Math. Z. (2011) 267:875–886
DOI 10.1007/s00209-009-0651-3

Mathematische Zeitschrift

Capillary surfaces in a convex cone

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Received: 19 March 2009 / Accepted: 2 November 2009 / Published online: 1 December 2009

Abstract We show that a compact embedded hypersurface $S \subset \mathbb{R}^{n+1}$ with constant higher-order mean curvature in a convex piecewise smooth cone C which is perpendicular to ∂C is part of a hypersphere. Also we prove that an embedded disk type constant mean curvature surface $S \subset \mathbb{R}^3$ in a convex polyhedral cone C which makes constant contact angles with ∂C is a spherical cap if C has at most five faces. This condition on the number of faces can be dropped if C is a right cone over a regular n-gon and the contact angles are the same on ∂S.

Mathematics Subject Classification (2000) 53A10 · 49Q10

0 Introduction

Spheres and the Delaunay surfaces have long been known as surfaces of constant mean curvature (CMC) in $\mathbb{R}^3$. The former is compact and round and the latter is noncompact and rotational. In 1950s Alexandrov [2] showed that a compact embedded CMC hypersurface in $\mathbb{R}^{n+1}$ is a round hypersphere, and Hopf [5] proved that an immersed CMC sphere in $\mathbb{R}^3$ is round. However, contrary to Hopf's conjecture, Wente [12] constructed an immersed CMC torus. Since then many compact immersed CMC surfaces have been found [1,6]. But Ros [9]

Work of J. Choe was supported in part by KRF-2007-313-C00057. Work of S.-H. Park was supported in part by KOSEF-R17-2008-001-01001-0.

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extended Alexandrov's result by showing that a compact embedded hypersurface of constant higher-order mean curvature in $\mathbb{R}^{n+1}$ is a round sphere.

The simplest compact CMC surface with nonempty boundary is a spherical cap. In fact Nitsche [7] showed that an immersed disk type CMC surface in a ball which makes a constant contact angle with the boundary sphere is a spherical cap. Moreover the first-named author [3] proved that given a domain $D \subset \mathbb{R}^3$ bounded by planes or spheres, every immersed disk type CMC surface in D which makes a constant contact angle with ∂D and has less than four vertices is part of a round sphere. The upper bound of three on the number of vertices in this result is critical in applying the Poincaré–Hopf theorem. Indeed there is a simple counterexample with four vertices: A rectangular domain bounded by two line segments and two circular arcs on a cylinder is a nonspherical capillary surface in a prism.

Now one can ask a natural question: When can the capillary surfaces with more than three vertices in a domain be necessarily part of a sphere? A capillary surface in a domain D is a CMC surface making a piecewise constant contact angle with ∂D, assuming no influence of gravity. In this paper we show that if $S \subset \mathbb{R}^{n+1}$ is a compact embedded hypersurface with constant higher order mean curvature in a convex piecewise smooth cone C which is perpendicular to ∂C, then S is part of a round hypersphere (Theorem 1). And we prove that every embedded disk type capillary surface in a convex polyhedral cone with at most 5 faces is part of a round sphere (Theorem 2). Furthermore, it is proved that if C is a right cone over a regular n-gon and S is a capillary surface in C with contact angle $\equiv$ const on ∂S, then S is round (Theorem 3). In the proof of Theorem 1, the Minkowski formula and the Reilly formula are used as in [9]. For Theorems 2 and 3 we use the Poincaré–Hopf theorem [3] and the parallel H-surface [10].

1 Minkowski formula

In order to prove Theorem 1 for hypersurfaces in $\mathbb{R}^{n+1}$, we need to extend the Minkowski formula in this section and introduce the Reilly formula in the next section. Throughout this paper the volume forms in the integrals will be dropped for notational convenience.

Let S be a compact immersed hypersurface in $\mathbb{R}^{n+1}$ and A its volume. When S is embedded, S encloses a domain whose volume is denoted by V. Even when S has self intersection and nonempty boundary one can naturally define the enclosed volume V with respect to $p \in \mathbb{R}^{n+1}$ to be the volume of the cone $p \divideontimes S$ by counting multiplicity. If η denotes the outward unit normal to S and $X(q) := \overrightarrow{pq}$, then we have

$$(n+1)V = \int_S \langle X, \eta \rangle. \tag{1}$$

One can easily get (1) by integrating $\bar{\Delta}|X|^2 = 2(n+1)$ on $p \divideontimes S$, where $\bar{\Delta}$ is the Laplacian on $\mathbb{R}^{n+1}$. On the other hand, if S is perpendicular to $p \divideontimes \partial S$ along ∂S, the first variation of S under the homothetic expansion in $\mathbb{R}^{n+1}$ with center at p gives

$$A = \int_S H\langle X, \eta \rangle, \tag{2}$$

where H is the mean curvature of S. (2) can also be obtained by integrating

$$\frac{1}{2n}\,\Delta|X|^2 = 1 - H\langle X, \eta\rangle \tag{3}$$

on S, where Δ is the Laplacian on S.

Minkowski generalized (1) and (2) as follows. Assume $\partial S = \phi$ and let S_t be a parallel surface of S, i.e., the set of all points with distance t from S in η direction. If dS and $\kappa_1, \ldots, \kappa_n$ denote the volume form and the principal curvatures of S, respectively, then the volume form of S_t is

$$dS_t = (1 + \kappa_1 t) \cdots (1 + \kappa_n t) dS = P_n(t) dS, \tag{4}$$

where

$$P_n(t) := (1 + \kappa_1 t) \cdots (1 + \kappa_n t) = 1 + \binom{n}{1} H_1\, t + \cdots + \binom{n}{n} H_n\, t^n.$$

Being the elementary symmetric polynomial of degree k in $\kappa_1, \ldots, \kappa_n$, H_k is called the k-th *order mean curvature* of S ($H_1 = H$). Furthermore the mean curvature of S_t is

$$H(t) = \frac{1}{n} \sum_i \frac{\kappa_i}{1 + \kappa_i t} = \frac{P_n{}'(t)}{n P_n(t)}.$$

Hence, integrating (3) on S_t, we get for all sufficiently small t

$$0 = \int_{S_t} \{1 - H(t)\langle X + t\eta, \eta\rangle\} = \int_S \left\{P_n(t) - \frac{t}{n} P_n{}'(t) - \frac{1}{n} P_n{}'(t)\langle X, \eta\rangle\right\}. \quad \text{(by (4))} \tag{5}$$

Equating the like terms in (5) yields the Minkowski formula:

$$\int_S (H_{k-1} - H_k\langle X, \eta\rangle) = 0, \quad k = 1, \ldots, n, \quad \text{with } H_0 = 1. \tag{6}$$

We now obtain the Minkowski formula for immersed hypersurfaces with nonempty boundary in the following.

Proposition 1 *Let C be a domain in $\mathbb{R}^{n+1}$ which is a cone with piecewise smooth boundary and with vertex at the origin. Let S be an immersed hypersurface in $\mathbb{R}^{n+1}$ with $\partial S \subset \partial C$ such that near ∂S S is inside C and perpendicular to ∂C. Then we have*

$$\int_S (H_{k-1} - H_k\langle X, \eta\rangle) = 0, \quad k = 1, \ldots, n. \tag{7}$$

Proof Integrating (3) on S_t and applying the Stokes theorem, we get as in (5)

$$0 = \int_S \left\{P_n(t) - \frac{t}{n} P_n{}'(t) - \frac{1}{n} P_n{}'(t)\langle X, \eta\rangle\right\} + \int_{\partial S_t} \frac{1}{n}\,|X + t\eta|\,\frac{\partial |X + t\eta|}{\partial \nu}, \tag{8}$$

where ν is the outward unit conormal to ∂S_t on S_t. Note that ν is perpendicular to the tangent plane $T_q\partial C$ and that $X(q) + t\eta$ is parallel to $T_q\partial C$. Hence $\partial|X + t\eta|/\partial\nu = 0$ and the second integral in (8) vanishes. Then (7) follows from the vanishing of the first integral in (8) for all small t. ◻

2 Reilly formula

A basic tool in tensor analysis is the Ricci identity: If X, Y, Z are vector fields and α is a 1-form on a Riemannian manifold M with curvature tensor R, then

$$((\nabla_X \nabla_Y - \nabla_Y \nabla_X)\alpha)(Z) = \alpha(R(X, Y)Z).$$

Given a smooth function f on M, one can obtain Bochner's formula by applying the Ricci identity to df and taking trace:

$$\langle \Delta df, df\rangle = |\nabla df|^2 + \frac{1}{2}\Delta|df|^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

If we integrate Bochner's formula on a domain D in M^{n+1} and use the Stokes theorem, we can get the Reilly formula [8]:

$$\int\limits_D \left\{(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \mathrm{Ric}(\bar{\nabla} f, \bar{\nabla} f)\right\} = \int\limits_{\partial D} \left\{\left(2\Delta f + nH\frac{\partial f}{\partial \eta}\right)\frac{\partial f}{\partial \eta} + \mathrm{II}(\nabla f, \nabla f)\right\},$$

where $\bar{\Delta} f, \bar{\nabla}^2 f, \bar{\nabla} f$ are the Laplacian, Hessian, gradient of f in M respectively, and Δf, $\nabla f, H, \eta, \mathrm{II}$ are the Laplacian of f on ∂D, gradient of f, mean curvature, outward unit normal, and second fundamental form of ∂D, respectively.

Ros [9] used the Reilly formula to prove $\int_S 1/H \geq (n+1)V$ for a compact embedded hypersurface $S \subset \mathbb{R}^{n+1}$. We extend his result as follows.

Proposition 2 *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary and with vertex at the origin. Let $S \subset C$ be an embedded hypersurface with $\partial S \subset \partial C$ such that S is perpendicular to ∂C along ∂S. Let H be the mean curvature of S and V the volume of the domain D enclosed by S and ∂C. If $H > 0$ on S, then*

$$\int\limits_S \frac{1}{H} \geq (n+1)V \tag{9}$$

and equality holds if and only if S is a spherical cap.

Proof To apply the Reilly formula it is necessary to make ∂D smooth. Let $D_\varepsilon \subset D$ be a domain with smooth boundary which is obtained from D by chipping away points of distance $< \varepsilon$ from $\partial S \cup \hat{C}$, $\hat{C} := \{q \in \partial C : \partial C$ has no tangent plane at $q\}$. Define $f \in C^\infty(D_\varepsilon)$ to be the solution to the problem with mixed boundary condition:

$$\bar{\Delta} f = 1 \text{ in } D_\varepsilon, \quad f = 0 \text{ on } \partial D_\varepsilon \backslash \partial C \quad \text{and} \quad \frac{\partial f}{\partial \eta} = 0 \text{ on } \partial D_\varepsilon \cap \partial C. \tag{10}$$

Then the Reilly formula gives

$$\int\limits_{D_\varepsilon} \left\{(\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2\right\} = \int\limits_{\partial D_\varepsilon \backslash \partial C} nH\left(\frac{\partial f}{\partial \eta}\right)^2 + \int\limits_{\partial D_\varepsilon \cap \partial C} \mathrm{II}(\nabla f, \nabla f).$$

The convexity of C implies $\int_{\partial D_\varepsilon \cap \partial C} \mathrm{II}(\nabla f, \nabla f) \geq 0$. Therefore the Cauchy–Schwarz inequality $(\bar{\Delta} f)^2 \leq (n+1)|\bar{\nabla}^2 f|^2$ gives

$$\frac{\mathrm{Vol}(D_\varepsilon)}{n+1} \geq \int\limits_{\partial D_\varepsilon \backslash \partial C} H\left(\frac{\partial f}{\partial \eta}\right)^2.$$

On the other hand,

$$\mathrm{Vol}(D_\varepsilon)^2 = \left(\int\limits_{D_\varepsilon} \bar{\Delta} f\right)^2 = \left(\int\limits_{\partial D_\varepsilon \setminus \partial C} \frac{\partial f}{\partial \eta}\right)^2$$
$$\leq \int\limits_{\partial D_\varepsilon \setminus \partial C} H \left(\frac{\partial f}{\partial \eta}\right)^2 \int\limits_{\partial D_\varepsilon \setminus \partial C} \frac{1}{H} \leq \frac{\mathrm{Vol}(D_\varepsilon)}{n+1} \int\limits_{\partial D_\varepsilon \setminus \partial C} \frac{1}{H}.$$

Hence letting $\varepsilon \to 0$ yields (9). Here it should be noted that $H \to \infty$ near $\partial S \cup \hat{C}$.

If equality holds, then the Cauchy–Schwarz inequality becomes equality and so $\bar{\nabla}^2 f$ is a multiple of the identity matrix. Since $\bar{\Delta} f = 1$ we have

$$f(X) = \frac{|X|^2 - a^2}{2n+2},$$

for some constant $a > 0$ and therefore S is a spherical cap of radius a. □

3 Hypersurfaces with constant H_ℓ

With Propositions 1 and 2 we are now ready to prove the first theorem.

Theorem 1 *Let C be a domain in $\mathbb{R}^{n+1}$ which is a convex cone with piecewise smooth boundary and with vertex at the origin. Let $S \subset C$ be an embedded hypersurface of constant ℓ-th order mean curvature with boundary in ∂C such that S is $C^{2,\alpha}$ and perpendicular to ∂C along ∂S. Then S is a spherical cap.*

Proof Gårding [4] showed that $H_m > 0$ if $H_n > 0$ and $m < n$. And we have

$$H_k^{k-1} \leq H_{k-1}^k,$$

provided $H_j > 0$ for some $j \geq k$, where equality holds only at the umbilic points of S. Note that the farthest point in S from the origin has positive principal curvatures, and hence $H_\ell \equiv \text{const} > 0$ and

$$0 < H_k^{k-1} \leq H_{k-1}^k, \quad k = 1, \ldots, \ell. \tag{11}$$

Therefore

$$0 < H_\ell^{1/\ell} \leq H_1. \tag{12}$$

Now by Proposition 1 we have

$$0 = \int\limits_S H_{\ell-1} - \int\limits_S H_\ell \langle X, \eta \rangle$$
$$= \int\limits_S H_{\ell-1} - (n+1) H_\ell V. \quad \text{(by (1))}$$

Then (11) implies

$$(n+1) H_\ell V = \int\limits_S H_{\ell-1} \geq A H_\ell^{\frac{\ell-1}{\ell}}, \quad A = \mathrm{Vol}(S). \tag{13}$$

Proposition 2 and (12) yield

$$(n+1)V \leq \int_S \frac{1}{H} \leq A H_\ell^{-\frac{1}{\ell}}. \tag{14}$$

It follows from (13) and (14) that equality holds in Proposition 2 and hence S is a spherical cap. □

4 Capillary surfaces in $\mathbb{R}^3$

In this section we give a sufficient condition for an embedded disk type capillary surface S (with nonzero constant mean curvature) in a convex polyhedral cone $C \subset \mathbb{R}^3$ to be spherical. In the following, we assume that S is $C^{2,\alpha}$ up to and including ∂S and each $C^{2,\alpha}$ component of ∂S is $C^{2,\alpha}$ up to the singular points which are called the vertices of S, and S is C^1 up to the vertices. Let us label the faces of C as F_i, $i = 1, \dots, n$, and assume that $\partial S \cap F_i$ is connected for each i, which is called an edge of S. Let $E_i = F_i \cap F_{i+1}$, $i = 1, \dots, n$, be the edges of C. Let v_i be the vertex of S on E_i. Joachimstahl's theorem [11] implies that each edge $\Gamma_i := \partial S \cap F_i$ is a curvature line of S. We denote by θ_i the contact angle between S and F_i toward the vertex of C. Henceforth we assume without loss of generality that S has mean curvature 1. Let ν be the unit normal to S such that $\nu = \vec{H}$, the mean curvature vector, and let $\tilde{\Gamma}_i = \Gamma_i + \nu$ be the parallel curve of Γ_i. Since S meets F_i at a constant angle θ_i, $\tilde{\Gamma}_i$ is contained in a plane P_i which is parallel to F_i with distance $\cos\theta_i$. Let κ_i be the curvature of Γ_i in F_i. Then the principal curvature of S in the direction of Γ_i is $\kappa_i \sin\theta_i$.

Theorem 2 *Let S be an embedded disk type capillary surface in a convex polyhedral cone C. Suppose the following*:

(i) *S has only one edge on each face of C,*
(ii) *the mean curvature vector $\vec{H}$ of S points away from the vertex of C, and*
(iii) *there is a point O such that $dist(O, \langle F_i \rangle) = \cos\theta_i$, for $i = 1, \dots, n$, where $\langle F_i \rangle$ is the plane containing F_i and the distance function is signed in such a way that $P_i \cap C = \emptyset$ if $\cos\theta_i < 0$.*

If the number of faces of C is at most 5, *then S is a spherical cap.*

Condition (iii) of the above theorem implies that each P_i passes through O. Hence P_i's define two special cones C^+ and C^- with vertex O: C^+ is a parallel translation of C and C^- is the reflection of C^+ about O. We denote by F_i^+ (F_i^-, respectively), $i = 1, \dots, n$, the face of C^+ (C^-, respectively) in P_i, and denote by E_i^+ (E_i^-, respectively) the edge of C^+ (C^-, respectively) such that $E_i^+ \cup E_i^- = P_i \cap P_{i+1}$. At first look Condition (iii) might seem very strong. However, without this condition we have a counterexample: an area minimizing capillary surface in a rectangular cone with contact angles $\theta_1 = \theta_2 = \theta_3 = \theta_4 \neq \pi/2$.

For a capillary surface S in a right cone over a regular n-gon we can eliminate the condition $n \leq 5$ if its contact angles are all the same, as follows.

Theorem 3 *Let C be a right cone over a regular n-gon and S an embedded disk type capillary surface in C satisfying* (*i*) *and* (*ii*) *of Theorem 2. If the contact angles of S are all the same, then S is a spherical cap.*

Our proof of the theorems will be based on the following: the parallel H-surface and the Poincaré–Hopf theorem. First let's introduce the parallel H-surface, which is essentially due to Bonnet (see, for example, [10]). Given a surface S of constant mean curvature H, the parellel H-surface of S, denoted by $\tilde{S}$, is the set of the points with position vector $\tilde{X}$ given by

$$\tilde{X} = X(p) + \frac{1}{|H|}\nu(p), \quad \nu = \frac{\vec{H}}{|H|}, \quad p \in S.$$

Choose a complex coordinate $w = u + iv$ on S and let K be the Gaussian curvature of S. It is straightforward to see that

$$|\tilde{X}_u|^2 = |\tilde{X}_v|^2 = \left(1 - \frac{K}{H^2}\right)|X_u|^2, \quad \langle \tilde{X}_u, \tilde{X}_v\rangle = 0, \tag{15}$$

For the intrinsic Laplacian Δ_S of S, we have

$$\Delta_S X = 2\vec{H} = 2|H|\nu, \quad \Delta_S\, \nu = -|\mathrm{II}|^2\nu,$$

where II is the second fundamental form of S. From the above equations, it follows that

$$\Delta_{\tilde{S}}\tilde{X} = -2|H|\nu. \tag{16}$$

Equation (15) says that $\tilde{S}$ is singular at the image of the umbilic points of S. In fact, $\tilde{S}$ is branched at the singular points. From (16), we see that $\tilde{S}$ has mean curvature H and the mean curvature vector of $\tilde{S}$ is $-|H|\nu$.

From now on, again, we assume that the capillary surface S has mean curvature 1. Let $\tilde{\Gamma}_i$, $\tilde{v}_i$ be the image of Γ_i, v_i in $\tilde{S}$, respectively. The regularity assumption on S guarantees that $\tilde{S}$ is well-defined at $\tilde{v}_i$ and that $\tilde{v}_i$ lies on the line $E_i^+ \cup E_i^- = P_i \cap P_{i+1}$ (recall that $v_i \in F_i \cap F_{i+1}$).

Second, in order to state the Poincaré–Hopf theorem, we introduce the rotation index for a family of curvature lines on S [5]. Let

$$\mathrm{II} = Ldu^2 + 2Mdudv + Ndv^2$$

be the second fundamental form of S. The Hopf differential Φdw^2 is a quadratic differential defined by

$$\Phi(w, \bar{w}) = L - N - 2iM.$$

Since the Hopf differential is holomorphic on CMC surfaces in $\mathbb{R}^3$ [5], the zeros of Φ, the umbilic points of S, are isolated unless Φ is identically zero. Let $\mathcal{F}$ be a family of curvature lines on S. The rotation index I of $\mathcal{F}$ at an interior umbilic point p is defined by

$$I = -\frac{1}{4\pi}\delta(\arg\Phi),$$

where δ denotes the variation along a small closed curve around p in the positive direction. In [5] Hopf showed that $I \le -\frac{1}{2}$ at an interior umbilic point.

The first-named author [3] defined the rotation index of a boundary umbilic point and that of a vertex by

$$I = -\frac{1}{8\pi}\delta(\arg\Phi)$$

after extending Φ holomorphically across ∂S. Lemma 2 of [3] says that (i) *the boundary umbilic points are isolated,* (ii) *the rotation index of a boundary umbilic point is* $\le -\frac{1}{4}$ and (iii) *the rotation index of a vertex with angle* $< \pi$ *is* $\le \frac{1}{4}$ (*if the angle of a vertex is* $> \pi$*, then the rotation index is* $\le -\frac{1}{4}$).

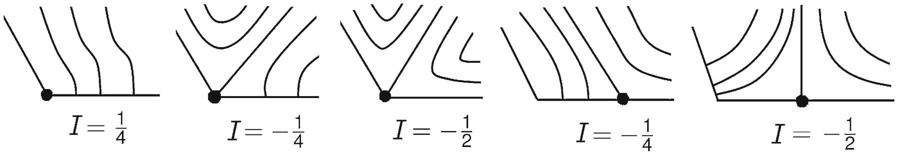

Fig. 1 Indices at the vertices and boundary umbilic points

Remark 1 (a) Suppose that a vertex v_i of S with angle $< \pi$ has rotation index $\frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, \ldots$. Then $\kappa \sin\theta - 1$ changes sign as we move along ∂S across v_i, i.e., either $\kappa_i \sin\theta_i > 1$ and $\kappa_{i+1} \sin\theta_{i+1} < 1$ or $\kappa_i \sin\theta_i < 1$ and $\kappa_{i+1} \sin\theta_{i+1} > 1$ near v_i.

(b) $\kappa \sin\theta - 1$ does not change sign at v_i when the rotation index at v_i is $-\frac{n}{2}$, for some nonnegative integer n.

(c) If an edge of S has no boundary umbilic point, obviously $\kappa \sin\theta - 1$ does not change sign on that edge.

(d) $\kappa \sin\theta - 1$ changes sign at a boundary umbilic point on an edge of S when $I = -n/4$, n: positive odd, and does not change sign when $I = -n/4$, n: positive even (see Fig. 1).

5 Lemmas

For later use, we introduce Euclidean coordinates on each P_i such that the positive y-axis bisects F_i^+ and the x-axis passes through O. Let $\varphi_i : F_i \to P_i$ be the orthogonal projection. Then $\varphi_i(F_i)$ opens toward the positive y direction and $\tilde{\Gamma}_i \subset P_i$ is a parallel curve of $\varphi_i(\Gamma_i)$ with distance $\sin\theta_i$. We assume that $\varphi_i(E_{i-1})$ is on the left of $\varphi_i(E_i)$ on P_i.

In the following lemmas we assume that S is not spherical and that the sign of κ_i is determined by $\vec{H}$.

Lemma 1 *If $\kappa_i \geq 1/\sin\theta_i$ on the edge Γ_i, then the endpoints of $\tilde{\Gamma}_i$ lie on $\partial F_i^- \backslash \{O\}$.*

Proof Let $c : [0, l] \to F_i$ be the arc-length parametrization of Γ_i with $c(0) = v_{i-1}$ and $c(l) = v_i$. Then $\tilde{\Gamma}_i$ is parameterized by

$$\tilde{c}(s) = \varphi_i(c(s)) + \sin\theta_i \ \vec{n}(s),$$

where $\vec{n}$ is the principal normal to c. Hence

$$\tilde{c}'(s) = (1 - \kappa_i \sin\theta_i)\, c'(s). \tag{17}$$

Note that $\kappa_i = 1/\sin\theta_i$ at the boundary umbilic points which are isolated. Hence by (17) $\tilde{c}'(s)$ and $c'(s)$ are opposite except at a finite number of points. Γ_i being concave up, and due to the choice of the coordinate axes as above, we have

$$\frac{d}{ds} x(c(s)) > 0.$$

Hence (i) $\tilde{c}(s)$ moves to the left as s increases; (ii) $\tilde{c}(s)$ is concave down; (iii) near $\tilde{v}_i$ $\tilde{\Gamma}_i$ is below the line $E_i^+ \cup E_i^-$ since Γ_i is above E_i near v_i. Therefore, remembering that $\tilde{\Gamma}_i$ joins $\tilde{v}_{i-1} \in E_{i-1}^+ \cup E_{i-1}^-$ to $\tilde{v}_i \in E_i^+ \cup E_i^-$, we conclude that $\tilde{\Gamma}_i$ connects $E_{i-1}^- \backslash \{O\}$ to $E_i^- \backslash \{O\}$. □

Lemma 2 *If $\kappa_j \leq 1/\sin\theta_j$ on the edge Γ_j, then at least one endpoint of $\tilde{\Gamma}_j$ lies on ∂F_j^+.*

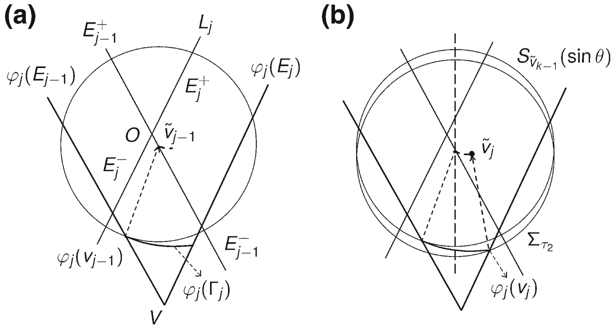

Fig. 2 Case (I) in the proof of Lemma 2

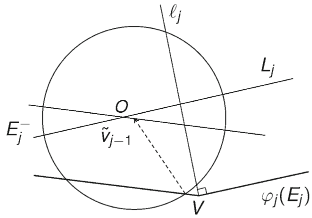

Fig. 3 Case (II) in the proof of Lemma 2

Proof Suppose that $\tilde{v}_{j-1} \in \partial F_j^- \backslash \{O\}$. On P_j the circle $S_{\tilde{v}_{j-1}}(\sin\theta_j)$ is tangent to $\varphi_j(\Gamma_j)$ at $\varphi_j(v_{j-1})$ and intersects $\varphi_j(E_{j-1})$ at two points. Among the two intersection points $\varphi_j(v_{j-1})$ is the point closer to the vertex V of $\varphi_j(F_j)$ because of the hypothesis of Theorem 2 that $\vec{H}$ points away from the vertex of C. Let $L_j = E_j^+ \cup E_j^-$. Then $\tilde{v}_j \in L_j$.

We consider two cases: (I) $dist(\tilde{v}_{j-1}, \varphi_j(E_j))$ is attained at some interior point of $\varphi_j(E_j)$ (cf. Fig. 2a); (II) $dist(\tilde{v}_{j-1}, \varphi_j(E_j)) = dist(\tilde{v}_{j-1}, V)$ (cf. Fig. 3).

We show that case (I) is impossible. (I) implies that $\angle(\overline{\tilde{v}_{j-1}V}, \varphi_j(E_j)) < \pi/2$. Since $\varphi_j(v_{j-1})$ is the point of $S_{\tilde{v}_{j-1}}(\sin\theta_j) \cap \varphi_j(E_{j-1})$ closer to V and $\tilde{v}_{j-1} \in \partial F_j^- \backslash \{O\}$, $\tilde{v}_{j-1}$ and $\varphi_j(E_j)$ are on the same side of L_j. Remember that $dist(\tilde{v}_j, \varphi_j(v_j)) = \sin\theta_j$ and that $\tilde{v}_j \in L_j$ and $\varphi_j(v_j) \in \varphi_j(E_j)$. Hence we have $dist(L_j, \varphi_j(E_j)) \leq \sin\theta_j$ and $S_{\tilde{v}_{j-1}}(\sin\theta_j) \cap \varphi_j(E_j) \neq \emptyset$.

Note that $\varphi_j(\Gamma_j)$ meets $S_{\tilde{v}_{j-1}}(\sin\theta_j)$ tangentially at $\varphi_j(v_{j-1})$ and stays outside $S_{\tilde{v}_{j-1}}(\sin\theta_j)$ near that point. We claim that

$$\varphi_j(\Gamma_j) \cap S_{\tilde{v}_{j-1}}(\sin\theta_j) = \{\varphi_j(v_{j-1})\}. \tag{18}$$

Suppose that $\varphi_j(\Gamma_j) \cap S_{\tilde{v}_{j-1}}(\sin\theta_j)$ contains more than one point. Define $\Sigma_\tau = S_{\tilde{v}_{j-1}-(0,\tau)}(\sin\theta_j)$ and let W_j be the bounded component of $\varphi_j(F_j) \backslash \bar{B}_{\tilde{v}_{j-1}}(\sin\theta_j)$. Since the arcs $\Sigma_\tau \cap W_j$ foliate W_j, there is some $\tau_1 > 0$ such that $\varphi_j(\Gamma_j)$ and Σ_{τ_1} are tangent at some point, say w, and $\varphi_j(\Gamma_j)$ stays inside Σ_{τ_1} near w. This implies that $\kappa_j \sin\theta_j \geq 1$ at w and $\kappa_j \sin\theta_j > 1$ near w, contradicting our hypothesis. So claim (18) follows.

We will now obtain a contradiction $\tilde{v}_j \notin L_j$ which will prove that case I) is impossible. Since $\varphi_j(v_j) \notin \bar{B}_{\tilde{v}_{j-1}}(\sin\theta_j)$ by (18), there exists $\tau_2 > 0$ such that $\varphi_j(v_j) \in \Sigma_{\tau_2}$. Arguing as in the proof of (18), we see that $\varphi_j(\Gamma_j)$ stays inside Σ_{τ_2} near $\varphi_j(v_j)$. Moreover, $\varphi_j(\Gamma_j)$ meets Σ_{τ_2} transversally at $\varphi_j(v_j)$. Therefore $\tilde{v}_j$, lying on the parallel curve of $\varphi_j(\Gamma_j)$ with distance of $\sin\theta_j$, is on the right side of the vertical line through $\tilde{v}_{j-1}$ (see Fig. 2b). Furthermore $\tilde{v}_j \notin L_j$, as desired, because $\tilde{v}_j$ is closer to $\varphi_j(E_j)$ than $\tilde{v}_{j-1} - (0, \tau_2)$ is.

Now suppose case (II) holds. We have $dist(\tilde{v}_{j-1}, \varphi_j(E_j)) = dist(\tilde{v}_{j-1}, V) > \sin\theta_j$. Let ℓ_j be the line perpendicular to $\varphi_j(E_j)$ at V (see Fig. 3). Then $\tilde{v}_{j-1}$ does not lie in the quadrant formed by ℓ_j and $\varphi_j(E_j)$. Hence we have $dist(O, \varphi_j(E_j)) = dist(O, V) > dist(\tilde{v}_{j-1}, \varphi_j(E_j)) > \sin\theta_j$ and, for all $q \in E_j^-$, $dist(q, \varphi_j(v_j)) > dist(q, \varphi_j(E_j)) > dist(O, \varphi_j(E_j)) > \sin\theta_j$. Thus $\tilde{v}_j$ should not lie on E_j^- but on E_j^+. □

Remark 2 Impossibility of case (I) in the proof of Lemma 2 implies that if $\kappa_j \sin\theta_j \leq 1$ on Γ_j and if $dist(\tilde{v}_{j-1}, \varphi_j(E_j))$ is attained at some interior point of $\varphi_j(E_j)$, then both end points of $\tilde{\Gamma}_j$ lie on ∂F_j^+.

6 Proofs of Theorems

The Poincaré–Hopf theorem [5] says that the sum of the rotation indices for a fixed family of curvature lines on a compact surface is equal to the Euler characteristic of the surface. Since $I \leq -1/2$ at an interior umbilic point and $I \leq -1/4$ at a boundary umbilic point [3], on a disk type surface S we have

$$1 = \chi(S) = \sum_{\bar{S}} I \leq \sum_{\partial S} I \leq \sum_{\text{vertices}} I. \tag{19}$$

Proof of Theorem 2 We will show that the total rotation index along ∂S is < 1 unless S is totally umbilic. Then we get a contradiction by (19).

It was shown in [3] that if C has at most 3 faces, S is a spherical cap. Suppose that C has 4 faces. The total rotation index along ∂S can be ≥ 1 only when each vertex of S has rotation index 1/4 and each edge of S has no boundary umbilic point (see Lemma 2 of [3]). This means that $\kappa_i \sin\theta_i - 1$ does not change sign on each edge Γ_i and changes sign at each vertex. Suppose without loss of generality that

$$\kappa_i \sin\theta_i \geq 1 \quad \text{for } i = 1, 3,$$

and

$$\kappa_i \sin\theta_i \leq 1 \quad \text{for } i = 2, 4.$$

Then $\tilde{v}_0, \tilde{v}_1 \in C^-\backslash\{O\}$ by Lemma 1 for Γ_1, $\tilde{v}_2 \in C^+$ by Lemma 2 for Γ_2, and $\tilde{v}_2 \in C^-\backslash\{O\}$ by Lemma 1 for Γ_3, which is a contradiction. Hence $\sum_{\partial S} I < 1$.

Suppose that C has 5 faces and $\Sigma_{\partial S} I \geq 1$. Then either all vertices of S have rotation index 1/4 and at most one edge of S has a boundary umbilic point or only four vertices have rotation index 1/4 and no edge has a boundary umbilic point. In either case there are four consecutive edges with no boundary umbilic point. On these four consecutive edges there are at least four consecutive vertices at which the rotation index is 1/4. Since the signs of $\kappa_i \sin\theta_i - 1$ are alternating, out of these edges we can choose three consecutive edges $\Gamma_1, \Gamma_2, \Gamma_3$ connecting four vertices v_0, v_1, v_2, v_3 such that $\kappa_i \sin\theta_i \geq 1$ on Γ_1, Γ_3 and $\kappa_i \sin\theta_i \leq 1$ on Γ_2. By Lemma 1 for Γ_1, $\tilde{v}_1 \in C^-$; by Lemma 2 for Γ_2, $\tilde{v}_2 \in C^+$; by Lemma 1 for Γ_3, $\tilde{v}_2 \in C^-\backslash\{O\}$: contradiction. Hence $\sum_{\partial S} I < 1$. □

Finally for Theorem 3 the contact angles of S are assumed to be constant on ∂S. Then for any k $\tilde{v}_{k-1}$ is on the plane bisecting the wedge formed by the planes $\langle F_{k-1}\rangle$ and $\langle F_k\rangle$. Moreover, Γ_k makes the same angle with E_{k-1} and with E_k. If $\theta > \pi/2$, then the distance $dist_S(\cdot, V_C)$ of the points of S to the vertex V_C of C has the maximum at some interior point

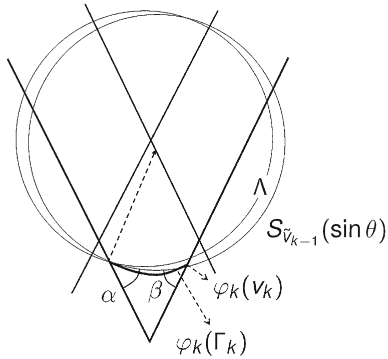

Fig. 4 The circle Λ in the proof of Lemma 3

m of S. This implies that $\vec{H}(m)$ points toward V_C, which we do not deal with in this paper. If $\theta \leq \pi/2$, then $C^+ \subset C$. We need the following lemma to prove Theorem 3.

Lemma 3 *Let Γ_k be an edge of S with $\tilde{v}_{k-1} \in E^-_{k-1}$ and $\tilde{v}_k \in E^+_k$. Then Γ_k contains at least one boundary umbilic point at which $\kappa_k \sin\theta - 1$ changes sign. If $\kappa_k \sin\theta < 1$ near v_{k-1}, then Γ_k contains at least two boundary umbilic points at which $\kappa_k \sin\theta - 1$ changes sign.*

Proof Since $\tilde{v}_{k-1} \in E^-_{k-1}$, $dist(\tilde{v}_{k-1}, \varphi_k(E_k))$ is attained at an interior point of $\varphi_k(E_k)$. Then by Remark 2 it may not happen that $\kappa_k \sin\theta \leq 1$ throughout Γ_k. On the other hand, by Lemma 1, we cannot have $\kappa_k \sin\theta \geq 1$ on Γ_k, either. Hence there should exist a boundary umbilic point at which $\kappa_k \sin\theta - 1$ changes sign.

Now suppose that $\kappa_k \sin\theta < 1$ near v_{k-1} and that Γ_k contains only one boundary umbilic point at which $\kappa_k \sin\theta - 1$ changes sign. The angle $\alpha \leq \pi/2$ that the arc $S_{\tilde{v}_{k-1}}(\sin\theta) \cap \varphi_k(F_k)$ makes with the line $\varphi_k(E_{k-1})$ should be smaller than the angle $\beta \leq \pi/2$ with the line $\varphi_k(E_k)$. Clearly $\varphi_k(\Gamma_k)$ makes α with $\varphi_k(E_{k-1})$ and with $\varphi_k(E_k)$. Let Λ be the circle of radius $\sin\theta$ passing through $\varphi_k(v_{k-1})$ such that the arc $\Lambda \cap \varphi_k(F_k)$ makes the angle α with $\varphi_k(E_k)$ (see Fig. 4). Λ can be obtained by rotating $S_{\tilde{v}_{k-1}}(\sin\theta)$ about $\varphi_k(v_{k-1})$ counterclockwise. Hence the center o of Λ cannot be in F^+_k. Note here that near $\varphi_k(v_{k-1})$ $\varphi_k(\Gamma_k)$ stays outside $S_{\tilde{v}_{k-1}}(\sin\theta)$ and hence outside Λ as well. Let Λ_t be the parallel translations of Λ along $\varphi_k(E_k)$ toward V, the vertex of $\varphi_k(F_k)$. Among Λ_t there should exist Λ_{t_0} that is the last one to intersect $\varphi_k(\Gamma_k)$. Let q be their intersection point. Actually Λ_{t_0} is tangent to $\varphi_k(\Gamma_k)$ at q since Λ_{t_0} makes the same angle α with $\varphi_k(E_k)$ as $\varphi_k(\Gamma_k)$ does. Then near q $\varphi_k(\Gamma_k)$ should lie inside Λ_{t_0} and so $\kappa_k \sin\theta \geq 1$ at q. Hence $\kappa_k \sin\theta \geq 1$ on $\varphi_k(\Gamma_k)$ between q and $\varphi_k(v_k)$ as $\kappa_k \sin\theta - 1$ changes sign only once. But then Λ_{t_0} should coincide with $\varphi_k(\Gamma_k)$ between q and $\varphi_k(v_k)$ since they make the same angle with $\varphi_k(E_k)$. Therefore $\tilde{v}_k$ must be the center of Λ_{t_0}. Note that $\overrightarrow{o\tilde{v}_k}$ is parallel to $\varphi_k(E_k)$ and points away from F^+_k. Hence $\tilde{v}_k \notin E^+_k \subset F^+_k$: contradiction. Thus there are more than one umbilic points at which $\kappa_k \sin\theta - 1$ changes sign. □

Proof of Theorem 3 Remember that on S a vertex has index $\leq 1/4$ and a boundary umbilic point has index $\leq -1/4$. Since there are an equal number of vertices and edges on S, the disk type surface S should have edges with no umbilic point. Each edge without an umbilic point together with adjacent edges with umbilic points form a chain c_i and ∂S can be cut into these chains: $\partial S = \bigcup_i c_i$, $c_i = \Gamma_i \cup \cdots \cup \Gamma_{i+l}$, Γ_i contains no umbilic point whereas each of $\Gamma_{i+1}, \ldots, \Gamma_{i+l}$ has at least one umbilic point. Assume that c_i connects the vertices $v_{i-1}, v_i, \ldots, v_{i+l}$. Note that v_{i+l} is the first vertex of the next chain c_{i+l+1}. Hence

$$1 = \chi(S) \le \sum_{\partial S} I = \sum_i \sum_{c_i \setminus \{v_{i-1}\}} I. \tag{20}$$

It should be remarked that the index sum on c_i, $\sum_{c_i \setminus \{v_{i-1}\}} I$, can have the maximum if and only if $I = 1/4$ at $v_i, \ldots, v_{i+l}$, each edge of $\Gamma_{i+1}, \ldots, \Gamma_{i+l}$ has only one umbilic point, and $I = -1/4$ at these umbilic points. Then the maximum index sum equals 1/4 on c_i. But we claim that the maximum equals 0, i.e.,

$$\sum_{c_i \setminus \{v_{i-1}\}} I \le 0. \tag{21}$$

Suppose $\kappa_i \sin\theta < 1$ on Γ_i. By Remark 2 both $\tilde{v}_{i-1}$ and $\tilde{v}_i$ lie in $\partial F_i^+ \subset \partial C^+$. Assume that all of $\tilde{v}_{i+1}, \ldots, \tilde{v}_{i+l}$ are in ∂C^+. Since $\kappa \sin\theta - 1$ changes sign at each vertex of c_i and at each boundary umbilic point on $\Gamma_{i+1}, \ldots, \Gamma_{i+l}$, we can see that $\kappa \sin\theta > 1$ on the first half of each edge and $\kappa \sin\theta < 1$ on the second half. Hence $\kappa \sin\theta > 1$ on the first edge Γ_{i+l+1} of the next chain c_{i+l+1}. Then our assumption, $\tilde{v}_{i+l} \in \partial C^+$, contradicts Lemma 1. Therefore there should exist an edge Γ_{i+m} in c_i such that $\tilde{v}_{i+m-1} \in \partial C^+$ and $\tilde{v}_{i+m} \in \partial C^-$. Since $\kappa \sin\theta > 1$ on the first half of Γ_{i+m} and $\kappa \sin\theta < 1$ on its second half, we have a contradiction because by Lemma 3 Γ_{i+m} has two umbilic points. Thus (21) follows. Similarly we can obtain (21) when $\kappa_i \sin\theta > 1$ on Γ_i. But (21) contradicts (20). Hence S is umbilic everywhere. □

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