boundary curves $\partial\hat{\Sigma}_{q_4} \setminus (\gamma_{q_4} \cup \gamma_1)$ are the parallel translates of one another. Therefore Σ_{q_4} is embedded. Now, we are going to use Krust's argument (see Section 3.3 of [1]) to prove that $\Sigma^*_{q_4}$ is also a graph. Let $X = (x_1, x_2, x_3)$ be the immersion of $[0, a] \times [0, \beta]$ into Σ_{q_4} and $X^* = (x^*_1, x^*_2, x^*_3)$ the immersion: $[0, a] \times [0, \beta] \to \Sigma^*_{q_4}$. We can write the orthogonal projections of X and X^* into the horizontal plane as, respectively

$$w(z) := x_1(z) + ix_2(z), \; w^*(z) := x^*_1(z) + ix^*_2(z), \; z = x + iy, \; (x, y) \in [0, a] \times [0, \beta].$$

Then w is a map from $[0, a] \times [0, \beta]$ onto the triangle Δ_4. Given two distinct points $z_1, z_2 \in (0, a] \times (0, \beta]$, we have $w(z_1) \neq w(z_2)$ because $X((0, a] \times (0, \beta])$ is a graph over $\Delta_4 \setminus \{q_4\}$. Let $\ell : [0,1] \to \Delta_4$ be the line segment connecting $p_1 := w(z_1)$ to $p_2 := w(z_2)$ with constant speed, that is, $\ell(0) = p_1$, $\ell(1) = p_2$ and $|\dot{\ell}(t)| = |p_2 - p_1|$ for all $t \in [0,1]$.

(1) Choosing a fundamental region $\hat{\Sigma}_{q_4}$ of Σ_{q_4} suitably, we may suppose ℓ is disjoint from $\pi(\partial\hat{\Sigma}_{q_4})$. Then there is a smooth curve $c : [0,1] \to (0,a] \times (0,2\beta]$ such that $\ell(t) = w(c(t))$. Clearly $|\dot{c}(t)| > 0$ for all $0 \le t \le 1$. Let $g : [0,a] \times \mathbb{R} \to \mathbb{C}$ be the Gauss map of Σ_{q_4}. Krust showed that the inner product W of the two vectors $p_2 - p_1$ and $i(w^*(z_2) - w^*(z_1))$ of $\mathbb{R}^2$ is written as

$$W := \langle p_2 - p_1, i(w^*(z_2) - w^*(z_1))\rangle = \int_0^1 \frac{1}{4}|\dot{c}(t)|^2 \left(|g(c(t))|^2 - \frac{1}{|g(c(t))|^2}\right) dt.$$

Since $\Sigma_{q_4} \setminus \gamma_{q_4}$ is a multi-graph, we have $|g| > 1$ on $(0,a] \times \mathbb{R}$. Hence $W > 0$ and therefore $w^*(z_1) \neq w^*(z_2)$.

(2) Suppose ℓ intersects $\pi(\partial\hat{\Sigma}_{q_4})$ at the point q_4. Then c is piecewise smooth and there exist $0 < d_1 < d_2 < 1$ such that $q_4 \notin w(c([0,d_1))) \cup w(c((d_2,1]))$, $w(c([d_1,d_2])) = \{q_4\}$, and $|\dot{c}(t)| > 0$ for $t \in [0,d_1) \cup (d_2,1]$. Clearly

$$|g(c(t))| = 1 \text{ for } t \in [d_1,d_2], \quad |g(c(t))| > 1 \text{ for } t \in [0,d_1) \cup (d_2,1].$$

Hence

$$W = \left(\int_0^{d_1} + \int_{d_2}^1\right) \frac{1}{4}|\dot{c}(t)|^2 \left(|g(c(t))|^2 - \frac{1}{|g(c(t))|^2}\right) dt > 0$$

and so $w^*(z_1) \neq w^*(z_2)$.

Thus we can conclude that $X^*((0,a] \times (0,\beta))$ is a graph over the $x_1^*x_2^*$-plane. Since $X^*([0,a] \times \{0\})$ coincides with $X^*([0,a] \times \{\beta\})$, $X^*((0,a] \times [0,\beta]) = \Sigma_{q_4}^* \setminus \gamma_{q_4}$ is also a graph over its projection into the $x_1^*x_2^*$-plane. This proves (e). □

6. Pyramid

It has been possible to construct free boundary minimal annuli in a tetrahedron T because T is the simplest polyhedron in $\mathbb{R}^3$. Generally, one cannot find a free boundary minimal annulus in a polyhedron like a quadrilateral pyramid. However, if P_y is a regular or a rhombic pyramid, we can show that P_y has a free boundary minimal annulus. As a result, we can also show that there exists a genus zero free boundary minimal surfaces in every Platonic solid.

Theorem 6.1. *Let P_y be a right pyramid whose base B is a regular n-gon. Then, there exists a free boundary minimal annulus A in P_y, which is a graph over B. A is invariant under the rotation by $2\pi/n$ about the line through the apex and the center of B. One component of ∂A is convex and closed in B, and the other is convex in each remaining face of P_y.*

Proof. Let $F_1, \ldots, F_n$ be the faces of P_y other than the base B. Denote by $\nu_0, \nu_1, \ldots, \nu_n$ the outward unit normals to $B, F_1, \ldots, F_n$, respectively. Then, there exists a unique positive constant c such that

$$c\nu_0 + \nu_1 + \cdots + \nu_n = 0.$$

Assume that B lies in the x_1x_2-plane with center at the origin. Let $\bar{\gamma}_0$ be a vertical line segment of length c on the x_3-axis and let $\bar{\gamma}_1$ be a connected piecewise linear curve determined by $\nu_1, \ldots, \nu_n$(i.e., ν_i is the i-th oriented line segment of $\bar{\gamma}_1$) such that the projection $\pi(\bar{\gamma}_1)$ of $\bar{\gamma}_1$ onto the x_1x_2-plane is a regular n-gon centered at the origin. Moreover, let's assume that the two endpoints of $\bar{\gamma}_0$ and $\bar{\gamma}_1$ have the same x_3-coordinates: 0 and c. $\bar{\gamma}_0 \cup \bar{\gamma}_1$ determines a complete helically periodic curve

Γ of which $\bar\gamma_0\cup\bar\gamma_1$ is a fundamental piece. Γ is translationally periodic as well. Then Theorem 2.2 guarantees a translationally periodic minimal surface Σ spanning Γ.

Define the screw motion σ by

$$\sigma(r\cos\theta, r\sin\theta, x_3) = \left(r\cos(\theta+\frac{2\pi}{n}), r\sin(\theta+\frac{2\pi}{n}), x_3+\frac{c}{n}\right).$$

Obviously Σ is invariant under σ^n. The point is that Σ is invariant under σ as well. This is because by Theorem 3.1 the periodic Plateau solution spanning Γ uniquely exists and $\sigma(\Sigma)$ also spans Γ. So evenly divide $\bar\gamma_0$ into n line segments $\bar\gamma_0^1,\dots,\bar\gamma_0^n$ such that

$$\bar\gamma_0^k := \{p\in\bar\gamma_0 : \frac{k-1}{n}c \le x_3(p) \le \frac{k}{n}c\},\ \ k=1\dots,n.$$

Similarly, set

$$\Sigma^k = \{p\in\Sigma : \frac{k-1}{n}c \le x_3(p) \le \frac{k}{n}c\},\ \ k=1,\dots,n.$$

It is clear that

$$\sigma(\bar\gamma_0^k)=\bar\gamma_0^{k+1},\ \sigma(\Sigma^k)=\Sigma^{k+1},\ k=1,\dots,n-1,\ \text{ and }\ \sigma(\Sigma^n)=\sigma^n(\Sigma^1).$$

Denote by $f_\gamma(\Sigma)$ the flux of Σ along $\gamma\subset\partial\Sigma$, that is,

$$f_\gamma(\Sigma)=\int_{p\in\gamma} n(p),$$

where $n(p)$ is the inward unit conormal to γ on Σ at $p\in\gamma$. Clearly

$$f_{\sigma(\gamma)}(\sigma(\Sigma))=\sigma(f_\gamma(\Sigma))\ \text{ and }\ f_{\bar\gamma_0}(\Sigma)=\sum_{k=1}^n f_{\bar\gamma_0^k}(\Sigma^k).$$

Hence

$$\begin{aligned}\sigma(f_{\bar\gamma_0}(\Sigma)) &= \sum_{k=1}^n \sigma(f_{\bar\gamma_0^k}(\Sigma^k)) = \sum_{k=1}^n f_{\sigma(\bar\gamma_0^k)}(\sigma(\Sigma^k))\\ &= \sum_{k=1}^{n-1} f_{\bar\gamma_0^{k+1}}(\Sigma^{k+1}) + f_{\sigma^n(\bar\gamma_0^1)}(\sigma^n(\Sigma^1))\\ &= \sum_{k=1}^n f_{\bar\gamma_0^k}(\Sigma^k) = f_{\bar\gamma_0}(\Sigma).\end{aligned}$$

But $\sigma(f_{\bar\gamma_0}(\Sigma)) = f_{\bar\gamma_0}(\Sigma)$ holds only when $f_{\bar\gamma_0}(\Sigma)=0$. In this case $f_{\bar\gamma_1}(\Sigma)$ also vanishes. Therefore Σ^* is a well-defined minimal annulus.

We now show that Σ^* is in P_y with free boundary. Choose a point $p\in\Sigma^k$ with coordinates

$$X(p)=(x_1(p),x_2(p),x_3(p)).$$

Denote by $X^*(p)$ the point of Σ^{k*} corresponding to $p\in\Sigma^k$,

$$X^*(p)=(x_1^*(p),x_2^*(p),x_3^*(p)).$$

The coordinates of $\sigma(p)$ are

$$X(\sigma(p))=\left((x_1(p),x_2(p))\cdot\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix},\ x_3(p)+\frac{c}{n}\right),\ \ \alpha=\frac{2\pi}{n}.$$

Then

$$\begin{aligned}X^*(\sigma(p)) &= \left((x_1^*(p),x_2^*(p))\cdot\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix},\ x_3^*(p)+0\right)\\ &= \sigma_0(X^*(p)),\end{aligned}$$

where σ_0 is the rotation in $\mathbb{R}^3$ defined by

$$\sigma_0(r\cos\theta, r\sin\theta, x_3) = \left(r\cos(\theta + \frac{2\pi}{n}), r\sin(\theta + \frac{2\pi}{n}), x_3\right).$$

Hence

$$(\Sigma^{k+1})^* = \sigma_0(\Sigma^{k*}),\ k = 1, \ldots, n \tag{6.1}$$

and so

$$\begin{aligned}\sigma_0(\Sigma^*) &= \sigma_0(\Sigma^{1*} \cup \cdots \cup \Sigma^{n*}) = \Sigma^{2*} \cup \cdots \cup \Sigma^{n*} \cup \sigma_0(\Sigma^{n*}) \\ &= \Sigma^{2*} \cup \cdots \cup \Sigma^{n*} \cup \sigma_0^n(\Sigma^{1*}) = \Sigma^*.\end{aligned}$$

Therefore Σ^* is invariant under the rotation σ_0. We know that the curve $X^*(\nu_1)$ is in the plane $\{y_1^* = d_1^*\}$ orthogonal to $\nabla y_1^* = \nu_1$ and Σ^* is perpendicular to that plane along $X^*(\nu_1)$. Therefore (6.1) implies that Σ^* is a free boundary minimal surface in the pyramid P_m bounded by a plane perpendicular to ν_{n+1} and by the n planes $\cup_{i=1}^n (\sigma_0)^i(\{y_1^* = d_1^*\})$. P_m is similar to P_y and a homothetic expansion A of Σ^* is a free boundary minimal annulus in P_y. By the same argument as in the proof of Theorem 5.1 we see that A is a graph over B and ∂A is convex on each face of P_y. □

Corollary 6.2. *Every Platonic solid with regular n-gon faces contains an embedded, genus zero, free boundary minimal surface.*

Proof. Given a Platonic solid P_s, let p be its center and F one of its faces. Then the cone from p over F is a right pyramid with a regular n-gon base, and hence P_s is tessellated into congruent pyramids. Each pyramid contains an embedded free boundary minimal annulus by Theorem 6.1. The union of all those minimal annuli in the congruent pyramids of the tessellation is the analytic continuation of each minimal annulus into an embedded, genus zero, free boundary minimal surface Σ_1 in P_s. □

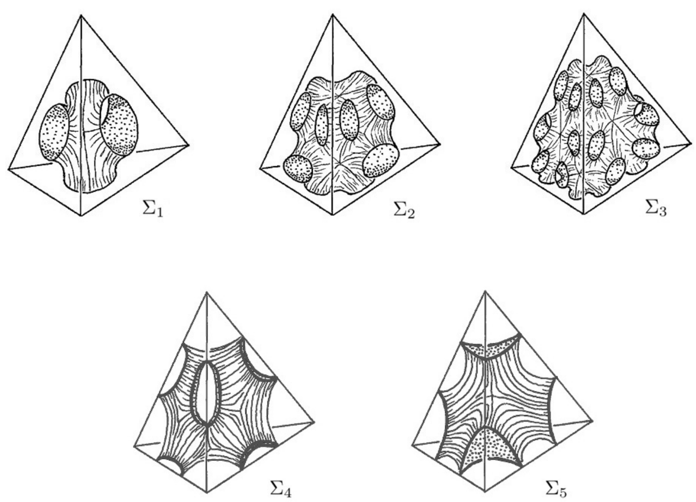

Figure 7. Five types of free boundary minimal surfaces in every Platonic solid

Remark 6.3. a) There are four more types of embedded, genus zero, free boundary minimal surfaces $\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$ in every Platonic solid. This fact results from various ways of tessellating its face into triangles. (See Figure 7.)

b) If P_r is a right pyramid with rhombic base B, there exists a free boundary minimal annulus A in P_r which is a graph over B.

We would like to conclude our paper by proposing the following interesting problems.

Problems.

(1) Let Γ be a Jordan curve in $\mathbb{R}^3$ bounding a minimal disk Σ. If the total curvature of Γ is $\leq 4\pi$, we know that Σ is unique [4]. Show that Σ^* is the unique minimal disk spanning $\partial\Sigma^*$.

(2) Assume that $\Gamma \subset \mathbb{R}^3$ is a Jordan curve with total curvature $\leq 4\pi$. It is proved that any minimal surface Σ spanning Γ is embedded [2]. If Σ is simply connected, show that Σ^* is also embedded.

(3) Let Γ be a complete translationally (or helically) periodic curve with a fundamental piece $\bar{\gamma}$. Assume that a translationally(or helically) periodic minimal surface Σ_Γ spans Γ. What is the maximum total curvature of $\bar{\gamma}$ that guarantees the uniqueness of Σ_Γ? What about the embeddedness of Σ_Γ?

References

[1] U. Dierkes, S. Hildebrandt, A. Küster, O. Wohlrab, *Minimal Surfaces I*, Springer-Verlag, Berlin Heidelberg 1992.

[2] T. Ekholm, B. White, D. Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most* 4π, Ann. of Math. **155** (2002), 209-234.

[3] R.D. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. **97** (1973), 275-305.

[4] J.C.C. Nitsche, *A new uniqueness theorem for minimal surfaces*, Arch. Rational Mech. Anal. **52** (1973), 319-329.

[5] J.C.C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal. **89** (1985), 1-19.

[6] T. Radó, *Some remarks on the problem of Plateau*, Proc. Natl. Acad. Sci. USA, **16** (1930), 242-248.

[7] H.A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Band I und II. Springer, Berlin 1890.

[8] B. Smyth, *Stationary minimal surfaces with boundary on a simplex*, Invent. math. **76** (1984), 411-420.

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