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Noncommutative Minimal Surfaces

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Abstract. We define noncommutative minimal surfaces in the Weyl algebra, and give a method to construct them by generalizing the well-known Weierstrass representation.

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Given the growing interest in noncommutative spaces, and zero-mean-curvature surfaces having been known for more than 250 years, it is rather astonishing that a general theory of noncommutative minimal surfaces seems to be lacking. Our note is a modest attempt to fill this gap.

1. Preliminaries

1.1. POISSON ALGEBRAIC GEOMETRY OF MINIMAL SURFACES

Not long ago, it was shown that the geometry of surfaces (or, in general, almost Kähler manifolds) can be expressed via Poisson brackets of the functions $x^1, \ldots, x^n$ which provide an isometric embedding into a given ambient manifold [2,3]. In noncommutative geometry, as well as quantum mechanics, there is an intimate relationship between an operator corresponding to the (commutative) function $\{f, g\}$, and the commutator of the operators that corresponds to f and g. Therefore, obtaining knowledge about geometrical quantities, as given in the Poisson algebra generated by $x^1, \ldots, x^n$, provides information about the corresponding noncommutative geometrical objects, and how to define them.

Assume that Σ is a 2-dimensional manifold, with local coordinates $u = u^1, v = u^2$, embedded in $\mathbb{R}^n$ via the embedding coordinates $x^1(u, v), x^2(u, v), \ldots, x^n(u, v)$, inducing on Σ the metric

$$g_{ab} = \partial_a \vec{x} \cdot \partial_b \vec{x} \equiv \sum_{i=1}^{n} \left(\partial_a x^i\right)\left(\partial_b x^i\right),$$

where $\partial_a = \frac{\partial}{\partial u^a}$. We adopt the convention that indices a, b, p, q take values in $\{1, 2\}$, and i, j, k, l run from 1 to n. For an arbitrary density ρ, one may introduce a Poisson bracket on $C^\infty(\Sigma)$ via

$$\{f, h\} = \frac{1}{\rho}\varepsilon^{ab}\left(\partial_a f\right)\left(\partial_b h\right),$$

and we define the function $\gamma = \sqrt{g}/\rho$, where g denotes the determinant of the metric g_{ab}. Setting $\theta^{ab} = \frac{1}{\rho}\varepsilon^{ab}$ (the Poisson bivector) one notes that

$$\theta^{ap}\theta^{bq} g_{pq} = \frac{1}{\rho^2}\varepsilon^{ap}\varepsilon^{bq} g_{pq} = \frac{g}{\rho^2} g^{ab} = \gamma^2 g^{ab} \tag{1.1}$$

since $\varepsilon^{ap}\varepsilon^{bq} g_{pq}$ is the cofactor expansion of the inverse of the metric. The fact that the geometry of the submanifold Σ can be expressed in terms of Poisson brackets follows from the trivial, but crucial, observation that the projection operator $\mathcal{D}$: $T\mathbb{R}^n \to T\Sigma$ (where one regards $T\Sigma$ as a subspace of $T\mathbb{R}^n$) can be written as

$$\mathcal{D}(X)^i = \frac{1}{\gamma^2}\sum_{j,k=1}^{n}\{x^i, x^k\}\{x^j, x^k\}X^j$$

for $X \in T\mathbb{R}^n$. Namely, one obtains

$$\begin{aligned}\mathcal{D}(X)^i &= \frac{1}{\gamma^2}\sum_{j,k=1}^{n}\theta^{ab}\theta^{pq}(\partial_a x^i)(\partial_b x^k)(\partial_p x^j)(\partial_q x^k)X^j \\ &= \frac{1}{\gamma^2}\sum_{j=1}^{n}\theta^{ab}\theta^{pq} g_{bq}(\partial_a x^i)(\partial_p x^j)X^j = \sum_{j=1}^{n} g^{ap}(\partial_a x^i)(\partial_p x^j)X^j,\end{aligned}$$

by using (1.1). From this expression one concludes that $\mathcal{D}^2 = \mathcal{D}$ and that $\mathcal{D}(X) = X$ and $\mathcal{D}(N) = 0$ if $X \in T\Sigma$ and $N \in T\Sigma^\perp$.

In this paper, we shall foremost be interested in the Laplace–Beltrami operator on Σ, defined as

$$\Delta(f) = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g} g^{ab}\partial_b f\right).$$

PROPOSITION 1.1. *For $f \in C^\infty(\Sigma)$ it holds that*

$$\Delta(f) = \gamma^{-1}\sum_{i=1}^{n}\{\gamma^{-1}\{f, x^i\}, x^i\}$$

$$\Delta(f) = \gamma^{-1}\{\gamma^{-1}\{f, u^a\}g_{ab}, u^b\}.$$

Proof. Let us prove the first formula; the second one is proven in an analogous way. One computes that

$$\begin{aligned}
&\frac{1}{\gamma}\sum_{i=1}^{n}\theta^{ab}\partial_a\big(\gamma^{-1}\theta^{pq}(\partial_p f)(\partial_q x^i)\big)\partial_b x^i\\
&=\frac{1}{\gamma}\sum_{i=1}^{n}\theta^{ab}\partial_a\big(\gamma^{-1}\theta^{pq}(\partial_p f)(\partial_q x^i)(\partial_b x^i)\big)\\
&=\frac{1}{\sqrt{g}}\partial_a\big(\gamma^{-1}\varepsilon^{ab}\theta^{pq}g_{bq}(\partial_p f)\big)=\frac{1}{\sqrt{g}}\partial_a\big(\gamma^{-1}\rho\theta^{ab}\theta^{pq}g_{bq}(\partial_p f)\big)\\
&=\frac{1}{\sqrt{g}}\partial_a\big(\gamma^{-1}\rho\gamma^2 g^{ap}\partial_p f\big)=\frac{1}{\sqrt{g}}\partial_a\big(\sqrt{g}g^{ap}\partial_p f\big)=\Delta(f),
\end{aligned}$$

by using (1.1). □

On a surface, one may always find *conformal coordinates*; i.e., coordinates with respect to which the metric becomes $g_{ab}=\mathcal{E}(u,v)\delta_{ab}$ for some (strictly positive) function $\mathcal{E}$. Furthermore, if we choose $\rho=1$ (giving $\gamma=\mathcal{E}$), the second formula in Proposition 1.1 can be written as

$$\Delta(f)=\frac{1}{\mathcal{E}}\{\{f,u^a\}\delta_{ab},u^b\}=\frac{1}{\mathcal{E}}\{\{f,u\},u\}+\frac{1}{\mathcal{E}}\{\{f,v\},v\}$$

if we assume the coordinates u,v to be conformal. For convenience, we shall also introduce $\Delta_0(f)=\{\{f,u\},u\}+\{\{f,v\},v\}=\mathcal{E}\Delta(f)$.

Minimal surfaces can be characterized by the fact that their embedding coordinates $x^1,\dots,x^n$ are harmonic with respect to the Laplace operator on the surface; i.e., $\Delta(x^i)=0$ for $i=1,\dots,n$. In local conformal coordinates, due to the above Poisson algebraic formulas, one may formulate this as follows: a surface $\vec{x}:D\subset\mathbb{R}^2\to\mathbb{R}^n$ is minimal if

$$\begin{aligned}
&\Delta_0(x^i)=\{\{x^i,u\},u\}+\{\{x^i,v\},v\}=0 \quad \text{for} \quad i=1,\dots,n\\
&\vec{x}_u\cdot\vec{x}_u=\vec{x}_v\cdot\vec{x}_v \quad \text{and} \quad \vec{x}_u\cdot\vec{x}_v=0,
\end{aligned}$$

where $\vec{x}_u$ and $\vec{x}_v$ denote the partial derivatives of $\vec{x}$ with respect to u and v. Note that the above choice of Poisson bracket implies that $\{u,v\}=1$. In Section 2 we will, in analogy with the above formulation, define noncommutative minimal surfaces in a (noncommutative) algebra with generators U,V satisfying $[U,V]\sim\mathbb{1}$; the universal algebra with these properties is commonly known as the *Weyl algebra*.

1.2. THE WEYL ALGEBRA AND ITS FIELD OF FRACTIONS

As mentioned in the previous section, the Weyl algebra provides us with a natural setting in which noncommutative minimal surfaces may be defined. In this section we recall some basic properties of the Weyl algebra (and its field of fractions), as well as introducing the notation which shall be used later.

DEFINITION 1.2. (*Weyl algebra*) Let $\mathbb{C}\langle U, V\rangle$ denote the free (associative) unital algebra generated by U, V. Furthermore, for $\hbar > 0$, let $I_\hbar$ denote the two-sided ideal generated by the relation

$$UV - VU = i\hbar\mathbb{1}.$$

The *Weyl algebra* is defined as $\mathcal{A}_\hbar = \mathbb{C}\langle U, V\rangle / I_\hbar$.

The Weyl algebra can be embedded in a skew field by a general procedure [9]. Let us briefly review the construction for the purposes of this paper.

Consider the Cartesian product $\mathcal{A}_\hbar \times \mathcal{A}_\hbar^\times$, i.e., ordered pairs (A, B) of elements in $A, B \in \mathcal{A}_\hbar$ with $B \neq 0$, which in the end will correspond to the expression AB^{-1}. The Weyl algebra satisfies the Ore condition; i.e., for each pair of elements A, B there exist $\beta_1, \beta_2 \in \mathcal{A}_\hbar$ such that

$$A\beta_1 = B\beta_2$$

(see [6,8] for a proof of this fact and many other properties of the Weyl algebra). This property allows one to define a relation on $\mathcal{A}_\hbar \times \mathcal{A}_\hbar^\times$. Namely, $(A, B) \sim (C, D)$ if there exist $\beta_1, \beta_2 \in \mathcal{A}_\hbar$ such that

$$A\beta_1 = C\beta_2$$
$$B\beta_1 = D\beta_2,$$

and it is straightforward to check that $\sim$ is an equivalence relation. The quotient $(\mathcal{A}_\hbar \times \mathcal{A}_\hbar^\times)/\sim$ is denoted by $\mathfrak{F}_\hbar$. Addition in $\mathfrak{F}_\hbar$ is defined as follows: let $\beta_1, \beta_2 \in \mathcal{A}_\hbar$ be such that $B\beta_1 = D\beta_2$. Then one sets

$$(A, B) + (C, D) = (A\beta_1 + C\beta_2, B\beta_1).$$

Likewise, when $\alpha_1, \alpha_2 \in \mathcal{A}_\hbar$ are such that $B\alpha_1 = C\alpha_2$, one defines

$$(A, B)(C, D) = (A\alpha_1, D\alpha_2).$$

It is straightforward (although tedious) to check that these are well-defined operations in $\mathfrak{F}_\hbar$ (i.e., they respect equivalence classes) and that they do not depend on the particular choice of $\beta_1, \beta_2, \alpha_1, \alpha_2$. Furthermore, both operations are associative, and they satisfy the distributive law. The unit element can be represented by $(\mathbb{1}, \mathbb{1})$ and the zero element by $(0, \mathbb{1})$. For every element $A \in \mathcal{A}_\hbar$ we identify A with $(A, \mathbb{1})$ and A^{-1} with $(\mathbb{1}, A)$ (for $A \neq 0$), and with this notation it follows that $AB^{-1} = (A, \mathbb{1})(\mathbb{1}, B) = (A, B)$. One easily checks that (B, A) is the (right and left) inverse of (A, B) and that $(AB)^{-1} = B^{-1}A^{-1}$. Moreover, if $[A, B] = 0$ it holds that $AB^{-1} = B^{-1}A$, i.e., $(A, B) = (\mathbb{1}, B)(A, \mathbb{1})$.

The Weyl algebra becomes a $*$-algebra upon setting $U^* = U$ and $V^* = V$, and as a consequence of the universal property of the fraction ring, the $*$-operation can be extended to $\mathfrak{F}_\hbar$. Thus, $\mathfrak{F}_\hbar$ is a $*$-algebra, and it follows that $(A, \mathbb{1})^* = (A^*, \mathbb{1})$

and $(\mathbb{1}, A)^* = (\mathbb{1}, A^*)$ (where the last equality can be written as $(A^{-1})^* = (A^*)^{-1}$) for all $A \in \mathcal{A}_\hbar$. Hence, it holds that

$$(AB^{-1})^* = (A, B)^* = \big((A, \mathbb{1})(\mathbb{1}, B)\big)^* = (\mathbb{1}, B^*)(A^*, \mathbb{1}) = (B^*)^{-1}A^*.$$

In the following, we shall drop the (cumbersome) notation (A, B) and simply write AB^{-1}; moreover, we do not distinguish between an element $A \in \mathcal{A}_\hbar$ and its corresponding image in $\mathfrak{F}_\hbar$. An element $A \in \mathfrak{F}_\hbar$ is called *hermitian* if $A^* = A$. The real and imaginary parts of an element are defined as

$$\mathrm{Re}(A) = \frac{1}{2}\big(A + A^*\big)$$
$$\mathrm{Im}(A) = \frac{1}{2i}\big(A - A^*\big),$$

and it is convenient to introduce the notation $U^1 = U$ and $U^2 = V$, as well as the derivations

$$\hat{\partial}_u(A) \equiv \hat{\partial}_1(A) = \frac{1}{i\hbar}[A, V]$$
$$\hat{\partial}_v(A) \equiv \hat{\partial}_2(A) = -\frac{1}{i\hbar}[A, U].$$

PROPOSITION 1.3. *For $A \in \mathfrak{F}_\hbar$ and $p(x) \in \mathbb{C}[x]$ it holds that*

(1) $\hat{\partial}_a A^{-1} = -A^{-1}\hat{\partial}_a(A)A^{-1}$,
(2) $\hat{\partial}_a\big(\hat{\partial}_b(A)\big) = \hat{\partial}_b\big(\hat{\partial}_a(A)\big)$,
(3) $\hat{\partial}_a\, p(U^a) = p'(U^a)$ *(no sum over a),*

for $a, b = 1, 2$, where $p'(x)$ denotes the derivative (w.r.t. x) of $p(x)$.

Proof. The first property is an immediate consequence of the fact that $\hat{\partial}_a(AA^{-1}) = \hat{\partial}_a(\mathbb{1}) = 0$. For the third property, one computes

$$\hat{\partial}_u\, p(U) = \frac{1}{i\hbar}\sum_{k=0}^{n}[a_k U^k, V] = \sum_{k=1}^{n} k a_k U^{k-1} = p'(U),$$

and similarly for $p(V)$. Finally, to show that the derivatives commute, one simply calculates

$$\hat{\partial}_u\big(\hat{\partial}_v(A)\big) = \frac{1}{\hbar^2}\big[[A, U], V\big] = -\frac{1}{\hbar^2}\big[[U, V], A\big] - \frac{1}{\hbar^2}\big[[V, A], U\big].$$

Since $[U, V] = i\hbar\mathbb{1}$ (and, hence, is in the center of the algebra) it follows that

$$\hat{\partial}_u\big(\hat{\partial}_v(A)\big) = \frac{1}{\hbar^2}\big[[A, V], U\big] = \hat{\partial}_v\big(\hat{\partial}_u(A)\big),$$

which proves the statement. $\square$

Furthermore, let us introduce $\Lambda = U + iV$ together with the operators

$$\partial(A) = \frac{1}{2}\left(\hat{\partial}_u(A) - i\hat{\partial}_v(A)\right) = \frac{1}{2\hbar}[A, \Lambda^*]$$
$$\bar{\partial}(A) = \frac{1}{2}\left(\hat{\partial}_u(A) + i\hat{\partial}_v(A)\right) = -\frac{1}{2\hbar}[A, \Lambda],$$

and it follows from Proposition 1.3 that $\partial\bar{\partial}A = \bar{\partial}\partial A$. It is useful to note that $[\Lambda, \Lambda^*] = 2\hbar\mathbb{1}$.

DEFINITION 1.4. An element $A \in \mathfrak{F}_\hbar$ is called *r-holomorphic*[1] if $\bar{\partial}A = 0$. An r-holomorphic element A is called *holomorphic* if $A \in \mathcal{A}_\hbar$.

By $\mathbb{C}[\Lambda]$ we denote the subalgebra of $\mathfrak{F}_\hbar$ generated by Λ and $\mathbb{1}$. It turns out that r-holomorphic elements can be characterized as elements of $\mathbb{C}[\Lambda]$ and their quotients.

LEMMA 1.5. *An element $A \in \mathfrak{F}_\hbar$ is holomorphic if and only if $A \in \mathbb{C}[\Lambda]$.*

Proof. Clearly, if $A \in \mathbb{C}[\Lambda]$ then $A \in \mathcal{A}_\hbar$ and $\bar{\partial}A = -\frac{1}{2\hbar}[A, \Lambda] = 0$. Now, assume that $\bar{\partial}A = 0$ and that $A \in \mathcal{A}_\hbar$. Every element $A \in \mathcal{A}_\hbar$ can be written in the following normal form

$$A = \sum_{k,l \geq 0} a_{kl}\Lambda^k(\Lambda^*)^l,$$

and one computes

$$\bar{\partial}A = \sum_{k \geq 0, l \geq 1} l a_{kl}\Lambda^k(\Lambda^*)^{l-1}.$$

The fact that $\bar{\partial}A = 0$ implies that $a_{kl} = 0$ for $l \geq 1$, which implies that A is a polynomial in Λ. Hence, $A \in \mathbb{C}[\Lambda]$. □

PROPOSITION 1.6. *An element $A \in \mathfrak{F}_\hbar$ is r-holomorphic if and only if there exist $B, C \in \mathbb{C}[\Lambda]$ such that $A = BC^{-1}$.*

Proof. Clearly, if $A = BC^{-1}$ with $B, C \in \mathbb{C}[\Lambda]$, then

$$\bar{\partial}A = (\bar{\partial}B)C^{-1} - BC^{-1}\left(\bar{\partial}C\right)C^{-1} = 0,$$

by Lemma 1.5. Now, assume that $A = BC^{-1}$, with $B \neq 0$, and that $\bar{\partial}A = 0$. From the above equation it follows that

$$\bar{\partial}B = BC^{-1}\left(\bar{\partial}C\right),$$

[1] "rational"-holomorphic.

and if $\bar{\partial}C=0$ then $\bar{\partial}B=0$ and Lemma 1.5 implies that $B,C\in\mathbb{C}[\Lambda]$. If $\bar{\partial}C\neq 0$ then $\bar{\partial}B\neq 0$ and one obtains

$$\big(\bar{\partial}B\big)(\bar{\partial}C)^{-1}=BC^{-1}=A.$$

It follows that $\bar{\partial}(\bar{\partial}B)(\bar{\partial}C)^{-1}=\bar{\partial}A=0$, and one may repeat the argument with respect to the representation $A=(\bar{\partial}B)(\bar{\partial}C)^{-1}$. Thus, as long as $\bar{\partial}^{n}C\neq 0$ (and, hence, $\bar{\partial}^{n}B\neq 0$) one obtains

$$A=(\bar{\partial}^{n}B)(\bar{\partial}^{n}C)^{-1}.$$

For every non-zero $B\in\mathcal{A}_{\hbar}$ there exists an integer n_0 such that $\bar{\partial}^{n_0}B\neq 0$ and $\bar{\partial}^{n_0+1}B=0$, since B can be written as a polynomial in Λ and Λ^*. The above argument implies that one can always find $\tilde{B}$ $(=\bar{\partial}^{n_0}B)$ and $\tilde{C}$ $(=\bar{\partial}^{n_0}C)$ such that $A=\tilde{B}\tilde{C}^{-1}$, fulfilling $\bar{\partial}\tilde{B}=\bar{\partial}\tilde{C}=0$. From Lemma 1.5 it follows that $\tilde{B},\tilde{C}\in\mathbb{C}[\Lambda]$. □

Note that r-holomorphic elements are the analogues of meromorphic functions in complex analysis. However, since there is no immediate concept of point in the noncommutative algebra, it holds that $\bar{\partial}A$ is identically 0 for a r-holomorphic element, and not only at points where the derivative exists. This distinction becomes important if one represents the Weyl algebra on a vector space, as in Section 3.3, where there are elements that are not invertible.

Let us continue by defining the Laplace operator, as well as harmonic elements and some of their properties.

DEFINITION 1.7. The *noncommutative Laplace operator* $\Delta_0:\mathfrak{F}_{\hbar}\to\mathfrak{F}_{\hbar}$ is defined as

$$\Delta_0(A)=\hat{\partial}_u^2(A)+\hat{\partial}_v^2(A)=-\frac{1}{\hbar^2}\big[[A,V],V\big]-\frac{1}{\hbar^2}\big[[A,U],U\big].$$

An element $A\in\mathfrak{F}_{\hbar}$ is called *harmonic* if $\Delta_0(A)=0$.

PROPOSITION 1.8. *For $A\in\mathfrak{F}_{\hbar}$ it holds that $\Delta_0(A)=4\partial\bar{\partial}(A)=4\bar{\partial}\partial(A)$.*

Proof. Let us prove that $\Delta_0(A)=4\partial\big(\bar{\partial}(A)\big)$; the second equality then follows from the fact that $\partial\bar{\partial}=\bar{\partial}\partial$. One computes

$$\begin{aligned}4\partial\big(\bar{\partial}(A)\big)&=\hat{\partial}_u\big(\hat{\partial}_u(A)+i\hat{\partial}_v(A)\big)-i\hat{\partial}_v\big(\hat{\partial}_u(A)+i\hat{\partial}_v(A)\big)\\&=\hat{\partial}_u^2(A)+\hat{\partial}_v^2(A)+i\hat{\partial}_u\big(\hat{\partial}_v(A)\big)-i\hat{\partial}_v\big(\hat{\partial}_u(A)\big)\\&=\hat{\partial}_u^2(A)+\hat{\partial}_v^2(A)=\Delta_0(A),\end{aligned}$$

by using Proposition 1.3. □

PROPOSITION 1.9. *Let* $A \in \mathfrak{F}_\hbar$ *be r-holomorphic. Then* $\operatorname{Re} A$ *and* $\operatorname{Im} A$ *fulfill*

$$\hat{\partial}_u \operatorname{Re} A = \hat{\partial}_v \operatorname{Im} A \quad \text{and} \quad \hat{\partial}_v \operatorname{Re} A = -\hat{\partial}_u \operatorname{Im} A,$$

and it follows that $\operatorname{Re} A$ *and* $\operatorname{Im} A$ *are harmonic.*

Proof. Since A is r-holomorphic, it holds that $\bar{\partial} A = 0$, which is equivalent to

$$0 = \big(\hat{\partial}_u + i\hat{\partial}_v\big)\big(\operatorname{Re} A + i \operatorname{Im} A\big) = \hat{\partial}_u \operatorname{Re} A - \hat{\partial}_v \operatorname{Im} A + i\big(\hat{\partial}_u \operatorname{Im} A + \hat{\partial}_v \operatorname{Re} A\big).$$

Since $\operatorname{Re} A$ and $\operatorname{Im} A$ are hermitian, it follows that

$$\hat{\partial}_u \operatorname{Re} A - \hat{\partial}_v \operatorname{Im} A = 0$$
$$\hat{\partial}_v \operatorname{Re} A + \hat{\partial}_u \operatorname{Im} A = 0,$$

which proves the first statement. Moreover, it is then easy to see that

$$\hat{\partial}_u^2 \operatorname{Re} A + \hat{\partial}_v^2 \operatorname{Re} A = \hat{\partial}_u \hat{\partial}_v \operatorname{Im} A - \hat{\partial}_v \hat{\partial}_u \operatorname{Im} A = 0,$$

since $\hat{\partial}_u$ and $\hat{\partial}_v$ commute, by Proposition 1.3. A similar computation is done to show that $\operatorname{Im} A$ is harmonic. $\square$

Integration of r-holomorphic elements is introduced as the inverse of the operator ∂; namely, if A and B are r-holomorphic elements, such that $\partial B = A$, then we call B a *primitive element of* A. Furthermore, we introduce the notation

$$\int A \, d\Lambda$$

to denote an arbitrary primitive element of A. Such r-holomorphic elements A, which have at least one primitive element, are called *integrable*. Clearly, holomorphic elements, being polynomials in Λ, are integrable, and primitive elements may readily be found.

2. Noncommutative Minimal Surfaces

We shall consider the free module $\mathfrak{F}_\hbar^n$ together with its canonical basis

$$e_k = (\underbrace{0, \dots, 0}_{k-1}, \mathbb{1}, 0, \dots, 0)$$

and one extends the action of $\hat{\partial}_a$ as

$$\hat{\partial}_a(\vec{X}) = \hat{\partial}_a(X^i) e_i$$

for $\vec{X} = X^i e_i$ and $a = 1, 2$. An element $\vec{X} \in \mathfrak{F}_\hbar^n$ is called *hermitian* if X^i is hermitian for $i = 1, \dots, n$, and an element $\vec{X} \in \mathfrak{F}_\hbar^n$ is called (r-)holomorphic if X^i is (r-)holomorphic for $i=1,\dots,n$. Moreover, for $\vec{X},\vec{Y}\in\mathfrak{F}_\hbar^n$ one introduces a symmetric bi-$\mathbb{C}$-linear form

$$\langle\vec{X},\vec{Y}\rangle=\sum_{i=1}^{n}\langle X^i,Y^i\rangle\equiv\frac{1}{2}\sum_{i=1}^{n}\left(X^iY^i+Y^iX^i\right).$$

The above form fulfills the following derivation property, with respect to $\hat{\partial}_1$ and $\hat{\partial}_2$:

PROPOSITION 2.1. *For $\vec{X},\vec{Y}\in\mathfrak{F}_\hbar^n$, with $\vec{X}=X^ie_i$ and $\vec{Y}=Y^ie_i$, it holds that*

$$[\langle\vec{X},\vec{Y}\rangle,A]=\langle[X^i,A]e_i,\vec{Y}\rangle+\langle\vec{X},[Y^i,A]e_i\rangle$$

for any $A\in\mathfrak{F}_\hbar$. In particular, it holds that

$$\hat{\partial}_a\langle\vec{X},\vec{Y}\rangle=\langle\hat{\partial}_a\vec{X},\vec{Y}\rangle+\langle\vec{X},\hat{\partial}_a\vec{Y}\rangle,$$

for $a=1,2$.

Proof. From the derivation property of the commutator it follows that

$$[AB+BA,C]=A[B,C]+[B,C]A+B[A,C]+[A,C]B,$$

which may be written as

$$[\langle A,B\rangle,C]=\langle A,[B,C]\rangle+\langle B,[A,C]\rangle. \tag{2.1}$$

Since (2.1) is linear in A and B, the desired result follows. □

We will now introduce noncommutative minimal surfaces in $\mathfrak{F}_\hbar^n$; this is done in analogy with the formulation in conformal coordinates, as given in Section 1.1. It turns out that most of the classical theory can be transferred to the noncommutative setting with essentially no, or only small, modifications.

DEFINITION 2.2. A hermitian element $\vec{X}\in\mathfrak{F}_\hbar^n$ is called a *noncommutative minimal surface* if

$$\Delta_0(X^i)=0 \quad \text{for} \quad i=1,2,\dots,n$$
$$\mathcal{E}=\mathcal{G} \quad \text{and} \quad \mathcal{F}=0,$$

where

$$\mathcal{E}=\langle\hat{\partial}_u\vec{X},\hat{\partial}_u\vec{X}\rangle,\quad \mathcal{G}=\langle\hat{\partial}_v\vec{X},\hat{\partial}_v\vec{X}\rangle,\quad \mathcal{F}=\langle\hat{\partial}_u\vec{X},\hat{\partial}_v\vec{X}\rangle.$$

Remark 2.3. Note that the above definition does, in principle, not rely on the fraction field $\mathfrak{F}_\hbar$, and is also valid in the Weyl algebra $\mathcal{A}_\hbar$. In fact, several results, in what follows, remain true in the Weyl algebra when r-holomorphic elements are replaced by holomorphic elements. We shall comment on this possibility as we proceed and develop the theory.

Remark 2.4. Let us also remark that the concept of noncommutative minimal surface is invariant with respect to automorphisms of $\mathfrak{F}_\hbar$, in analogy with coordinate transformations in classical geometry. Namely, let $\vec{X} \in \mathfrak{F}_\hbar^n$ be a noncommutative minimal surface and let $\phi : \mathfrak{F}_\hbar \to \mathfrak{F}_\hbar$ be a unital $*$-automorphism. Define

$$\tilde{X} = \phi(\vec{X}) = \tilde{X}^i e_i := \phi(X^i) e_i$$

as well as

$$\tilde{U} = \phi(U), \quad \tilde{V} = \phi(V), \quad \hat{\partial}_{\tilde{u}}(A) = \frac{1}{i\hbar}[A, \tilde{V}], \quad \hat{\partial}_{\tilde{v}}(A) = -\frac{1}{i\hbar}[A, \tilde{U}].$$

It follows immediately that $[\tilde{U}, \tilde{V}] = i\hbar\mathbb{1}$ and that

$$\begin{aligned}
&\hat{\partial}_{\tilde{u}}(A) = \phi\big(\hat{\partial}_u \phi^{-1}(A)\big), \quad \hat{\partial}_{\tilde{v}}(A) = \phi\big(\hat{\partial}_v \phi^{-1}(A)\big), \\
&\tilde{\Delta}_0(A) := \hat{\partial}_{\tilde{u}}^2(A) + \hat{\partial}_{\tilde{v}}^2(A) = \phi\big(\Delta_0 \phi^{-1}(A)\big)
\end{aligned}$$

for $A \in \mathfrak{F}_\hbar$. Hence, if $\vec{X}$ is a minimal surface and $\tilde{X} = \phi(\vec{X})$, then

$$\begin{aligned}
(\tilde{X}^i)^* &= \phi(X^i)^* = \phi\big((X^i)^*\big) = \phi(X^i) = \tilde{X}^i \\
\tilde{\Delta}_0(\tilde{X}^i) &= \tilde{\Delta}_0\big(\phi(X^i)\big) = \phi\big(\Delta_0(X^i)\big) = 0 \\
\langle \hat{\partial}_{\tilde{u}} \tilde{X}, \hat{\partial}_{\tilde{u}} \tilde{X} \rangle &= \phi\big(\langle \hat{\partial}_u \vec{X}, \hat{\partial}_u \vec{X} \rangle\big) = \phi(\mathcal{E}) \\
\langle \hat{\partial}_{\tilde{v}} \tilde{X}, \hat{\partial}_{\tilde{v}} \tilde{X} \rangle &= \phi\big(\langle \hat{\partial}_v \vec{X}, \hat{\partial}_v \vec{X} \rangle\big) = \phi(\mathcal{G}) = \phi(\mathcal{E}) \\
\langle \hat{\partial}_{\tilde{u}} \tilde{X}, \hat{\partial}_{\tilde{v}} \tilde{X} \rangle &= \phi\big(\langle \hat{\partial}_u \vec{X}, \hat{\partial}_v \vec{X} \rangle\big) = 0,
\end{aligned}$$

which shows that $\tilde{X}$ fulfills the definition of a noncommutative minimal surface with respect to the generators $\tilde{U}$ and $\tilde{V}$.

Let us now define $\Phi \in \mathfrak{F}_\hbar^n$ as

$$\Phi = \Phi^i e_i = 2\partial(X^i) e_i = \big(\hat{\partial}_u(X^i) - i\hat{\partial}_v(X^i)\big) e_i$$

and prove the following:

PROPOSITION 2.5. *It holds that*

$$\langle \Phi, \Phi \rangle = \mathcal{E} - \mathcal{G} - 2i\mathcal{F}.$$

Proof. One computes

$$\begin{aligned}(\Phi^i)^2 &= \big(\hat{\partial}_u(X^i) - i\hat{\partial}_v(X^i)\big)\big(\hat{\partial}_u(X^i) - i\hat{\partial}_v(X^i)\big)\\ &= \hat{\partial}_u(X^i)^2 - \hat{\partial}_v(X^i)^2 - i\hat{\partial}_u(X^i)\hat{\partial}_v(X^i) - i\hat{\partial}_v(X^i)\hat{\partial}_u(X^i),\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i=1}^n (\Phi^i)^2 &= \sum_{i=1}^n \hat{\partial}_u(X^i)^2 - \sum_{i=1}^n \hat{\partial}_v(X^i)^2\\ &\quad - 2i\sum_{i=1}^n \frac{1}{2}\Big(\hat{\partial}_u(X^i)\hat{\partial}_v(X^i) + \hat{\partial}_v(X^i)\hat{\partial}_u(X^i)\Big)\\ &= \mathcal{E} - \mathcal{G} - 2i\mathcal{F},\end{aligned}$$

which is the desired result. □

PROPOSITION 2.6. $\langle\Phi, \Phi\rangle = 0$ *if and only if* $\mathcal{E} = \mathcal{G}$ *and* $\mathcal{F} = 0$.

Proof. Clearly, if $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$ then Proposition 2.5 gives $\langle\Phi, \Phi\rangle = 0$. Now, assume that $\langle\Phi, \Phi\rangle = 0$. Since $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are hermitian, the $*$-conjugate of the equation $\langle\Phi, \Phi\rangle = 0$ (via Proposition 2.5) gives $\mathcal{E} - \mathcal{G} + 2i\mathcal{F} = 0$ which, together with $\mathcal{E} - \mathcal{G} - 2i\mathcal{F} = 0$ implies that $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$. □

PROPOSITION 2.7. *Assume that* $\vec{X} \in \mathfrak{F}_\hbar^n$ *is hermitian and set* $\Phi = 2\partial(\vec{X})$. *Then the following are equivalent:*

(1) $\vec{X}$ *is a minimal surface,*
(2) Φ *is r-holomorphic and* $\langle\Phi, \Phi\rangle = 0$.

Proof. First, assume that $\vec{X}$ is a minimal surface (which directly implies, by Proposition 2.6, that $\langle\Phi, \Phi\rangle = 0$). By definition, it holds that $\Delta_0(X^i) = 0$, and one computes

$$0 = \Delta_0(X^i) = 4\bar{\partial}\big(\partial(X^i)\big) = 2\bar{\partial}(\Phi^i),$$

which proves that Φ is r-holomorphic. For the other implication, assume that Φ is r-holomorphic and that $\langle\Phi, \Phi\rangle = 0$. From Proposition 2.6 it follows that $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$. Moreover, since Φ^i is r-holomorphic one gets

$$0 = \bar{\partial}(\Phi^i) = 2\bar{\partial}\big(\partial(X^i)\big) = \frac{1}{2}\Delta_0(X^i).$$

Hence, $\vec{X}$ is a minimal surface. □

Note that the theorem remains true if $\vec{X} \in \mathcal{A}_\hbar^n$ and Φ is a assumed to be holomorphic. Hence, the equivalence also holds in the Weyl algebra.

One may straightforwardly define conjugate minimal surfaces; namely, we will call a hermitian $\tilde{X} \in \mathfrak{F}_\hbar^n$ conjugate to the minimal surface $\vec{X} \in \mathfrak{F}_\hbar^n$ if

$$\hat{\partial}_u(\vec{X}) = \hat{\partial}_v(\tilde{X}) \quad \text{and} \quad \hat{\partial}_v(\vec{X}) = -\hat{\partial}_u(\tilde{X}).$$

PROPOSITION 2.8. *Let $\vec{X} \in \mathfrak{F}_\hbar^n$ be a minimal surface. If a hermitian $\tilde{X} \in \mathfrak{F}_\hbar^n$ satisfies*

$$\hat{\partial}_u(\vec{X}) = \hat{\partial}_v(\tilde{X}) \quad \textit{and} \quad \hat{\partial}_v(\vec{X}) = -\hat{\partial}_u(\tilde{X}),$$

then $\tilde{X}$ is a minimal surface.

Proof. One computes

$$\begin{aligned}\Delta_0(\tilde{X}^i) &= \hat{\partial}_u\big(\hat{\partial}_u(\tilde{X}^i)\big) + \hat{\partial}_v\big(\hat{\partial}_v(\tilde{X}^i)\big)\\ &= -\hat{\partial}_u\big(\hat{\partial}_v(X^i)\big) + \hat{\partial}_v\big(\hat{\partial}_u(X^i)\big) = 0,\end{aligned}$$

by using Proposition 1.3. Moreover, it holds that

$$\begin{aligned}\tilde{\mathcal{E}} &= \sum_{i=1}^n \hat{\partial}_u(\tilde{X}^i)^2 = \sum_{i=1}^n \hat{\partial}_v(X^i)^2 = \mathcal{G},\\ \tilde{\mathcal{G}} &= \sum_{i=1}^n \hat{\partial}_v(\tilde{X}^i)^2 = \sum_{i=1}^n \hat{\partial}_u(X^i)^2 = \mathcal{E} \quad (= \mathcal{G} = \tilde{\mathcal{E}}),\\ \tilde{\mathcal{F}} &= \frac{1}{2}\sum_{i=1}^n \Big(\hat{\partial}_u(\tilde{X}^i)\hat{\partial}_v(\tilde{X}^i) + \hat{\partial}_v(\tilde{X}^i)\hat{\partial}_u(\tilde{X}^i)\Big)\\ &= -\frac{1}{2}\sum_{i=1}^n \Big(\hat{\partial}_v(X^i)\hat{\partial}_u(X^i) + \hat{\partial}_u(X^i)\hat{\partial}_v(X^i)\Big) = -\mathcal{F} = 0,\end{aligned}$$

since $\vec{X}$ is assumed to be a minimal surface. Hence, $\tilde{X}$ is a minimal surface. □

Note that the presentation of a noncommutative surface as an element of a free module has also been considered in the context of star products [4], where the focus lies on metric aspects (which does in principle also apply to our case), but there is no discussion of minimal embeddings. However, the Riemannian aspects of noncommutative minimal surfaces are certainly very interesting and deserve further attention.

2.1. NONCOMMUTATIVE WEIERSTRASS REPRESENTATION

The classical theory of minimal surfaces in $\mathbb{R}^3$ is an old and very rich subject. For such minimal surfaces, there are several representation formulas available; i.e., explicit formulas for the parametrization of an arbitrary minimal surface (see e.g.,

[5]). It turns out that one can prove analogous statements in the noncommutative setting.

PROPOSITION 2.9. *Assume that* $\Phi\in\mathfrak{F}_\hbar^3$ *is r-holomorphic, fulfilling* $\langle\Phi,\Phi\rangle=0$ *and* $\Phi^1-i\Phi^2\neq 0$. *Then there exist r-holomorphic* $f,g\in\mathfrak{F}_\hbar$ *such that*

$$\Phi^1=\frac{1}{2}f\left(\mathbb{1}-g^2\right),\quad \Phi^2=\frac{i}{2}f\left(\mathbb{1}+g^2\right),\quad \Phi^3=fg.$$

Moreover, if Φ *is holomorphic then* f *can be chosen to be holomorphic.*

Proof. First, since Φ^1,Φ^2,Φ^3 are r-holomorphic, they commute; thus, one need not be careful with the ordering in what follows. If one sets

$$\begin{aligned}
&f=\Phi^1-i\Phi^2\\
&g=\Phi^3(\Phi^1-i\Phi^2)^{-1}
\end{aligned}$$

then f and g are r-holomorphic (since $\Phi^1-i\Phi^2\neq 0$), and one computes

$$-fg^2=-\left(\Phi^3\right)^2(\Phi^1-i\Phi^2)^{-1}=\Phi^1+i\Phi^2$$

where the last equality follows from $\langle\Phi,\Phi\rangle=0$ (written in the form $(\Phi^1+i\Phi^2)(\Phi^1-i\Phi^2)+(\Phi^3)^2=0$). Now, from $f=\Phi^1-i\Phi^2$ and $-fg^2=\Phi^1+i\Phi^2$, the desired expressions for Φ^1, Φ^2 and Φ^3 follow. Finally, we note that if Φ is holomorphic, then clearly $f=\Phi^1-i\Phi^2$ is holomorphic. □

As a corollary we get an analogue of the Weierstrass representation theorem.

THEOREM 2.10. *Let* $\vec{X}=X^ie_i\in\mathfrak{F}_\hbar^3$ *be a minimal surface for which it holds that* $\partial(X^1-iX^2)\neq 0$. *Then there exist r-holomorphic elements* $f,g\in\mathfrak{F}_\hbar$ *together with* $x^i\in\mathbb{R}$ *(for* $i=1,2,3$*), such that*

$$\begin{aligned}
X^1&=x^1\mathbb{1}+\mathrm{Re}\int\tfrac{1}{2}f(\mathbb{1}-g^2)d\Lambda\\
X^2&=x^2\mathbb{1}+\mathrm{Re}\int\tfrac{i}{2}f(\mathbb{1}+g^2)d\Lambda\\
X^3&=x^3\mathbb{1}+\mathrm{Re}\int fg\,d\Lambda.
\end{aligned}\tag{2.2}$$

Conversely, for any r-holomorphic f *and* g *such that* $f(1-g^2)$, $f(1+g^2)$ *and* fg *are integrable, equation* (2.2) *defines a minimal surface.*

Proof. Assume that $\vec{X}$ is a minimal surface. Setting $\Phi=2\partial\vec{X}$ it follows from Proposition 2.7 that Φ is r-holomorphic and $\langle\Phi,\Phi\rangle=0$. The assumption $\partial(X^1-iX^2)\neq 0$ is equivalent to $\Phi^1-i\Phi^2\neq 0$. Therefore, Proposition 2.9 gives the existence of r-holomorphic f and g such that

$$\Phi^1=\frac{1}{2}f\left(\mathbb{1}-g^2\right),\quad \Phi^2=\frac{i}{2}f\left(\mathbb{1}+g^2\right),\quad \Phi^3=fg.$$

These equations may be integrated as in (2.2), and since $\partial\mathrm{Re}(A)=\partial A/2$ when A is r-holomorphic, they satisfy $\Phi=2\partial\vec{X}$. Now, assume that f and g are r-holomorphic and that the integrals in (2.2) are defined. It is easy to check that (2.2) gives r-holomorphic $\Phi=2\partial\vec{X}$ such that $\langle\Phi,\Phi\rangle=0$. From Proposition 2.7 it follows that $\vec{X}$ is a minimal surface. □

There is another classical representation formula, which assigns a minimal surface to an arbitrary holomorphic function F. The theorem below does not rely on r-holomorphic elements, and therefore also holds in the Weyl algebra when F is chosen to be holomorphic.

THEOREM 2.11. *Let $F\in\mathfrak{F}_\hbar$ be r-holomorphic and assume that*

$$\Phi^1=\left(1-\Lambda^2\right)F,\quad \Phi^2=i\left(1+\Lambda^2\right)F,\quad \Phi^3=2\Lambda F$$

are integrable. Then $\vec{X}=X^ie_i\in\mathfrak{F}_\hbar^3$, defined by

$$X^i=x^i\mathbb{1}+\mathrm{Re}\int\Phi^i\mathrm{d}\Lambda,$$

is a minimal surface for arbitrary $x^1,x^2,x^3\in\mathbb{R}$.

Proof. By definition, X is hermitian, and one computes that

$$\begin{aligned}2\partial(X^i)&=\partial\int\Phi^i\mathrm{d}\Lambda+\partial\left(\left(\int\Phi^i\mathrm{d}\Lambda\right)^*\right)\\&=\partial\int\Phi^i\mathrm{d}\Lambda=\Phi^i,\end{aligned}$$

since ∂ applied to a quotient of polynomials in Λ^* gives zero. Moreover, a simple computation shows that $\langle\Phi,\Phi\rangle=0$ for every r-holomorphic $F\in\mathfrak{F}_\hbar$. Finally, since Φ^i is r-holomorphic, it follows from Proposition 2.7 that $\vec{X}$ is a minimal surface. □

In the geometric setting, a minimal surface constructed via Theorem 2.11 has a normal vector given by

$$N=\frac{1}{1+u^2+v^2}\left(2u,2v,u^2+v^2-1\right).$$

Let us show that, with respect to the symmetric form $\langle\cdot,\cdot\rangle$, a noncommutative normal can be constructed.

PROPOSITION 2.12. *Let $\vec{X}\in\mathfrak{F}_\hbar^3$ be a minimal surface given by an r-holomorphic element $F\in\mathfrak{F}_\hbar$, as in Theorem 2.11. Then $\vec{N}=N^ie_i\in\mathfrak{F}_\hbar^3$, given by*

$$N^1 = \Lambda + \Lambda^*, \quad N^2 = -i(\Lambda - \Lambda^*), \quad N^3 = \frac{1}{2}\big(\Lambda\Lambda^* + \Lambda^*\Lambda\big) - \mathbb{1}$$

satisfies $\langle \hat{\partial}_u \vec{X}, \vec{N} \rangle = \langle \hat{\partial}_v \vec{X}, \vec{N} \rangle = 0$.

Proof. The proof consists of a straightforward computation. The statement that $\langle \hat{\partial}_u \vec{X}, \vec{N} \rangle = \langle \hat{\partial}_v \vec{X}, \vec{N} \rangle = 0$ is equivalent to $\langle \partial \vec{X}, \vec{N} \rangle = \langle \bar{\partial} \vec{X}, \vec{N} \rangle = 0$, which in turn is equivalent to $\langle \Phi, \vec{N} \rangle = \langle \Phi^*, \vec{N} \rangle = 0$. Since $\vec{N}$ is hermitian, it is enough to prove that $\langle \Phi, \vec{N} \rangle = 0$. With Φ as in Theorem 2.11, one computes that

$$\begin{aligned} N^1\Phi^1 + N^2\Phi^2 + N^3\Phi^3 &= \Lambda\Lambda^*\Lambda F - \Lambda^*\Lambda^2 F \\ &= [\Lambda, \Lambda^*]\Lambda F = 2\hbar\Lambda F, \end{aligned}$$

as well as

$$\begin{aligned} \Phi^1 N^1 + \Phi^2 N^2 + \Phi^3 N^3 &= F\Lambda\Lambda^*\Lambda - F\Lambda^2\Lambda^* \\ &= F\Lambda[\Lambda^*, \Lambda] = -2\hbar F\Lambda, \end{aligned}$$

which implies that $\langle \Phi, \vec{N} \rangle = 0$. □

Let us end this section by noting that the "mean curvature" of a minimal surface vanishes. As in differential geometry, given a normal element $\vec{N} \in \mathfrak{F}_\hbar^3$, one may define the mean curvature (in conformal coordinates) as

$$H(\vec{N}) = -\frac{1}{2\mathcal{E}} \langle \hat{\partial}_u \vec{X}, \hat{\partial}_u \vec{N} \rangle - \frac{1}{2\mathcal{E}} \langle \hat{\partial}_v \vec{X}, \hat{\partial}_v \vec{N} \rangle \equiv \frac{1}{\mathcal{E}} H_0(\vec{N}).$$

Hence, if $\Delta_0(\vec{X}) = 0$ then it follows that

$$\begin{aligned} 2H_0(\vec{N}) &= \langle \hat{\partial}_u^2 \vec{X}, \vec{N} \rangle - \hat{\partial}_u \langle \hat{\partial}_u \vec{X}, \vec{N} \rangle + \langle \hat{\partial}_v^2 \vec{X}, \vec{N} \rangle - \hat{\partial}_v \langle \hat{\partial}_v \vec{X}, \vec{N} \rangle \\ &= \langle \Delta_0(\vec{X}), \vec{N} \rangle = 0. \end{aligned}$$

Conversely, if $H_0(\vec{N}) = 0$ then $\langle \Delta_0 \vec{X}, \vec{N} \rangle = 0$, and it follows from $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = 0$ that

$$\langle \hat{\partial}_u \vec{X}, \Delta_0(\vec{X}) \rangle = \langle \hat{\partial}_v \vec{X}, \Delta_0(\vec{X}) \rangle = 0.$$

However, since $\langle \cdot, \cdot \rangle$ is not $\mathfrak{F}_\hbar$-linear, these equations do not necessarily imply that $\Delta_0(\vec{X}) = 0$.

3. Examples

3.1. ALGEBRAIC MINIMAL SURFACES

A holomorphic element F may be integrated an arbitrary number of times. Hence, choosing a holomorphic element $\tilde{F}$ such that $\partial^3\tilde{F}=F$, the representation formula in Theorem 2.11 may be integrated (via partial integration) to yield

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + \mathrm{Re}\left((\mathbb{1}-\Lambda^2)\partial^2\tilde{F} + 2\Lambda\partial\tilde{F} - 2\tilde{F}\right) \equiv x^1\mathbb{1} + \mathrm{Re}\left(\Omega^1\right)\\
X^2 &= x^2\mathbb{1} + \mathrm{Re}\left(i(\mathbb{1}+\Lambda^2)\partial^2\tilde{F} - 2i\Lambda\partial\tilde{F} + 2i\tilde{F}\right) \equiv x^2\mathbb{1} + \mathrm{Re}\left(\Omega^2\right)\\
X^3 &= x^3\mathbb{1} + \mathrm{Re}\left(2\Lambda\partial^2\tilde{F} - 2\partial\tilde{F}\right) \equiv x^3\mathbb{1} + \mathrm{Re}\left(\Omega^3\right).
\end{aligned} \tag{3.1}$$

In other words, every holomorphic $\tilde{F}(\Lambda)$ gives rise to a minimal surface via (3.1). As an example, let us choose $\tilde{F}(\Lambda)=\Lambda^n$ (with $n\geq 2$), which gives

$$\begin{aligned}
\Omega^1 &= (n-1)\left(n\Lambda^{n-2} - (n-2)\Lambda^n\right)\\
\Omega^2 &= i(n-1)\left(n\Lambda^{n-2} + (n-2)\Lambda^n\right)\\
\Omega^3 &= 2n(n-2)\Lambda^{n-1}.
\end{aligned}$$

We note that the real part of Λ^n consists of the total symmetrization of all monomials with an even (total) power of V. That is,

$$\mathrm{Re}(\Lambda^n) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\,\mathrm{Sym}(U^{n-2k}V^{2k}),$$

where $\mathrm{Sym}(U^kV^l)$ denotes the sum of all terms of different permutations of k U's and l V's, and $\lfloor r\rfloor$ denotes the integer part of $r\in\mathbb{R}$. Likewise, it holds that

$$\mathrm{Re}(i\Lambda^n) = \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-2k+1}V^{2k-1}\right),$$

and one obtains the following explicit representation formulas

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + \sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-2(k+1)}V^{2k}\right) - \frac{n-2}{n}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-2k}V^{2k}\right)\\
X^2 &= x^2\mathbb{1} + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-1-2k}V^{2k-1}\right)\\
&\quad + \frac{n-2}{n}\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-2k+1}V^{2k-1}\right)\\
X^3 &= x^3\mathbb{1} + \frac{2(n-2)}{n-1}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n-1-2k}V^{2k}\right).
\end{aligned}$$

Thus, one may construct a noncommutative minimal surface from the classical one by completely symmetrizing the polynomials. As an illustration, let us consider the first two non-trivial minimal surfaces arising in this way.

For $\tilde{F}(\Lambda)=\Lambda^3$ (corresponding to $F(\Lambda)=6$), one obtains the noncommutative Enneper surface

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + U - \frac{1}{3}U^3 + \frac{1}{3}\,\mathrm{Sym}\left(UV^2\right)\\
X^2 &= x^2\mathbb{1} - V + \frac{1}{3}V^3 - \frac{1}{3}\,\mathrm{Sym}\left(U^2V\right)\\
X^3 &= x^3\mathbb{1} + U^2 - V^2
\end{aligned}$$

which, using that $[U,V]=i\hbar\mathbb{1}$, can be written as

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + U + UV^2 - \frac{1}{3}U^3 - i\hbar V\\
X^2 &= x^2\mathbb{1} - V - U^2V + \frac{1}{3}V^3 + i\hbar U\\
X^3 &= x^3\mathbb{1} + U^2 - V^2.
\end{aligned}$$

For $\tilde{F}(\Lambda)=\Lambda^4$ (corresponding to $F(\Lambda)=24\Lambda$) one obtains

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + U^2 - V^2 - \frac{1}{2}\left(U^4+V^4\right) + \frac{1}{2}\,\mathrm{Sym}\left(U^2V^2\right)\\
X^2 &= x^2\mathbb{1} - UV - VU - \frac{1}{2}\,\mathrm{Sym}(U^3V) + \frac{1}{2}\,\mathrm{Sym}(UV^3)\\
X^3 &= x^3\mathbb{1} + \frac{4}{3}U^3 - \frac{4}{3}\,\mathrm{Sym}(UV^2),
\end{aligned}$$

which may be written as

$$\begin{aligned}
X^1 &= \left(x^1 - \frac{3}{2}\hbar^2\right)\mathbb{1} + U^2 - V^2 - \frac{1}{2}\left(U^4+V^4\right) + 3U^2V^2 - 6i\hbar U\\
X^2 &= (x^2 + i\hbar)\mathbb{1} - 2UV - 2U^3V + 2UV^3 - 3i\hbar V^2 + 3i\hbar U^2\\
X^3 &= x^3\mathbb{1} + \frac{4}{3}U^3 - 4UV^2 + 4i\hbar V.
\end{aligned}$$

Algebraic surfaces can also be obtained from Theorem 2.10, some of which cannot be constructed as in Theorem 2.11. For instance, choosing $f=2$ and $g=\Lambda^n$ gives the higher order Enneper surfaces as

$$\begin{aligned}
X^1 &= x^1\mathbb{1} + U - \frac{1}{2n+1}\sum_{k=0}^{n}(-1)^k\,\mathrm{Sym}\left(U^{2n+1-2k}V^{2k}\right)\\
X^2 &= x^2\mathbb{1} - V + \frac{1}{2n+1}\sum_{k=1}^{n+1}(-1)^k\,\mathrm{Sym}\left(U^{2n+2-2k}V^{2k-1}\right)\\
X^3 &= x^3\mathbb{1} + \frac{2}{n+1}\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^k\,\mathrm{Sym}\left(U^{n+1-2k}V^{2k}\right),
\end{aligned}$$

which, for $n=2$, becomes

$$X^1 = x^1\mathbb{1} + U + 2U^3V^2 - UV^4 - \frac{1}{5}U^5 - 6i\hbar U^2V + 2i\hbar V^3 - 3\hbar^2 U$$
$$X^2 = x^2\mathbb{1} - V + 2U^2V^3 - U^4V - \frac{1}{5}V^5 - 6i\hbar UV^2 + 2i\hbar U^3 - 3\hbar^2 V$$
$$X^3 = x^3\mathbb{1} - 2UV^2 + \frac{2}{3}U^3 + 2i\hbar V.$$

3.2. MINIMAL SURFACES IN $\mathfrak{F}_\hbar^4$

For two holomorphic functions $f(z)$ and $g(z)$, it is well known (cp. [7]) that one can construct a minimal surface in $\mathbb{R}^4$ by setting

$$\vec{x} = (\operatorname{Re} f(z), \operatorname{Im} f(z), \operatorname{Re} g(z), \operatorname{Im} g(z)).$$

This extends to noncommutative minimal surfaces:

PROPOSITION 3.1. *Let $f, g \in \mathfrak{F}_\hbar$ be r-holomorphic and set $\vec{X} = X^i e_i \in \mathfrak{F}_\hbar^4$ with*

$$\left(X^1, X^2, X^3, X^4\right) = \left(\operatorname{Re} f, \operatorname{Im} f, \operatorname{Re} g, \operatorname{Im} g\right).$$

Then $\vec{X}$ is a minimal surface.

Proof. Defining $\Phi = 2\partial\vec{X}$ yields

$$\left(\Phi^1, \Phi^2, \Phi^3, \Phi^4\right) = \left(\partial f, -i\partial f, \partial g, -i\partial g\right),$$

which implies that $\langle \Phi, \Phi \rangle = 0$. From Proposition 2.7 it follows that $\vec{X}$ is a minimal surface (since Φ^i is clearly r-holomorphic). □

As an example, let us choose $f(\Lambda) = \Lambda^n$ and $g(\Lambda) = \Lambda^m$, which implies that

$$X^1 = \operatorname{Re}(\Lambda^n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \operatorname{Sym}\left(U^{n-2k}V^{2k}\right)$$
$$X^2 = \operatorname{Im}(\Lambda^n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^{k+1} \operatorname{Sym}\left(U^{n-2k+1}V^{2k-1}\right)$$
$$X^3 = \operatorname{Re}(\Lambda^m) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \operatorname{Sym}\left(U^{m-2k}V^{2k}\right)$$
$$X^4 = \operatorname{Im}(\Lambda^m) = \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^{k+1} \operatorname{Sym}\left(U^{m-2k+1}V^{2k-1}\right),$$

and for $f=\Lambda$ and $g=\Lambda^2$ one obtains

$$(X^1, X^2, X^3, X^4) = (U, V, U^2 - V^2, 2UV - i\hbar\mathbb{1}).$$

3.3. NONCOMMUTATIVE CATENOIDS

The minimal surfaces in the preceding section are algebraic in the sense that they arise from (finite) polynomials. Several classical minimal surfaces, such as the catenoid, are constructed in terms of analytic functions, which are, a priori, not defined in the algebra. However, as we shall see, one may construct particular representations in which certain power series are well defined. (A different approach to the catenoid was taken in [1].)

Let $\mathcal{V}$ be the vector space consisting of infinite sequences of complex numbers

$$\mathcal{V} = \{(x_0, x_1, x_2, \ldots) : x_i \in \mathbb{C} \text{ for } i \in \mathbb{N}_0\},$$

and we denote the canonical basis vectors by $|n\rangle$, $n \in \mathbb{N}_0$. For convenience, we shall write an element $x = (x_0, x_1, x_2, \ldots) \in \mathcal{V}$ as a formal sum

$$x = \sum_{k=0}^{\infty} x_k |k\rangle.$$

The space of linear operators $\mathcal{V} \to \mathcal{V}$ is denoted by $L(\mathcal{V})$. Moreover, we introduce the subspace $\mathcal{V}_0 \subset \mathcal{V}$ of finite linear combinations

$$\mathcal{V}_0 = \{x \in \mathcal{V} : |i : x_i \neq 0| < \infty\},$$

and denote the set of linear operators with domain $\mathcal{V}_0$ by $L(\mathcal{V}_0, \mathcal{V})$. As is well known, the Weyl algebra can be represented on $\mathcal{V}$ by introducing operators $a, a^\dagger \in L(\mathcal{V})$, defined by

$$\begin{aligned} a|0\rangle &= 0 \\ a|n\rangle &= \sqrt{n}|n-1\rangle \quad \text{for} \quad n \geq 1 \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \end{aligned}$$

fulfilling $[a, a^\dagger]|n\rangle = |n\rangle$, and then setting

$$\begin{aligned} U &= \sqrt{\frac{\hbar}{2}}(a^\dagger + a) \\ V &= i\sqrt{\frac{\hbar}{2}}(a^\dagger - a), \end{aligned}$$

from which it follows that $\Lambda = U + iV = \sqrt{2\hbar}a$ and $\Lambda^\dagger \equiv \Lambda^* = U - iV = \sqrt{2\hbar}a^\dagger$. We note that the operators U and V leave the subspace $\mathcal{V}_0$ invariant. Let us recall two useful formulas:

LEMMA 3.2.

$$a^k|n\rangle = \begin{cases} \sqrt{\frac{n!}{(n-k)!}}|n-k\rangle & \textit{if } k \leq n \\ 0 & \textit{if } k > n \end{cases} \tag{3.2}$$

$$(a^\dagger)^k|n\rangle = \sqrt{\frac{(n+k)!}{n!}}|n+k\rangle. \tag{3.3}$$

For arbitrary $\lambda \in \mathbb{C}$, we define linear operators $e^{\lambda a}, e^{\lambda a^\dagger} \in L(\mathcal{V}_0, \mathcal{V})$ as

$$e^{\lambda a}|n\rangle = \sum_{k=0}^{\infty} \frac{(\lambda a)^k}{k!}|n\rangle = \sum_{k=0}^{n} \frac{\lambda^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle$$
$$e^{\lambda a^\dagger}|n\rangle = \sum_{k=0}^{\infty} \frac{(\lambda a^\dagger)^k}{k!}|n\rangle = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\sqrt{\frac{(n+k)!}{n!}}|n+k\rangle.$$

Furthermore, let us introduce $\hat{\partial}_u, \hat{\partial}_v, \partial, \bar{\partial}, \Delta_0 : L(\mathcal{V}) \to L(\mathcal{V})$, defined via commutators, as in Section 1.2. Since U and V leave $\mathcal{V}_0$ invariant, the aforementioned maps can be considered as maps $L(\mathcal{V}_0, \mathcal{V}) \to L(\mathcal{V}_0, \mathcal{V})$.

The classical catenoid may be parametrized as ($z = u + iv$)

$$x^1(u, v) = \mathrm{Re}(\cosh z) = \cosh u \cos v$$
$$x^2(u, v) = \mathrm{Re}(-i \sinh z) = \cosh u \sin v$$
$$x^3(u, v) = \mathrm{Re}(z) = u$$

arising from the Weierstrass data $f(z) = -e^{-z}$ and $g(z) = -e^z$ (cp. Theorem 2.10).[2] In analogy, we set

$$X^1 = \frac{1}{4}\left(e^{\Lambda} + e^{-\Lambda} + e^{\Lambda^\dagger} + e^{-\Lambda^\dagger}\right)$$
$$X^2 = -\frac{i}{4}\left(e^{\Lambda} - e^{-\Lambda} - e^{\Lambda^\dagger} + e^{-\Lambda^\dagger}\right)$$
$$X^3 = U$$

which implies that $X^1, X^2, X^3 \in L(\mathcal{V}_0, \mathcal{V})$; we will now show that $\Delta_0(X^i) = 0$ for $i = 1, 2, 3$.

LEMMA 3.3. *For $\lambda \in \mathbb{C}$, it holds that*

$$[e^{\lambda a}, a^\dagger]|n\rangle = \lambda e^{\lambda a}|n\rangle \tag{3.4}$$
$$[e^{\lambda a^\dagger}, a]|n\rangle = -\lambda e^{\lambda a^\dagger}|n\rangle. \tag{3.5}$$

[2] Note that there exist other possibilities for f and g.

From the above result, one easily deduces

$$\partial e^{\lambda\Lambda}|n\rangle = \lambda e^{\lambda\Lambda}|n\rangle \qquad \bar{\partial} e^{\lambda\Lambda}|n\rangle = 0$$
$$\bar{\partial} e^{\lambda\Lambda^\dagger}|n\rangle = \lambda e^{\lambda\Lambda^\dagger}|n\rangle \qquad \partial e^{\lambda\Lambda^\dagger}|n\rangle = 0$$

for arbitrary $\lambda \in \mathbb{C}$. Since $\Delta_0(X^i) = 4\bar{\partial}\partial X^i$ one obtains

$$\Delta_0(X^1)|n\rangle = \bar{\partial}\Big(e^{\Lambda} - e^{-\Lambda}\Big)|n\rangle = 0$$
$$\Delta_0(X^2)|n\rangle = -i\bar{\partial}\Big(e^{\Lambda} + e^{-\Lambda}\Big)|n\rangle = 0$$
$$\Delta_0(X^3)|n\rangle = 2\bar{\partial}\partial\big(\Lambda + \Lambda^\dagger\big)|n\rangle = 2\bar{\partial}(\mathbb{1})|n\rangle = 0.$$

Hence, $\Delta_0 X^i$, for $i = 1, 2, 3$, are 0 as operators in $L(\mathcal{V}_0, \mathcal{V})$.

What about the condition that the parametrization is conformal? That is

$$\langle \hat{\partial}_u \vec{X}, \hat{\partial}_u \vec{X}\rangle = \langle \hat{\partial}_v \vec{X}, \hat{\partial}_v \vec{X}\rangle \text{ and } \langle \hat{\partial}_u \vec{X}, \hat{\partial}_v \vec{X}\rangle = 0.$$

Since X^1 and X^2 do not preserve $\mathcal{V}_0$, their composition is a priori not well defined. However, algebraically, the above is equivalent to $\langle \Phi, \Phi\rangle = 0$ (cp. Proposition 2.6); with

$$\Phi^1|n\rangle = 2\partial X^1|n\rangle = \frac{1}{2}\Big(e^{\Lambda} - e^{-\Lambda}\Big)|n\rangle$$
$$\Phi^2|n\rangle = 2\partial X^2|n\rangle = -\frac{i}{2}\Big(e^{\Lambda} + e^{-\Lambda}\Big)|n\rangle$$
$$\Phi^3|n\rangle = |n\rangle$$

the expression $\langle \Phi, \Phi\rangle$ is well defined, since $e^{\pm\Lambda}$ maps $\mathcal{V}_0$ into $\mathcal{V}_0$, and one readily checks that $\langle \Phi, \Phi\rangle = 0$.

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