IN SEARCH OF NONEXISTENT MINIMAL SURFACES

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Abstract

In this note we introduce four open problems in the minimal surface theory: i) Hoffman and Meeks asked whether there exists a complete embedded minimal surface of genus g and with e ends such that $g+2<e$. ii) Does there exist a complete embedded minimal surface other than the plane and the helicoid which intersects horizontal planes along an open curve? iii) Does there exist a complete minimal surface which lies inside a cylinder? iv) Show that if a compact minimal surface is bounded by two convex curves lying in two parallel planes then the surface has genus zero.

1. The Weierstrass representation formula

Let Σ be a surface immersed in $\mathbf{R}^3$. For the rectangular coordinates $X=(x_1,x_2,x_3)$,

$$\triangle X \;=\; (\triangle_\Sigma x_1, \triangle_\Sigma x_2, \triangle_\Sigma x_3) \;=\; \vec{H},$$

where the Laplacian is taken with respect to the metric of Σ. The first variation formula implies that the mean curvature vector field $\vec{H}$ is the gradient of the area functional on the space of immersions into $\mathbf{R}^3$. So if we define a minimal surface to be a locally least area surface, then the minimal surface satisfies

$$\vec{H} \;=\; 0.$$

Therefore x_1,x_2,x_3 are harmonic functions on Σ. Let x_1^*,x_2^*,x_3^* be the conjugate harmonic functions of x_1,x_2,x_3, respectively. Then

$$\psi_j \;=\; dx_j \;+\; i dx_j^*$$

is a well defined holomorphic 1-form on Σ and so the minimal immersion $X:\Sigma\to\mathbf{R}^3$ is written as

$$X \;=\; \mathrm{Re}\int\Psi \;=\; (\mathrm{Re}\int\psi_1,\ \mathrm{Re}\int\psi_2,\ \mathrm{Re}\int\psi_3). \tag{1}$$

Choose isothermal coordinates x and y on Σ and set $z=x+iy$. The conformality of the immersion X implies that

$$< \frac{dX}{dz}, \frac{dX}{dz} > \;=\; 0.$$

Hence

$$\psi_1^2+\psi_2^2+\psi_3^2 \;=\; 0.$$

We then define a holomorphic 1-form ω and a meromorphic function g on Σ by

$$\omega \;=\; \psi_1 - i\psi_2 \quad \text{and} \quad g=\frac{\psi_3}{\psi_1-i\psi_2}.$$

Then we have

$$\psi_1 \;=\; \frac{1}{2}(1-g^2)\omega, \quad \psi_2 \;=\; \frac{i}{2}(1+g^2)\omega, \quad \psi_3 \;=\; g\omega. \tag{2}$$

Consequently we get the famous *Weierstrass representation formula*

$$\begin{aligned} x_1 &= \mathrm{Re}\int \frac{1}{2}(1-g^2)\omega \\ x_2 &= \mathrm{Re}\int \frac{i}{2}(1+g^2)\omega \\ x_3 &= \mathrm{Re}\int \omega. \end{aligned}$$

From this formula the original minimal surface can be recaptured. Conversely, let $g : \Sigma \to \mathbf{C}$ be a meromorphic function and let ω be a holomorphic 1-form on Σ such that whenever g has a pole of order m at a point $p \in \Sigma$, ω has a zero at p of order $2m$. Then, if the ψ_j's defined by (2) have no real periods, the Weierstrass representation formula defines a minimal immersion $\psi : \Sigma \to \mathbf{R}^3$.

Let $N : \Sigma \to S^2 \subset \mathbf{R}^3$ be the map assigning to each point $p \in \Sigma$ the unit normal vector

$$N(p) = \frac{\psi_x \wedge \psi_y}{|\psi_x \wedge \psi_y|}(p)$$

and suppose $\pi : S^2 \sim \{(0,0,1)\} \to \mathbf{R}^2$ is a stereographic projection. Then

$$g = \pi \circ N.$$

So g is called the Gauss map of Σ. The minimality of Σ implies that g is conformal.

The Weierstrass representation formula makes it easy to write down an enormous number of complete minimal surfaces in $\mathbf{R}^3$. For example, if we set $\Sigma = \mathbf{C}$, $\omega = dz$ and $g(z) = z$, we get Enneper's surface. If we set $\Sigma = \mathbf{C} \sim \{0\}$, $\omega = dz/z^2$ and $g(z) = z$ we get the catenoid. We get the helicoid for $\Sigma = \mathbf{C} \sim \{0\}$, $\omega = idz/z^2$ and $g(z) = z$.

If Σ has finite total curvature, then there exist $p_1, \cdots, p_l$ in the closure $\overline{\Sigma}$ of the Riemann surface Σ with $\Sigma = \overline{\Sigma} \sim \{p_1, \cdots, p_l\}$ such that the Gauss map $g : \Sigma \to S^2$ extends to a holomorphic map $g : \overline{\Sigma} \to S^2$.

Given the immersion X as in (1), we get the immersion

$$X^* = \mathrm{Im}\int \Psi$$

called the *conjugate* minimal immersion. While the conjugate minimal surfaces are locally isometric, they are usually not congruent. The classic example of two conjugate surfaces in $\mathbf{R}^3$ are the catenoid and the helicoid.

Assume that dw is a holomorphic one-form on M arising from the z-coordinate function on M, or precisely, from $z \circ \psi$:

$$z(p) = \mathrm{Re}\int^p dw, \quad p \in M.$$

From the Weierstrass representation we have

$$\psi(p) = \mathrm{Re}\int^p \left(\frac{1}{2}(-g+g^{-1})dw,\ \frac{i}{2}(g+g^{-1})dw,\ dw\right).$$

Let $\Gamma(s)$ be a curve on M with arclength parameter s and $\nu(s)$ a unit conormal to $\Gamma(s)$ such that $\{\nu(s), \Gamma'(s)\}$ determines the orientation of M. The flux of M along Γ is defined by

$$\mathrm{Flux}(\Gamma) = \int_\Gamma \nu(s)ds.$$

It is well known (see [F2]) that

$$\mathrm{Flux}(\Gamma) \;=\; \mathrm{Im}\int_{\Gamma}\left(\frac{1}{2}(-g+g^{-1})dw,\ \frac{i}{2}(g+g^{-1})dw,\ dw\right).$$

2. The Hoffman-Meeks Conjecture

Although one can easily construct minimal surfaces analytically from the Weierstrass represention formula, it is very hard to figure out the geometric shape of the minimal surface from the weierstrass data g and ω. Also showing that ψ_j's have no real period involves very complicated computations. It is due to these reasons that only a handful of minimal surfaces had been known for over 200 years. Moreover the plane, the catenoid and the helicoid had been the only three complete embedded minimal surfaces with finite topology until 1982. Therefore it came as a big surprise when Costa constructed a complete minimal surface with genus one and three ends in 1982 and subsequently Hoffman and Meeks showed that the Costa surface is embedded. Furthermore they constructed complete embedded minimal surfaces with three ends and arbitrary genus in 1985. One could imagine that the Costa example is constructed in the following heuristic manner. Take a catenoid and intersect it by its waist plane. Remove a tubular neighborhood of the intersection circle and smoothly join up the four boundary circles by a genus one minimal surface. The higher-genus examples by Hoffman-Meeks can be thought of as improved approximations.

It has been conjectured by Hoffman and Meeks that for a complete embedded nonplanar minimal surface of finite total curvature,

$$g \geq\; e-2, \tag{3}$$

where g is the genus and e is the number of ends. So far (3) has been confirmed only when $g=0$: In 1991 Lopez and Ros [LR] proved that if a complete embedded minimal surface has genus zero, then it is the catenoid. Before Hoffman and Meeks conjectured (3) Schoen [Sc] showed in 1983 that the equality should hold in (3) when $e=2$, i.e., every complete immersed minimal surface with two ends and finite total curvature is the catenoid. All the examples constructed so far satisfy the inequality (3). The heuristic idea behind the conjecture (3) is that to increase the number of ends e by adding a flat end and then looking for a Costa style surface which is nearby and embedded forces the genus g to increase. Related to this conjecture is the question of whether the order of the Gauss map at a flat end of a complete embedded minimal surface can be equal to two. The order must be at least two, and in all known embedded examples of finite total curvature the order is at least three. However, there are complete embedded periodic minimal surfaces with flat ends where the Gauss map has degree two. The most famous of these is the example of Riemann.

3. A characterization of the helicoid

In 1962 Nitsche [Ni] conjectured that if a complete minimal surface intersects horizontal planes along a Jordan curve, then it is the catenoid. In [Ni] he gave a partial answer showing that his conjecture is true if the intersection curve is star-shaped. Recently Collin [Co] proved that Nitsche's conjecture is true. In fact he obtained a stronger theorem that a complete properly embedded minimal surface with at least two ends has finite topology if and only if it has finite total curvature. Since the helicoid is the conjugate minimal surface of the catenoid, the following natural question arises from the Nitsche conjecture: Is there any complete minimal surface other than the plane and the helicoid that intersects horizontal planes along an open curve? So far we have only a partial answer to this question that if the minimal surface has bounded Gaussian curvature, then it is either the plane or the helicoid.

4. A minimal surface inside a cylinder

Calabi asked whether there exists a complete minimal surface inside a ball. In 1980 Jorge and Xavier [JX] constructed a complete minimal surface in a slab, that is, in the region between two parallel planes. in 1996 Nadirashvili [Na] showed that there exists a complete minimal surface inside a ball. In fact the surface he constructed has negative Gaussian curvature. Now we can ask the following question: Does there exist a minimal surface inside a cylinder? Nadirashvili's minimal surface has three bounded coordinates x_1, x_2, x_3. But Jorge-Xavier's minimal surface has only one bounded coordinate x_3. The minimal surface inside a cylinder should have two bounded coordinates x_2 and x_3.

5. The Meeks conjecture

In 1956 Shiffman [S] proved a beautiful theorem on minimal surfaces lying between two horizontal planes as follows. Let M be a minimal annulus in $\mathbf{R}^3$, P_1, P_2 horizontal planes such that $\partial M = C_1 \cup C_2,\ C_i \subset P_i,\ i = 1, 2$. He showed that for any horizontal plane P between P_1 and P_2, i) $M \cap P$ is a circle whenever $C_1,\ C_2$ are circles and ii) if $C_1,\ C_2$ are convex curves, so is $M \cap P$. For this theorem Shiffman may have been inspired by Riemann's minimal surface. That is a complete periodic minimal surface R whose fundamental piece F, homeomorphic to an annulus and bounded by two parallel lines $l_1,\ l_2$, lies between two horizontal planes containing l_1, l_2 such that R is an infinite number of stacks of the translated copies of F. Riemann's minimal surface has the characterizing property that it intersects every horizontal plane along a circle or a line. Indeed, Hoffman-Karcher-Rosenberg [HKR] showed that a properly embedded minimal annulus bounded by a pair of parallel lines in a slab is the fundamental piece F of Riemann's minimal surface. Moreover Fang [F1] proved that a properly embedded minimal annulus bounded by a line and a circle, each lying in a horizontal plane, is also part of Riemann's example. It is not by accident that the lines in Riemann's minimal surface are all parallel: Toubiana [T] showed that the two lines bounding a properly embedded minimal annulus (genus 0) in a slab must be parallel.

In Shiffman's theorem he assumed that the minimal surface has genus zero. But Meeks conjectured that the genus condition is redundant. The only partial result in this regard is by Schoen [Sc] that if the compact minimal surface bounded by two convex curves is symmetric with respect to two vertical planes, then the surface has genus zero.

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