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The isoperimetric inequality for minimal surfaces in a Riemannian manifold

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Abstract. It is proved that every minimal surface with one or two boundary components in a simply connected Riemannian manifold with sectional curvature bounded above by a nonpositive constant K satisfies the sharp isoperimetric inequality $4\pi A \leqq L^2 + KA^2$. Here equality holds if and only if the minimal surface is a geodesic disk in a surface of constant Gaussian curvature K. □

Let D be a domain in a simply connected surface of constant Gaussian curvature K. The area A of D and the perimeter L satisfy the isoperimetric inequality

$$4\pi A \leqq L^2 + KA^2, \tag{1}$$

where equality holds if and only if D is a geodesic disk. The case $K = 0$ was proved by Steiner in 1842 [S], $K > 0$ by Bernstein in 1905 [B], and $K < 0$ by Schmidt in 1940 [Sc].

Let M be a simply connected Riemannian manifold of constant sectional curvature K. The isoperimetric inequality (1) holds for any domain on a totally geodesic surface in M. Since a totally geodesic surface is minimal in M, it has been naturally conjectured that (1) should hold for every minimal surface in M.

The first result of this nature is due to Carleman [C], who showed in 1921 that (1) holds for a simply connected domain on a minimal surface in $\mathbb{R}^n$. So far (1) has been proved only for minimal surfaces with one or two boundary components in $\mathbb{R}^n$ [LSY], [Ch] and in H^n [CG1].

In this paper we consider minimal surfaces in a simply connected Riemannian manifold N of varying sectional curvature. Suppose the sectional curvature of N is bounded above by a constant K. We prove that (1) holds for minimal surfaces with one or two boundary components in N when $K \leqq 0$. Also, with no restrictions on the topology of minimal surfaces in N and on the sign of K, we obtain a weaker isoperimetric inequality

$$2\pi A \leqq L^2 + KA^2.$$

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The Gauss equation says that the Gaussian curvature of a minimal surface is at most the sectional curvature of the ambient space. Accordingly it should be mentioned that for the case of simply connected surfaces with Gaussian curvature bounded above by K, (1) was proved for $K=0$ by Weil [W] in 1926 and by Beckenbach and Radó [BR] in 1933, and for $K \neq 0$ by Bol [Bo] in 1941.

1. The Laplacian of functions of distance

The distance function on a Riemannian manifold M, being the simplest geometric function on M, implicitly gives us many pieces of information on the geometry of M. Indeed all the results of this paper can be derived from the Laplacians of functions of distance.

Let $r(x)$ be the distance from a fixed point p to x in M and denote the Hessian of r by $\bar{\nabla}^2 r$. Assume that γ is a geodesic from p to q and v is a vector at q perpendicular to γ. Then $\bar{\nabla}^2 r(v,v)$ is the second variation of the length of γ associated with the Jacobi field X along γ satisfying $X(p)=0$ and $X(q)=v$. The Jacobi field minimizes the second variation among all vector fields along γ with the same boundary conditions. Therefore if the sectional curvature of M^n is bounded from above by that of a Riemannian manifold $\bar{M}^n$ which has a distance function $\bar{r}$ with $\bar{r}(\cdot)=\mathrm{dist}(\bar{p},\cdot)$ and the connection also denoted $\bar{\nabla}$, then one gets the Hessian comparison

$$\bar{\nabla}^2 r(v,v) \geqq \bar{\nabla}^2 \bar{r}(u,u), \tag{2}$$

where u is a vector at $\bar{q} \in \bar{M}$ with $|u|=|v|$ and $\bar{r}(\bar{q})=r(q)$, which is perpendicular to the geodesic $\bar{\gamma} \subset \bar{M}$ from $\bar{p}$ to $\bar{q}$.

Let Σ^m be a submanifold of M with the Riemannian connection ∇ and the mean curvature vector H. Given a smooth function f on M, there are two types of Laplacians of f on Σ, $\bar{\Delta} f$ and Δf: for an orthonormal basis $\{e_1, \ldots, e_m\}$ of Σ define

$$\bar{\Delta} f = \sum_{i=1}^{m} \bar{\nabla}^2 f(e_i, e_i), \quad \Delta f = \sum_{i=1}^{m} \nabla^2 f(e_i, e_i).$$

One easily sees that

$$\bar{\Delta} f = \Delta f - Hf. \tag{3}$$

Therefore if Σ is minimal in M, the intrinsic Laplacian Δf can be replaced with $\bar{\Delta} f$ which is more extrinsic and easier to compute. With (2) and (3) we are now ready to compute the Laplacians of functions of distance.

Lemma 1. *Let Σ^2 be a minimal surface in a simply connected Riemannian manifold M^n of nonpositive sectional curvature. Define $r(x)=\mathrm{dist}(p,x)$ for a fixed point $p \in M$. On Σ we have*

(a) $\Delta r^2 \geqq 4$;

(b) $\Delta r \geqq \frac{1}{r}(2 - |\nabla r|^2)$;

(c) *if* $p \in \Sigma$, *then* $\frac{1}{2\pi}\Delta \log r \geqq \delta_p$, *the Dirac function centered at* p.

Proof. Define a distance function $\bar{r}$ on $\mathbb{R}^n$ by $\bar{r}(x) = |x|$, $x \in \mathbb{R}^n$. $\bar{\nabla}$ and g denote the Euclidean connection and metric, respectively. Then a straightforward computation in orthonormal coordinates of $\mathbb{R}^n$ gives

$$\bar{\nabla}^2 \bar{r}^2 = 2g. \tag{4}$$

Since

$$\bar{\nabla}^2 r^2 = 2r\bar{\nabla}^2 r + 2\bar{\nabla} r \otimes \bar{\nabla} r \quad \text{and} \quad \bar{\nabla}^2 \bar{r}^2 = 2\bar{r}\,\bar{\nabla}^2 \bar{r} + 2\bar{\nabla}\bar{r} \otimes \bar{\nabla}\bar{r},$$

(2) and (4) imply

$$\bar{\nabla}^2 r^2 \geqq 2g_M, \tag{5}$$

where g_M is the metric tensor of M. Hence (a) follows from (3) and (5). For (b) we compute

$$\Delta r = \operatorname{div}\nabla(r^2)^{1/2} = \operatorname{div}\frac{1}{2r}\nabla r^2 = \frac{1}{2r}\Delta r^2 - \frac{1}{2r^2}\langle \nabla r, 2r\nabla r\rangle \geqq \frac{1}{r}(2 - |\nabla r|^2).$$

Similarly

$$\Delta \log r = \operatorname{div}\frac{1}{r}\nabla r = \frac{1}{r}\Delta r - \frac{1}{r^2}|\nabla r|^2 \geqq \frac{2}{r^2}(1 - |\nabla r|^2) \geqq 0.$$

Near p, however, Σ can be identified with $T_p\Sigma$, the tangent plane of Σ at p, on which $\Delta \log r = 2\pi\delta_p$ with respect to the Euclidean metric. Therefore on Σ, $\Delta \log r \geqq 2\pi\delta_p$.

Lemma 2. *Suppose that* Σ^2 *is a minimal surface in a simply connected Riemannian manifold* M^n *with sectional curvature bounded above by a negative constant* $K = -k^2$. *On* Σ

(a) $\Delta r \geqq k(2 - |\nabla r|^2)\coth kr$;

(b) $\Delta \log(1 + \cosh kr) \geqq -K$;

(c) $\Delta \log \frac{\sinh kr}{1 + \cosh kr} \geqq 2\pi\delta_p$, *if* $p \in \Sigma$;

(d) $\Delta \log \sinh kr \geqq 2\pi\delta_p - K$, *if* $p \in \Sigma$.

Proof. Let $\bar{M}^n$ be a complete simply connected Riemannian manifold of constant sectional curvature K. Take a point $\bar{p} \in \bar{M}$ and define $\bar{r}(x) = \operatorname{dist}(\bar{p}, x)$, $x \in \bar{M}$. For any circle $C \subset \bar{M}$ of radius a with center at $\bar{p}$, the length of C equals $l(a) = \frac{2\pi}{k}\sinh ka$. So the geodesic curvature of C is $l'(a)/l(a) = k\coth ka$. Hence the principal curvature of the geodesic sphere $S \subset \bar{M}$ of radius a with center at $\bar{p}$ is $k\coth ka$ everywhere on S in any direction. Note here that both the tangent space and the normal line to S are the eigenspaces of the Hessian of $\bar{r}$, $\bar{\nabla}^2\bar{r}$. Therefore we can see that

$$\bar{\nabla}^2\cosh k\bar{r} = (k^2\cosh k\bar{r})\,g, \tag{6}$$

where g is the metric of $\bar{M}$, and hence

$$\bar{\nabla}^2\bar{r} = k\coth k\bar{r}\,(g - \bar{\nabla}\bar{r}\otimes\bar{\nabla}\bar{r}). \tag{7}$$

Thus (2), (3), (7), and the fact that ∇r is an eigenvector of $\bar{\nabla}^2 r$ with eigenvalue zero prove (a). Then

$$\begin{aligned}\Delta\log(1+\cosh kr) &= \operatorname{div}\frac{k\sinh kr}{1+\cosh kr}\nabla r = \frac{k^2}{1+\cosh kr}|\nabla r|^2 + \frac{k\sinh kr}{1+\cosh kr}\Delta r\\ &\geqq \frac{2k^2\cosh kr + k^2|\nabla r|^2(1-\cosh kr)}{1+\cosh kr} \geqq k^2,\end{aligned}$$

which gives (b). Now we have

$$\begin{aligned}\Delta\log\frac{\sinh kr}{1+\cosh kr} &= \operatorname{div}\frac{k}{\sinh kr}\nabla r = -\frac{k^2\cosh kr}{\sinh^2 kr}|\nabla r|^2 + \frac{k}{\sinh kr}\Delta r\\ &\geqq \frac{2k^2\cosh kr(1-|\nabla r|^2)}{\sinh^2 kr} \geqq 0.\end{aligned}$$

However, $f(r) = \frac{1}{2\pi}\log\frac{\sinh kr}{1+\cosh kr}$ is a fundamental solution of Δ on $H^2(K)$ since $\frac{1}{2\pi f'(r)} = \frac{1}{k}\sinh kr$ is the length of a Jacobi field. So (c) follows. Adding (b) to (c) gives (d).

Lemma 3. *If Σ^2 is a minimal surface in a Riemannian manifold M^n with sectional curvature bounded above by a positive constant $K = k^2$, then on Σ we have*

(a) $\Delta r \geqq k(2 - |\nabla r|^2)\cot kr$;

(b) $\Delta\log\frac{\sin kr}{1+\cos kr} \geqq 2\pi\delta_p$, *if $p \in \Sigma$ and $r \leqq \frac{\pi}{2k}$*;

(c) $\Delta\log\sin kr \geqq 2\pi\delta_p - K$, *if $p \in \Sigma$ and $r \leqq \frac{\pi}{2k}$*.

Proof. As in the proof of Lemma 2, one can show that in $\bar{M}^n$ of constant sectional curvature K

$$\bar{\nabla}^2 \cos k\bar{r} = -(k^2 \cos k\bar{r})g$$

and

$$\bar{\nabla}^2 \bar{r} = k \cot k\bar{r}(g - \bar{\nabla}\bar{r} \otimes \bar{\nabla}\bar{r}),$$

from which (a) follows. And then

$$\Delta \log \frac{\sin kr}{1 + \cos kr} = \operatorname{div} \frac{k}{\sin kr} \nabla r = -\frac{k^2 \cos kr}{\sin^2 kr} |\nabla r|^2 + \frac{k}{\sin kr} \Delta r$$
$$\geqq \frac{2k^2 \cos kr}{\sin^2 kr}(1 - |\nabla r|^2) \geqq 0.$$

As in Lemma 2(c), $f(r) = \frac{1}{2\pi} \log \frac{\sin kr}{1 + \cos kr}$ is a fundamental solution of Δ on $S^2(K)$ since $\frac{1}{2\pi f'(r)} = \frac{1}{k} \sin kr$ is the length of a Jacobi field. Thus (b) follows. For (c) we compute

$$\Delta \log \sin kr = \operatorname{div}\left(\frac{k \cos kr}{\sin kr} \nabla r\right) = -k^2 \csc^2 kr\, |\nabla r|^2 + k \cot kr\, \Delta r$$
$$\geqq k^2 \csc^2 kr [2 \cos^2 kr - (1 + \cos^2 kr) |\nabla r|^2] \geqq -k^2.$$

Note that

$$\lim_{r \to 0} \frac{\frac{d}{dr} \log \sin kr}{\frac{d}{dr} \log \frac{\sin kr}{1 + \cos kr}} = 1,$$

which proves (c).

Lemma 4. *Let $\Gamma = \bar{p} \ast C$ be the cone from $\bar{p}$ over a curve C (that is, the union of the geodesic segments from $\bar{p}$ to the points of C) in a Riemannian manifold $\bar{M}$ of nonpositive constant sectional curvature $K = -k^2$ and let $\bar{r}(x) = \operatorname{dist}(\bar{p}, x)$, $x \in \bar{M}$. Then on Γ*

(a) $\Delta \bar{r}^2 = 4$, *if* $K = 0$; $\Delta \log(1 + \cosh k\bar{r}) = -K$, *if* $K < 0$;

(b) $\Delta \log \bar{r} = \alpha \delta_{\bar{p}}$, *if* $K = 0$; $\Delta \log \frac{\sinh k\bar{r}}{1 + \cosh k\bar{r}} = \alpha \delta_{\bar{p}}$, *if* $K < 0$,

where $\alpha = \operatorname{Angle}(C, \bar{p})$.

Proof. On Γ $\bar{\nabla}\bar{r}$ is perpendicular to H, the mean curvature vector of Γ; hence (3) implies that for any function f of distance $\bar{r}$, $\bar{\Delta} f = \Delta f$. Moreover $|\bar{\nabla}\bar{r}| \equiv 1$ on Γ. It follows from (4), (6) that all the inequalities in Lemma 1(a), (c) and Lemma 2(b), (c) become equalities. This proves the lemma except for the constant α. The constant 2π that appears in the Laplacian of the fundamental solution on $\mathbb{R}^2$ and H^2 comes from the limit as $a \to 0$ of the circumference of the circle of radius a with center at $\bar{p}$ divided by a. Similarly, if $S_a(\bar{p})$ denotes the geodesic sphere of radius a with center at $\bar{p}$, α equals

$$\lim_{a \to 0} \frac{1}{a} \operatorname{Length}(\Gamma \cap S_a(\bar{p})),$$

which is called the *angle* of C viewed from $\bar{p}$ and denoted $\operatorname{Angle}(C, \bar{p})$.

2. Sharp isoperimetric inequality

The sharp isoperimetric inequalities for minimal surfaces Σ in $\mathbb{R}^n$ and H^n have been derived in [Ch], [CG1] from the area and angle estimates

$$\operatorname{Area}(\Sigma) \leqq \operatorname{Area}(p \divideontimes \partial\Sigma) \quad \text{and} \quad \text{if } p \in \Sigma,\ \operatorname{Angle}(\partial\Sigma, p) \geqq 2\pi.$$

Unfortunately for a minimal surface Σ in a Riemannian manifold M of varying sectional curvature $\leqq K$ it is impossible to get these area and angle estimates. In this section, however, we will construct in a Riemannian manifold $\bar{M}$ of constant sectional curvature K a suitable cone $\bar{p} \divideontimes \bar{C}$ associated with Σ and derive similar estimates for Σ and $\bar{p} \divideontimes \bar{C}$.

A curve $\gamma \subset M$ is said to be *radially connected* from a point $p \in M$ if $\{\operatorname{dist}(p, q) : q \in \gamma\}$ is a connected interval.

Theorem 1. *Let Σ^2 be a minimal surface in a complete simply connected Riemannian manifold M with sectional curvature bounded above by a nonpositive constant K. If $\partial\Sigma$ is radially connected from a point $p \in \Sigma$, then Σ satisfies the isoperimetric inequality*

$$4\pi A \leqq L^2 + KA^2,$$

where equality holds if and only if Σ is a geodesic disk in a surface of constant Gaussian curvature K.

Proof. First let us assume $K = -k^2 < 0$. Define $r(x) = \operatorname{dist}(p, x)$, $x \in M$. By integrating Lemma 2(b) over Σ we get

$$-K \operatorname{Area}(\Sigma) \leqq \int_\Sigma \Delta \log(1 + \cosh kr) = \int_{\partial\Sigma} \frac{k \sinh kr}{1 + \cosh kr} \frac{\partial r}{\partial v}, \tag{8}$$

where v is the outward unit conormal vector to $\partial\Sigma$ on Σ. Let η be the unit vector normal to $\partial\Sigma$ that makes the smallest angle with ∇r, that is, the unit normal vector to $\partial\Sigma$ that lies in the two-dimensional plane spanned by ∇r and the tangent line of $\partial\Sigma$ such that $\dfrac{\partial r}{\partial \eta} \geqq 0$. Clearly

(9) $$\frac{\partial r}{\partial v} \leqq \frac{\partial r}{\partial \eta} = \sqrt{1 - \langle \nabla r, \tau \rangle^2},$$

where τ is a unit tangent to $\partial \Sigma$. It follows from (8) and (9) that

(10) $$-K \operatorname{Area}(\Sigma) \leqq \int_{\partial \Sigma} \frac{k \sinh kr}{1 + \cosh kr} \sqrt{1 - \langle \nabla r, \tau \rangle^2}.$$

Now the key step in the proof of Theorem 1 is to carry the integral term in (10) over to the simply connected space form $\bar{M}$ of curvature K. Let $C_1, \ldots, C_l$ be the components of $\partial \Sigma$. Fix $\bar{p} \in \bar{M}$, define $\bar{r}(y) = \operatorname{dist}(\bar{p}, y)$, $y \in \bar{M}$, and choose $q_i \in C_i$ for each $i = 1, \ldots, l$. Then choose $\bar{q}_1, \ldots, \bar{q}_l \in \bar{M}$ in such a way that $r(q_i) = \bar{r}(\bar{q}_i)$. Suppose each curve C_i is parametrized by $c_i(s)$ with arclength parameter s such that $q_i = c_i(0) = c_i(\lambda_i)$, $\lambda_i = \operatorname{Length}(C_i)$. Then we construct a curve $\bar{C}_i$ in $\bar{M}$ starting from $\bar{q}_i$ and parametrized by $\bar{c}_i(s)$ with arclength parameter $s \in [0, \lambda_i]$ and $\bar{c}_i(0) = \bar{q}_i$ such that the unit tangent vector $\bar{c}_i'(s)$ makes an angle of $\cos^{-1}\langle \nabla r, c_i'(s) \rangle$ with $\nabla \bar{r}$. Of course the curve $\bar{C}_i$ is not unique; but given a two-dimensional infinite cone $\bar{p} \times C$ containing $\bar{q}_i$, one can uniquely determine a curve $\bar{C}_i$ on $\bar{p} \times C$ with the prescribed properties. Note that any two curves C_a, C_b of equal length on the geodesic sphere $S(\bar{r}, \bar{p})$ of radius $\bar{r}$ centered at $\bar{p}$ are isometric. Note further that $\bar{p} \times C_a$ and $\bar{p} \times C_b$ are also isometric. Thus, by isometrically perturbing $(\bar{p} \times \bar{C}_i) \cap S(\bar{r}, \bar{p})$ on $S(\bar{r}, \bar{p})$ if necessary, one can construct $\bar{C}_i$ in such a way that $\bar{C}_i$ is closed, or equivalently, $\bar{c}_i(0) = \bar{c}_i(\lambda_i)$. Now r on C_i coincides with $\bar{r}$ on $\bar{C}_i$ in the sense that

$$r(c_i(s)) = \bar{r}(\bar{c}_i(s)) \quad \text{and} \quad \langle \nabla r, c_i'(s) \rangle = \langle \nabla \bar{r}, \bar{c}_i'(s) \rangle.$$

Hence (10) becomes

$$\begin{aligned} -K \operatorname{Area}(\Sigma) &\leqq \sum_{i=1}^{l} \int_{C_i} \frac{k \sinh kr}{1 + \cosh kr} \sqrt{1 - \langle \nabla r, c_i'(s) \rangle^2} \\ &= \sum_{i=1}^{l} \int_{\bar{C}_i} \frac{k \sinh k\bar{r}}{1 + \cosh k\bar{r}} \sqrt{1 - \langle \nabla \bar{r}, \bar{c}_i'(s) \rangle^2}. \end{aligned}$$

Let $\bar{\eta}$ be the outward unit conormal to $\bar{C}_i$ on $\bar{p} \times \bar{C}_i$. Then

$$\sqrt{1 - \langle \nabla \bar{r}, \bar{c}_i'(s) \rangle^2} = \frac{\partial \bar{r}}{\partial \bar{\eta}}.$$

Therefore

(11) $$\begin{aligned} \operatorname{Area}(\Sigma) &\leqq -\frac{1}{K} \sum_{i=1}^{l} \int_{\bar{C}_i} \frac{k \sinh k\bar{r}}{1 + \cosh k\bar{r}} \frac{\partial \bar{r}}{\partial \bar{\eta}} = \sum_{i=1}^{l} \int_{\bar{p} \times \bar{C}_i} \frac{1}{-K} \Delta \log(1 + \cosh k\bar{r}) \\ &= \sum_{i=1}^{l} \operatorname{Area}(\bar{p} \times \bar{C}_i) \quad \text{(by Lemma 4(a))} \\ &= \operatorname{Area}(\bar{p} \times \bar{C}), \quad \bar{C} = \bigcup_{i=1}^{l} \bar{C}_i. \end{aligned}$$

Also it follows from the definition of $\bar{C}_i$ that

$$\text{(12)} \qquad \operatorname{Length}(\partial\Sigma) = \operatorname{Length}(\bar{C}).$$

On the other hand, integrating Lemma 2(c) over Σ gives

$$\begin{aligned} 2\pi &\leqq \int_\Sigma \Delta \log \frac{\sinh kr}{1+\cosh kr} = \int_{\partial\Sigma} \frac{k}{\sinh kr} \frac{\partial r}{\partial \nu} \leqq \int_{\partial\Sigma} \frac{k}{\sinh kr} \frac{\partial r}{\partial \eta} \\ &= \int_{\bar{C}} \frac{k}{\sinh k\bar{r}} \frac{\partial \bar{r}}{\partial \bar{\eta}} = \int_{\bar{p} \divideontimes \bar{C}} \Delta \log \frac{\sinh k\bar{r}}{1+\cosh k\bar{r}} \\ &= \operatorname{Angle}(\bar{C}, \bar{p}) \quad \text{(by Lemma 4(b))}. \end{aligned}$$

Moreover, since $r|_{\partial\Sigma}$ coincides with $\bar{r}|_{\bar{C}}$, $\bar{C}$ is also radially connected from $\bar{p}$. Hence by [CG1], Lemma 4, we get

$$4\pi \operatorname{Area}(\bar{p} \divideontimes \bar{C}) \leqq \operatorname{Length}(\bar{C})^2 + K \operatorname{Area}(\bar{p} \divideontimes \bar{C})^2.$$

Therefore using (11) and (12), we obtain the desired isoperimetric inequality for Σ in case $K < 0$.

If equality holds in the isoperimetric inequality, then

$$\operatorname{Area}(\Sigma) = \operatorname{Area}(\bar{p} \divideontimes \bar{C})$$

and therefore equality should hold in Lemma 2(b). Consequently equality holds in (2) and $|\nabla r| \equiv 1$ on Σ as we easily see in the proof of Lemma 2(b). It follows that $\Sigma = p \divideontimes \partial\Sigma$ and, by Index Lemma, Σ is constantly curved and hence totally geodesic. Thus Schmidt's theorem [Sc] completes the proof in case $K < 0$.

Now suppose $K = 0$. Take $\bar{M} = \mathbb{R}^3$, fix $\bar{p} \in \mathbb{R}^3$ and construct $\bar{C} \subset \mathbb{R}^3$ from $\partial\Sigma \subset M$ as above. Lemma 1(a) and Lemma 4(a) give

$$\operatorname{Area}(\Sigma) \leqq \operatorname{Area}(\bar{p} \divideontimes \bar{C});$$

Lemma 1(c) and Lemma 4(b) give

$$2\pi \leqq \operatorname{Angle}(\bar{C}, \bar{p}).$$

Thus the desired result follows from [Ch].

Remark. If $\partial\Sigma$ is connected, it is radially connected from any point of Σ. If $\partial\Sigma$ has two components C_1 and C_2, one can find a point $p \in \Sigma$ with $\operatorname{dist}(p, C_1) = \operatorname{dist}(p, C_2)$, and then $\partial\Sigma$ is radially connected from p. Therefore one obtains the above isoperimetric inequality for Σ in case $\partial\Sigma$ has one or two components.

3. Weak isoperimetric inequality

In the preceding section we could not get the sharp isoperimetric inequality for a minimal surface whose boundary is not radially connected. But in this section, by contrast, we will obtain an isoperimetric inequality which, though not sharp, holds for any minimal surface; see also [CG2], Theorem 5.

Theorem 2. *Let Σ^2 be a minimal surface in a complete simply connected Riemannian manifold with sectional curvature bounded above by a constant K. If $K \leqq 0$, then*

$$2\pi A \leqq L^2 + KA^2. \tag{13}$$

If $K > 0$, (13) holds under the additional assumption $\operatorname{diam}(\Sigma) \leqq \dfrac{\pi}{2\sqrt{K}}$.

Proof. (i) $K = -k^2 < 0$. Integrating Lemma 2(d) over Σ for fixed $p \in \Sigma$, we get

$$2\pi - KA \leqq \int_{\Sigma} \Delta \log \sinh kr \leqq \int_{\partial\Sigma} k \coth kr. \tag{14}$$

Since (14) holds for all $p \in \Sigma$ we can integrate it over Σ and apply Fubini's theorem to obtain

$$\begin{aligned} 2\pi A - KA^2 &\leqq \int_{\Sigma}\int_{\partial\Sigma} k \coth kr = \int_{\partial\Sigma}\int_{\Sigma} k \coth kr \\ &\leqq \int_{\partial\Sigma}\int_{\Sigma} \Delta r \text{ (by Lemma 2(a))} \\ &= \int_{\partial\Sigma}\int_{\partial\Sigma} \frac{\partial r}{\partial \nu} \leqq L^2. \end{aligned}$$

(ii) $K = 0$. Integrate Lemma 1(c) twice and apply Lemma 1(b) as in (i).

(iii) $K > 0$. Integrate Lemma 3(c) twice and apply Lemma 3(a).

References

[BR] *E. F. Beckenbach* and *T. Rado*, Subharmonic functions and surfaces of negative curvature, Trans. Amer. Math. Soc. **35** (1933), 662–674.

[B] *F. Bernstein*, Über die isoperimetrische Eigenschaft des Kreises auf der Kugeloberfläche und in der Ebene, Math. Ann. **60** (1905), 117–136.

[Bo] *G. Bol*, Isoperimetrische Ungleichung für Bereiche auf Flächen, Jber. Deutsch. Math.-Verein. **51** (1941), 219–257.

[C] *T. Carleman*, Zur Theorie der Minimalflächen, Math. Z. **9** (1921), 154–160.

[Ch] *J. Choe*, The isoperimetric inequality for a minimal surface with radially connected boundary, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), 583–593.

[CG1] *J. Choe* and *R. Gulliver*, The sharp isoperimetric inequality for minimal surfaces with radially connected boundary in hyperbolic space, Invent. Math. **109** (1992), 495–503.

[CG2] *J. Choe* and *R. Gulliver*, Isoperimetric inequalities on minimal submanifolds of space forms, manuscr. math. **77** (1992), 169–189.

[LSY] *P. Li, R. Schoen*, and *S.-T. Yau*, On the isoperimetric inequality for minimal surfaces, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 237–244.

[Sc] *E. Schmidt*, Über die isometrische Aufgabe im n-dimensionalen Raum konstanter negativer Krümmung. I. Die isoperimetrischen Ungleichungen in der hyperbolischen Ebene und für Rotationskörper im n-dimensionalen hyperbolischen Raum, Math. Z. **46** (1940), 204–230.

[S] *J. Steiner*, Sur le maximum et le minimum des figures dans le plan sur la sphère et dans l'espace en général, J. reine angew. Math. **24** (1842), 93–152.

[W] *A. Weil*, Sur les surfaces à courbure négative, C. R. Acad. Sci., Paris **182** (1926), 1069–1071.

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