# RIGIDITY THEOREMS FOR POSITIVELY CURVED MANIFOLDS WITH SYMMETRY

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## 1. Introduction

Our main issue of this paper is to classify or determine the structure of positively or non-negatively curved manifolds with isometry groups. An isometry of a connected $n$-dimensional manifold with a Riemannian metric $g$ is a transformation of $M$ which leaves the metric $g$ invariant. It can be shown that the Lie algebra of the isometry group $\mathrm{Isom}\,(M,g)$ is of dimension at most $\frac{1}{2}n(n+1)$ (e.g., see [20]). Moreover, if the dimension exactly equals $\frac{1}{2}n(n+1)$ then $M$ is a space of constant curvature. It is a prototype of various rigidity theorems for Riemannian manifolds. In a similar vein, the concept of the symmetry rank, denoted sym-rank $(M)$, of a Riemannian manifold $(M,g)$ was first introduced by Grove and Searle in [11] in order to measure the amount of symmetry of $M$. Here the symmetry rank is defined as the rank of the isometry group $\mathrm{Isom}\,(M,g)$. Equivalently, it can be defined as the largest number $r$ such that a $r$-dimensional torus acts effectively and isometrically on $M$.

In [11], Grove and Searle showed that the symmetry rank sym-rank $(M)$ of a positively curved manifold is less than or equal to $\left[\frac{\dim M+1}{2}\right]$ and that the maximal rank case holds if and only if the manifold $M$ is diffeomorphic to the unit sphere or the complex projective space or a lens space. In fact, this result seemed to be motivated by the homeomorphism classification of 4-dimensional positively curved manifold with an isometric effective $S^1$-action by Hsiang and Kleiner in [16]. In his paper [27], Rong later gave various classification results of positively curved manifolds with almost maximal symmetry rank case. The first aim of this paper is to show that an orientable closed positively curved 4-manifold $M$ with an effective isometric $S^1$-action is *diffeomorphic* to $S^4$ or $\mathbf{CP}^2$.

We also deal with another kind of positively curved manifolds. Namely, a compact quaternionic Kähler manifold is defined to be a Riemannian manifold of real dimension $4m$ whose holonomy group is contained in the Lie group $Sp(m)Sp(1)$ in $SO(4m)$ for $m \geq 2$. Such a manifold is called *positive* if it has the positive scalar curvature. It is known that every quaternionic Kähler manifold is Einstein. So it is customary to define a 4-dimensional quaternionic Kähler manifold to be both Einstein with non-zero scalar curvature and self-dual.

The second aim of this paper is to show that if $M$ is a positive quaternionic Kähler manifold of dimension $4m$ with the symmetry rank sym-rank $(M)$ greater than or equal to $\left[\frac{m}{2}\right]+3$, then $M$ is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. This result is very sharp, since the symmetry rank of the real oriented Grassmannian $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is $\frac{m+3}{2}$ and $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is not on the list.

We organize this paper as follows. In Section 2, we explain our main results in more details. Some further questions will be addressed in the talk.

## 2. Main Results

### 2.1. Diffeomorphism classification of positively curved 4-manifolds with $S^1$-symmetry:

In their paper [16], Hsiang and Kleiner investigated the question which orientable closed positively curved 4-manifolds admit a positively curved metric with an effective isometric $S^1$-action. They showed that if $M$ is a compact oriented positively curved 4-manifold with an effective isometric $S^1$-action, then $M$ is homeomorphic to $S^4$ or $\mathbf{CP}^2$. This classification is remarkable and sparked

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off many current research activities in this field. In particular, see the works of Grove–Searle and B. Wilking in [12] and [28, 29] respectively. However, it is a topological classification rather than a diffeomorphism one. In fact, Hsiang and Kleiner asked in the same paper whether or not the following conjecture was true:

**Conjecture 2.1.** *An orientable closed positively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to $S^4$ or $\mathbf{CP}^2$.*

In order to explain the main ideas of the proof of Conjecture 2.1, we let $F(S^1, M)$ be the fixed point set of such an $S^1$-action on $M$. It is well known as in [19] that the Euler characteristic of $F(S^1, M)$ is equal to that of $M$ in the presence of an effective $S^1$-action. Thus the Euler characteristic of the fixed point set $F(S^1, M)$ is greater than or equal to 2. One of the main steps in the paper of Hsiang and Kleiner is to prove that the Euler characteristic of the fixed point set $F(S^1, M)$ is at most three. Hence, since each component of $F(S^1, M)$ is a totally geodesic submanifold of $M$, $F(S^1, M)$ has the following four possibilities only:

(1) One 2-sphere.
(2) The disjoint union of one 2-sphere and one isolated fixed point.
(3) Two isolated fixed points.
(4) Three isolated fixed points.

Hsiang and Kleiner obtained the above information about the fixed point set $F(S^1, M)$ by essentially using the existence of a positively curved metric with an effective isometric $S^1$-action. But it seems to need more serious consideration on the topological properties of effective circle actions on simply connected 4-manifolds. Thus our starting point of this paper is to investigate all possible topological configurations of effective circle actions on simply connected 4-manifolds. However, the existence of a positively curved metric with an effective isometric $S^1$-action is also used crucially. Moreover, we need to use the recent resolution of the Poincaré conjecture by Perelman as in [24] and [25]. (See also [22], [4, 5] and [18].) So our proof is differential-topological in nature. We remark that it is inspired by the work [1] of S. Baldridge.

In case that the fixed point set $F(S^1, M)$ is either one 2-sphere or the union of one 2-sphere and one isolated fixed point, the effective isometric $S^1$-action on $M$ has the fixed point set whose codimension is 2. It then follows from Theorem 1.2 of Grove and Searle in [12] that $M$ is diffeomorphic to $S^4$ or $\mathbf{CP}^2$. So it is enough to consider the remaining two cases: $F(S^1, M)$ consists of either two isolated fixed points or three isolated fixed points. In the present paper, we prove that even in this case $M$ is diffeomorphic to $S^4$ or $\mathbf{CP}^2$.

One of the new ingredients of this paper that is not present in the paper of Hsiang and Kleiner is to use the classification results of circle actions on simply connected 4-manifolds by R. Fintushel in [8] and [9]. According to Fintushel, smooth $S^1$-actions on simply connected 4-manifolds can be classified in terms of their legally weighted orbit spaces. Applying this classification to our situation, we will have at least four (resp. three) possibilities for legally weighted orbit spaces, if the fixed point set is three isolated fixed points (resp. two isolated fixed points).

In the proof we also need the "replacement trick" of P. Pao. This trick makes the given 4-manifold admit many different $S^1$-actions by replacing a weighted arc with a simpler one. To be precie, we use the following theorem of P. Pao in [23] (or Proposition 13.1 in [9]):

**Theorem 2.2.** *Let $X$ be a closed oriented 4-manifold with an effective $S^1$-action whose weighted orbit space contains a weighted circle $C^*$ with exactly two isolated fixed points. Then $M$ admits a a different $S^1$-action whose weight space is $M^*$ with $C^*$ replaced by two isolated fixed points or $M^*$ minus the interior of 3-ball with $C^*$ removed.*

This implies that in the second case the orbit space for the new $S^1$-action will have one more boundary component instead of the weight circle.

Finally we can show the following result by some case-by-case analysis:

**Theorem 2.3.** *An orientable closed positively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to $S^4$ or $\mathbf{CP}^2$.*

This completes the classification of closed oriented positively curved manifolds of dimension 4 with effective isometric $S^1$-actions, up to diffeomorphism. We recently prove that the technique of this paper answers the following result in [16] (e.g., see [17]):

**Theorem 2.4** (jointly with Hee-Kwon Lee)**.** *An orientable closed simply connected nonnegatively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to either $S^4$, $\mathbf{CP}^2$, $\mathbf{CP}^2 \# \pm \mathbf{CP}^2$, or $S^2 \times S^2$.*

2.2. **Positive quaternionic Kähler manifolds with certain symmetry rank:** Recall that a compact quaternionic Kähler manifold is defined to be a Riemannian manifold of real dimension $4m$ whose holonomy group is contained in the Lie group $Sp(m)Sp(1)$ in $SO(4m)$ for $m \geq 2$. Hitchin proved in [15] that every positive quaternionic Kähler 4-manifold must be isometric to $\mathbf{CP}^2$ and $S^4$. In case of dimension 8, Poon and Salamon showed in [26] that every positive quaternionic Kähler manifold should be isometric to $\mathbf{HP}^2$, $Gr_2(\mathbf{C}^4)$ or $G_2/SO(4)$, i.e., Wolf spaces. Moreover, in [14] Herrera and Herrera gave the classification of positive quaternionic Kähler 12-dimensional manifolds. In particular, according to a result of Lebrun and Salamon in [21], every positive quaternionic Kähler manifold $M$ is simply connected and the second homotopy group $\pi_2$ is a finite group with 2-torsion, trivial or $\mathbf{Z}$. More precisely, $M$ is isometric to $\mathbf{HP}^m$ (resp. $Gr_2(\mathbf{C}^{m+2})$) if $\pi_2(M) = 0$ (resp. $\pi_2(M) = \mathbf{Z}$). (See [6] and the references therein for more results.)

On the other hand, it is also one of the interesting problems to classify positive quaternionic Kähler manifolds in terms of the rank of its isometry group. In particular, in [3] Bielawski classified positive quaternionic Kähler manifolds of dimension $4m$ with isometry rank equal to $m+1$. Using an extension of the connected theorem of Wilking, and independently Fang, Mendonça, and Rong for positively curved manifolds, in [7] Fang gave an interesting classification result of positive quaternionic Kähler manifolds with symmetry as follows.

**Theorem 2.5.** *Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. Then the isometry group* $\mathrm{Isom}(M)$ *has the symmetry rank* sym-rank$(M)$ *at most $m+1$. Moreover, if the symmetry rank* sym-rank$(M)$ *is greater than or equal to $\frac{m+6}{2}$, then $M$ is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

If $m$ is even, the theorem is sharp, since the symmetry rank of the oriented real Grassmannian $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is $\frac{m+4}{2}$ and $\widetilde{Gr_4}(\mathbf{R}^{m+4})$ is not on the list of Theorem 2.5. Note also that the symmetry rank equals $m+1$ if $M$ is $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$. In the same paper Fang conjectured that Theorem 2.5 would be improved a little bit further, if $m$ is odd.

The main strategy for the conjecture is to use the connected theorem in the presence of a Lie group acting isometrically and fixing a quaternionic Kähler submanifold pointwise. Recall that a map $f : N \to M$ between two manifolds is called *h-connected* if the induced map $f_* : \pi_i(N) \to \pi_i(M)$ is an isomorphism for all $i < h$ and an epimorphism for $i = h$. If $f$ is an imbedding this is equivalent to saying that up to homotopy $M$ can be obtained from $f(N)$ by attaching cells of dimension $\geq h+1$.

**Theorem 2.6.** *Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. If $N$ is a quaternionic Kähler submanifold of dimension $4n$, then the inclusion $N \hookrightarrow M$ is $(2n-m+1)$-connected. Furthermore, if there is a Lie group $G$ acting isometrically on $M$ and fixing $N$ pointwise, then the inclusion map is $(2n-m+1+\delta(G))$-connected, where $\delta(G)$ is the dimension of the principal orbit of $G$.*

We remark that in [6] Fang has already established the connected theorem for positive quaternionic manifolds without the presence of a Lie group action. We extend his connected theorem to the case that admits a Lie group action, following the line of Wilking in [28].

Next we need to deal with positive quaternionic Kähler manifold $M$ with an effective isometric $S^1$-action. To do so, we need a concept of the moment map on a positive quaternionic Kähler manifold $M$ with an isometric $S^1$-action. (See [2] for more details.) For the existence of quaternion-Kähler moment map, see [10].

Let $X$ be the Killing vector field generated by the $S^1$-action, and let $\overline{X}$ be its dual 1-form with respect to the Riemannian metric. Let $S^2H$ be the bundle given by the adjoint representation of $Sp(1)$. Then the bundle $S^2H$ has the Lie aglebra $\mathbf{sp}(1)$ as the fiber and has the local basis $\{I_1.I_2, I_3\}$, corresponding to the three elements $i, j, k \in Sp(1)$, that are three almost complex structures such that $I_1 I_2 = -I_2 I_1 = I_3$. Let $\omega_1$, $\omega_2$, and $\omega_3$ be the locally defined 2-forms associated to the three almost complex structures $I_1$, $I_2$, and $I_3$, respectively. Let $\Omega$ be the closed non-degenerate 4-form defined by

$$\Omega = \sum_{i=1}^{3} \omega_i \wedge \omega_i.$$

Then the moment map $\mu$ on $M$ is defined to be a section of $S^2H$ solving the following equation

$$\nabla \mu = \sum_{i=1}^{3} \overline{I_i X} \otimes I_i \ (\text{or } d\mu = \iota_X \Omega), \tag{2.1}$$

where $\nabla$ denotes the Levi-Civita connection of the Riemannian metric. Since the moment map takes its values in a fiber bundle, $\mu^{-1}(0)$ means the inverse image of the zero section. Then the following proposition will play a key role in the proof of Theorem 2.8.

**Proposition 2.7.** *Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$ with an effective isometric $S^1$-action ($m \geq 3$). Let $\mu$ be the moment map defined by (2.1). Let $N$ be a fixed point component of codimension 4 in $M$ of the $S^1$-action which is contained in $\mu^{-1}(0)$. Then we have the following two cases:*

(1) *If $b_2(M) = 0$ then $M$ is isometric to $\mathbf{HP}^m$.*
(2) *If $b_2(M) \neq 0$ then $M$ is isometric to $Gr_2(\mathbf{C}^{m+2})$.*

Finally, we can prove the following theorem by combining Theorem 2.6 and Proposition 2.7 with the induction arguments on the dimension of the manifold:

**Theorem 2.8.** *Let $M$ be a positive quaternionic Kähler manifold of dimension $4m$. If the symmetry rank* sym-rank$(M)$ *is greater than or equal to $\left[\frac{m}{2}\right] + 3$, then $M$ is isometric to $\mathbf{HP}^m$ or $Gr_2(\mathbf{C}^{m+2})$.*

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