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The sharp isoperimetric inequality for minimal surfaces with radially connected boundary in hyperbolic space*

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Given a plane domain D bounded by a curve C, it has long been known that the area A of D and the length L of C are related by the classical isoperimetric inequality

$$4\pi A \leqq L^2 ,$$

where equality holds if and only if C is a circle. Many mathematicians have also sought isoperimetric inequalities for a domain in a curved space. An interesting one for a domain in the sphere was obtained by Bernstein in 1905 [B]:

$$4\pi A \leqq L^2 + A^2 .$$

Then Schmidt [S] proved in 1940 the analogue for the hyperbolic plane:

$$4\pi A \leqq L^2 - A^2 .$$

In each case, equality holds if and only if the domain is a geodesic disk. In fact, these three isoperimetric inequalities can all be expressed in one inequality as follows:

$$4\pi A \leqq L^2 + KA^2 ,$$

where K is the Gauss curvature of the simply connected space form in which D lies.

On the other hand, it has been a long-standing conjecture that the classical isoperimetric inequality $4\pi A \leqq L^2$ should hold for an arbitrary domain in a minimal surface in $\mathbf{R}^n$. Until now this inequality has been proved only for minimal surfaces with one or two boundary components, or more generally, with weakly or radially connected boundary [C, OS, LSY, Ch]. In view of this conjecture and the work of Bernstein and Schmidt, one may ask whether their inequalities hold for domains on a minimal surface in S^n or H^n. In this paper we show that any two-dimensional minimal surface Σ^2 in H^n such that $\partial\Sigma$ is *radially connected* from

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some point p of Σ, i.e. such that $\{r = \text{dist}(p, q),\ q \in \partial\Sigma\}$ is a connected interval, satisfies the sharp isoperimetric inequality

$$4\pi A \leqq L^2 - A^2 .$$

But the isoperimetric inequality $4\pi A \leqq L^2 + A^2$ for a minimal surface in S^n still remains open.

In our companion paper [CG] we obtain two different types of isoperimetric inequalities: First, we introduce a modified area $M(D)$ of a domain D, and show that

$$4\pi M(D) \leqq L(\partial D)^2 ,$$

where D is a domain on a minimal surface in S^n_+ or H^n, whose boundary is radially connected or weakly connected in analogy with [LSY]. Second, weaker isoperimetric inequalities

$$2\pi A \leqq L^2 + KA^2$$

are obtained for *any* minimal surface Σ in S^n_+ or in H^n, where $K = 1$ or -1 depending on whether Σ is in S^n_+ or in H^n. Surprisingly, while the modified-area inequality is valid for S^n_+ or for $\mathbf{R}^n$, the result of this paper is valid for H^n or for $\mathbf{R}^n$; compare Remark 1 below.

1 Estimates for the volume and angle of a cone

Every minimal surface considered in this paper is assumed to be differentiable up to its boundary.

Blaschke, earlier than [Ch], pointed out the value of comparing a minimal surface Σ in $\mathbf{R}^n$ with the cone over its boundary [Bl, p. 247]. Estimates for the volume of the cone $p \times\!\!\!\times\, \partial\Sigma$ and for the angle of $\partial\Sigma$ viewed from an interior point of Σ play crucial roles in the proof of the sharp isoperimetric inequality for Σ with radially connected boundary in [Ch]. In this section we obtain the analogous estimates for minimal surfaces in H^n. In fact, this will require a more exacting choice of test function: compare Proposition 1 and Proposition 2 of [Ch] with Proposition 2 and Proposition 1 below.

Lemma 1 *Suppose $h'(r) = r\varphi(r)$ for some smooth φ: $[0, \infty) \to \mathbf{R}$, and write $h(q) = h(r_p(q))$ where $r = r_p(q) = \text{dist}(p, q)$ for a fixed $p \in H^n$. If $\Sigma^k \subset H^n$ is either minimal or a cone over p, then*

$$\Delta h = r\varphi' + \varphi Q + (1 - |\nabla r|^2)(r\varphi \coth r - \varphi - r\varphi')$$

where $Q(r) = 1 + (k - 1)r \coth r$.

Proof. One shows that the Hessian in H^n,

$$\bar{\nabla}^2 \cosh r = (\cosh r)g ,$$

from which it follows that the Laplacian on Σ^k,

$$\Delta r = \coth r(k - |\nabla r|^2)$$

when Σ is either minimal or a cone over p. See Lemma 5(b) of [CG]. Lemma 1 follows by direct computation.

The following lemma addresses the case where $h(r)$ is the solution of $\Delta h \equiv 1$ on the totally geodesic submanifold $\Sigma = H^k \subset H^n$. The conclusions may also be found on p. 483 of [A].

Lemma 2 *Let $\varphi(r) = \alpha(r)/(r\alpha'(r))$, where $\alpha(r)$ is the volume of the geodesic ball of radius r in k-dimensional hyperbolic space H^k; thus $\alpha(0) = 0$ and $\alpha'(r) = k\omega_k \sinh^{k-1} r$. Define $Q(r)$ as in Lemma 1. Then*

(a) *for all $r > 0$, $\varphi'(r) < 0$ and $0 < \varphi(r) < \varphi(0) = 1/k$;*

and

(b) $r\varphi'(r) + \varphi(r)Q(r) \equiv 1$.

Proof. Differentiation of $r\varphi(r) = \alpha/\alpha'$ yields

$$r\varphi' + \varphi = 1 + r\varphi\alpha''/\alpha' = 1 - (k-1)\varphi r \coth r ,$$

from which (b) follows. Elementary asymptotic analysis shows that $\varphi(0) = 1/k$ and $\varphi'(0) = 0$. Since $\sinh r \cosh r > r$, we find $Q'(r) > 0$, so that $Q(r) > Q(0) = k$, for all $r > 0$. The derivative of (b) now yields $r\varphi'' + (1 + Q)\varphi' < 0$, or $(\varphi'(r)\exp P(r))' < 0$ where $P'(r) = (1 + Q)/r$. Since $\varphi'(0) = 0$, we conclude that $\varphi'(r) < 0$ for positive r.

Definition. Let $C \subset H^n$ be a $(k-1)$-dimensional rectifiable set and p a point in H^n. The $(k-1)$-dimensional *angle* $A^{k-1}(C, p)$ of C viewed from p is defined by setting

$$A^{k-1}(C, p) = \sinh^{1-k} t \cdot \text{Volume}\,[(p \times\!\!\times C) \cap S(p, t)] ,$$

where $S(p, t)$ is the geodesic sphere of radius $t < \text{dist}(p, C)$ centered at p, and the volume is measured counting multiplicity. Clearly, the angle does not depend on t.

Note that

$$A^{k-1}(C, p) = k\omega_k \Theta^k(p \times\!\!\times C, p) ,$$

where $\Theta^k(p \times\!\!\times C, p)$ is the k-dimensional density of $p \times\!\!\times C$ at p.

Proposition 1 *Let Σ be a k-dimensional compact minimal submanifold with boundary in H^n, and let p be an interior point of Σ. Then*

$$A^{k-1}(\partial\Sigma, p) \geqq k\omega_k .$$

Equality holds if and only if Σ is a domain on a totally geodesic H^k that is star-shaped with respect to p.

Proof. We use the Green's function $G(r)$ of H^k: $G'(r) = \sinh^{1-k} r$. Writing $G'(r) = r\varphi(r)$, we see that $r\varphi' + \varphi Q \equiv 0$ and

$$r\varphi \coth r - \varphi - r\varphi' = k \sinh^{-k} r \cosh r > 0$$

for $r > 0$, where $Q = 1 + (k-1) r \coth r$. Thus by Lemma 1, G is subharmonic on Σ, and harmonic on the cone $p \times\!\!\times \partial\Sigma$. Let v be the exterior unit normal vector to Σ and

η the exterior unit normal vector to the cone along $\partial\Sigma$. Then

$$\frac{\partial r}{\partial \nu} \leqq \frac{\partial r}{\partial \eta},$$

implying

$$k\omega_k \leqq k\omega_k + \lim_{t\to 0} \int_{\Sigma - B(p,t)} \Delta G = \int_{\partial\Sigma} G'(r)\frac{\partial r}{\partial \nu}$$

$$\leqq \int_{\partial\Sigma} \sinh^{1-k} r \cdot \frac{\partial r}{\partial \eta} = A^{k-1}(\partial\Sigma, p).$$

Equality holds if and only if $\Delta G(r) = 0$, $\Theta^k(\Sigma, p) = 1$, and $\nu = \eta$ if and only if Σ is a star-shaped minimal cone with density at the center equal to 1. Since S^{k-1} is the only $(k-1)$-dimensional minimal submanifold in S^{n-1} with volume $k\omega_k$, we conclude that Σ lies in a totally geodesic H^k.

The next proposition will allow us to replace a minimal submanifold Σ^k in H^n by the cone over its boundary, relying on the monotone dependence of the isoperimetric inequality on the volume of Σ. This proposition and Lemma 2 are closely related to the monotonicity formula of Anderson [A, p. 481].

Proposition 2 *Let Σ be a k-dimensional immersed compact minimal submanifold with boundary in hyperbolic space H^n, and let p be any point of H^n., Then*

$$\text{Volume}(\Sigma) \leqq \text{Volume}(p \times\!\!\!\!\times\, \partial\Sigma);$$

if equality holds, then $p \in \Sigma$, and Σ must be totally geodesic and star-shaped with respect to p.

Proof. Let $h(q) = h(r_p(q))$, where $h'(r) = \alpha(r)/\alpha'(r)$ as in Lemma 2. Let ν be the outward unit normal vector to $\partial\Sigma$, which is tangent to Σ, and η the unit vector tangent to $p \times\!\!\!\!\times\, \partial\Sigma$; as in the proof of Proposition 1, we have $\partial r/\partial\nu \leqq \partial r/\partial\eta$. This implies

$$\int_\Sigma \Delta h = \int_{\partial\Sigma} \frac{\partial h}{\partial \nu} \leqq \int_{\partial\Sigma} \frac{\partial h}{\partial \eta} = \int_{p \times\!\!\!\!\times\, \partial\Sigma} \Delta h,$$

since $h'(r) > 0$ for all $r > 0$. But according to Lemmas 1 and 2,

$$\Delta h = 1 + (1 - |\nabla r|^2)[(r\coth r - 1)\varphi - r\varphi']$$

either on Σ or on $p \times\!\!\!\!\times\, \partial\Sigma$, where $\varphi(r) > 0$ and $\varphi'(r) < 0$ for $r > 0$. In particular, $\Delta h \geqq 1$; and further, $\Delta h > 1$ unless $|\nabla r| = 1$ or $r = 0$. On the cone $p \times\!\!\!\!\times\, \partial\Sigma$, we have $|\nabla r| = 1$. Therefore,

$$\text{Volume}(\Sigma) \leqq \int_\Sigma \Delta h \leqq \int_{p \times\!\!\!\!\times\, \partial\Sigma} \Delta h = \text{Volume}(p \times\!\!\!\!\times\, \partial\Sigma).$$

Equality would imply $|\nabla r| = 1$ a.e. on Σ, which is to say that Σ coincides with a subset of the cone $p \times\!\!\!\!\times\, \partial\Sigma$. Equality also requires $\partial h/\partial\nu = \partial h/\partial\eta$, hence for every

$q \in \partial\Sigma$ the entire geodesic segment from p to q lies in Σ. At p, each such segment is tangent to the tangent plane to Σ. This implies that Σ is totally geodesic.

Remark 1 Proposition 2 is false when H^n is replaced by the hemisphere S^n_+, even for $n = 3$ and $k = 2$. For example, let Σ be half of the *Clifford torus*:

$$\Sigma = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| = |y| = 1/\sqrt{2}, x_1 > 0\},$$

and $p = (1, 0, 0, 0)$. Then $\text{Area}(\Sigma) = \pi^2$, which is greater than $\text{Area}(p \mathbin{⨳} \partial\Sigma) = 2\sqrt{2}\pi$. Nonetheless, for domains $\Omega \subset \Sigma$ we have an isoperimetric inequality $L^2 \geqq \min\{4\pi A, 8\pi^2\}$ which implies the sharp S^2-isoperimetric inequality

$$4\pi A \leqq L^2 + A^2 .$$

It is an interesting question whether this last inequality is valid for every two-dimensional minimal surface in the hemisphere S^n_+.

2 Approximation lemma

In light of Proposition 2 we would like to prove that certain hyperbolic cones satisfy the isoperimetric inequality $4\pi A \leqq L^2 - A^2$. This inequality was proved in great generality by Bol, namely, for any smooth, simply connected, two-dimensional manifold with Gauss curvature $K \leqq -1$. The following approximation lemma may be interpreted as stating in a precise way that a hyperbolic cone has generalized Gauss curvature $\leqq -1$ if the angle at its vertex is at least 2π. It is well known that a two-dimensional hyperbolic cone has Gauss curvature $\equiv -1$ away from its vertex.

Lemma 3 *Let $\Sigma_0 = (\mathbf{R}^2, ds^2)$ be the singular Riemannian 2-manifold (a hyperbolic cone) with metric given in geodesic polar coordinates (r, θ) by*

$$ds^2 = dr^2 + (a_0/2\pi)^2 \sinh^2 r \, d\theta^2 .$$

If $a_0 \geqq 2\pi$, then ds^2 may be approximated in $C^1_{\text{loc}}(\mathbf{R}^2 \backslash \{0\})$ by smooth metrics ds^2_δ having Gauss curvature $K_\delta \leqq -1$.

Proof. If $a_0 = 2\pi$, then $ds^2_\delta = ds^2$ suffices. For any angle $a_0 > 2\pi$, we shall construct ds^2_δ in the form

$$ds^2_\delta = dr^2 + g_\delta(r)^2 \, d\theta^2$$

for an appropriate function g_δ: $[0, \infty) \to [0, \infty)$. Similarly, write $g(r) = (a_0/2\pi)\sinh r$. The Gauss curvature K_δ of $(\mathbf{R}^2, ds^2_\delta)$ is determined by the Jacobi equation

$$\text{(J)} \qquad g''_\delta(r) + K_\delta(r) g_\delta(r) = 0 .$$

The C^∞ function g_δ will be smooth approximation to a $C^{1,1}$ function g_0 defined by

$$g_0(r) = \beta^{-1} \sinh \beta r, \quad 0 \leqq r \leqq r_1;$$

$$g_0(r) = g(r - \varepsilon), \quad r \geqq r_1;$$

where $\varepsilon > 0$, $r_1 > \varepsilon$, and $\beta > 1$ are appropriately chosen parameters. Continuity of g_0'/g_0 at r_1 is equivalent to

$$(*) \qquad \beta \coth \beta r_1 = \coth (r_1 - \varepsilon) .$$

This plus the continuity of g_0 at r_1 imply that

$$(a_0/2\pi)^2 = 1 + (1 - \beta^{-2}) \sinh^2 \beta r_1 ,$$

which determines r_1 uniquely as a function of $\beta \in (1, \infty)$ since $a_0 > 2\pi$. Now let $\varepsilon = \varepsilon(\beta) < r_1(\beta)$ be defined by Eq. $(*)$. Then the $C^{1,1}$ metric

$$ds_0^2 = dr^2 + g_0(r)^2 \, d\theta^2$$

has Gauss curvature $K_0 \equiv -\beta^2$ on the disk $B_{r_1}(0)$ and $K_0 \equiv -1$ on $\mathbf{R}^2 \backslash B_{r_1}(0)$. Note also that the mapping given in polar coordinates by $(r, \theta) \mapsto (r - \varepsilon, \theta)$ is an isometry from $\mathbf{R}^2 \backslash \bar{B}_{r_1}(0)$ with the metric ds_0^2 to $\Sigma_0 \backslash \bar{B}_{r_1 - \varepsilon}(0)$. Since $\coth \beta r_1 > 1$, it follows from $(*)$ that $r_1(\beta) - \varepsilon(\beta) \to 0$ as $\beta \to \infty$, so that the complement of an arbitrarily small neighborhood of the singularity in Σ_0 is isometric to a subset of $(\mathbf{R}^2, ds_0^2)$. Further, it may be seen from the definition of $r_1(\beta)$ that $r_1(\beta) \to 0$ as $\beta \to +\infty$, and hence also $\varepsilon(\beta) \to 0$.

We may now construct the smooth approximation g_δ by smoothing the Gauss curvature K_0 of ds_0^2: we choose $K_\delta \in C_0^\infty([0, \infty))$ with $K_\delta(r) \equiv -\beta^2 (0 \leqq r \leqq r_1 - \delta)$, $K_\delta(r) \equiv -1$ $(r \geqq r_1 + \delta)$ and $K_\delta'(r) \geqq 0$ for all r. We then solve the Jacobi equation (J) with $g_\delta(0) = 0$, $g_\delta'(0) = 1$. Since $-\beta^2 \leqq K_\delta(r) \leqq -1$, this initial value problem has a unique solution g_δ: $[0, \infty) \to [0, \infty)$ which is moreover positive on $(0, \infty)$. For any exponent $1 < p < \infty$, we have $K_\delta \to K_0$ in $L^p([0, \infty))$. This implies that $g_\delta \to g_0$ in $W^{2,p}$ on any bounded interval, and hence also in $C^{1,\alpha}$ for any $\alpha < 1$ on any bounded interval. By choosing β sufficiently large, we make $r_1(\beta)$ and $\varepsilon(\beta)$ as small as desired; choosing also δ sufficiently close to 0 results in a metric ds_δ^2 arbitrarily close to ds^2 in $C^{1,\alpha}_{\mathrm{loc}}(\mathbf{R}^2 \backslash \{0\})$.

3 The sharp isoperimetric inequality

As was hinted in the preceding section, we shall prove the sharp isoperimetric inequality for cones in H^n by combining Bol's theorem and the approximation lemma. The analogous result for cones in $\mathbf{R}^n$ was proved in [Ch, Lemma 1] by a substantially different method of developing the cone into a planar domain.

Lemma 4 *Choose $p \in H^n$, and let C be a compact 1-dimensional submanifold of H^n such that C is radially connected from p and $A^1(C, p) \geqq 2\pi$. Then the length L of C and the area A of the cone $p \times\!\!\!\times C$ satisfy the sharp isoperimetric inequality of domains in H^2:*

$$4\pi A \leqq L^2 - A^2 .$$

Proof. Write $r(q) = \operatorname{dist}(p, q)$, as usual, for the distance in H^n. We shall first show that on any radially connected 1-manifold C, there are a finite number of points $q_1, \ldots, q_m, p_1, \ldots, p_m = p_0$ such that

(i) $r(q_i) = r(p_i)$ for all $1 \leqq i \leqq m$;
(ii) p_i and q_{i+1} lie in the same component of C for all $0 \leqq i \leqq m-1$; and
(iii) C may be oriented so that the union of the m closed arcs of C from p_i to q_{i+1} in the positive sense, $0 \leqq i \leqq m-1$, covers C exactly once.

The proof is by induction on the number J of connected components of C. If $J = 1$, the assertion is obvious with $m = 1$. Now suppose the assertion holds for 1-manifolds in H^n with $(J-1)$ connected components. Write the connected components of C as $\Gamma_1, \ldots, \Gamma_J$, where $\min\{r(q): q \in \Gamma_1\} \geqq \min\{r(q): q \in \Gamma_j\}$ for all $2 \leqq j \leqq J$. Then $\Gamma_2 \cup \ldots \cup \Gamma_J$ is radially connected from p. Applying the induction hypothesis, we may write $\{Q_1, \ldots, Q_M, P_1, \ldots, P_M = P_0\}$ for a set of points satisfying (i), (ii) and (iii) with $\Gamma_2 \cup \ldots \cup \Gamma_J$ in place of C. Since C is radially connected, there are points $P \in \Gamma_1$ and $Q \in \Gamma_2 \cup \ldots \cup \Gamma_J$ with $r(P) = r(Q)$ (for example, $r(P) = \min\{r(q): q \in \Gamma_1\}$). Let P_k and Q_{k+1} be the endpoints of the interval in which Q falls, according to (iii). Define $p_l = P_l$ and $q_l = Q_l$ for $1 \leqq l \leqq k$; $q_{k+1} = Q = p_{k+2}$; $p_{k+1} = P = q_{k+2}$; and $q_l = Q_{l-2}$, $p_l = P_{l-2}$ for $k+3 \leqq l \leqq m = M+2$. Then $\{q_1, \ldots, q_m, p_1, \ldots, p_m = p_0\}$ satisfy (i), (ii) and (iii) as claimed. (Incidentally, one may note that $m + 1 = 2J$.)

Write $a_0 = A^1(C, p)$. We may now show that $p ⋙ C$ may be mapped discontinuously, but locally isometrically, into an abstract hyperbolic cone $\Sigma_0 = (\mathbf{R}^2, ds^2)$ with the singular Riemannian metric

$$ds^2 = dr^2 + (a_0/2\pi)^2 \sinh^2 r \, d\theta^2 ,$$

so that $r = \operatorname{dist}(p, \cdot)$ is preserved. Namely, let $\{q_1, \ldots, q_m, p_1, \ldots, p_m = p_0\}$ be a set of points in C such that properties (i), (ii) and (iii) are valid. For $0 \leqq i \leqq m-1$, write $C(p_i, q_{i+1})$ for the closed oriented arc of C from p_i to q_{i+1}. Then $p ⋙ C(p_0, q_1)$ may be mapped isometrically into Σ_0 so that for all $q \in C(p_0, q_1)$ the H^n-geodesic from p to q is mapped onto a geodesic segment $\theta =$ const. starting at the vertex $0 \in \Sigma_0$. The next sector $p ⋙ C(p_1, q_2)$ of $p ⋙ C$ is then mapped isometrically onto an adjacent sector of Σ_0, so that the geodesics from p to q_1 and from p to p_1 are mapped to the same radial geodesic segment. This process continues until $p ⋙ C(p_{m-1}, q_m)$ is mapped isometrically into Σ_0, so that the geodesics from p to q_{m-1} and from p to p_{m-1} are identified, and the geodesics from p to q_m and from p to $p_m = p_0$ are identified. This process closes up exactly since the angle at the vertex of Σ_0 is $a_0 = A^1(C, p) = \sum_{i=0}^{m-1} A^1(C(p_i, q_{i+1}), p)$. Observe that $p ⋙ C$ is mapped, almost everywhere one-to-one, onto a star-shaped domain $\Omega \subset \Sigma_0$ of area A, such that $\partial\Omega$ has length L. We may assume that $p \notin C$, since Area$(p ⋙ C)$ varies continuously with p, and since $A^1(C, p)$ is lower semi-continuous. Then Ω is a star-shaped neighborhood of 0 in Σ_0. Applying Lemma 3 we see that for each δ near 0 there is a smooth Riemannian surface $(\mathbf{R}^2, ds_\delta^2)$, with Gaussian curvature $K_\delta \leqq -1$, which converges locally uniformly to Σ_0, and which converges $C^{1,\alpha}$ to Σ_0 on compact sets in $\mathbf{R}^2 \setminus \{0\}$. Then with respect to ds_δ^2, $\partial\Omega$ has length $L(\delta) \to L$ and Ω has area $A(\delta) \to A$ as $\delta \to 0$. By Bol's theorem [Bol, p. 230] the isoperimetric inequality

$$4\pi A(\delta) \leqq L(\delta)^2 - A(\delta)^2$$

holds, and the conclusion of Lemma 4 follows.

Remark 2 Lemma 4 is false for submanifolds of dimension $k \geqq 3$ in H^n or even in $\mathbf{R}^n$. In $\mathbf{R}^n$, we may choose the reference point p near $p_0 = 0$. Given $R > 1, 0 < \varepsilon \ll 1$ and a point $q_1 \in \mathbf{R}^n$ with $|q_1|^2 = R^2 - 1$, let the $(k-1)$-submanifold C be formed from the two unit $(k-1)$-spheres $S_R^{n-1}(0) \cap S_1^{n-1}(\pm q_1) \cap \mathbf{R}^{k+1}$ plus a thin "bridge" of the form $[-R, R] \times S_\varepsilon^{k-2}$ connecting points q_2 and $-q_2$ on the unit spheres, and smoothed. Then for sufficiently small ε, there is an immersed minimal k-submanifold Σ with boundary C, which is uniformly close to the union of the two flat unit k-dimensional balls with a thin "bridge" of the form $[-R, R] \times B_\varepsilon^{k-1}$, by a theorem of Smale [Sm]. Choose $p \in \Sigma$ with $\operatorname{dist}(p, p_0) < \varepsilon$. Then the angle

$$A^{k-1}(C, p) \geqq k\omega_k$$

by Proposition 1. Thus C satisfies conditions analogous to all hypotheses of Lemma 4. But

$$\text{Volume}(C) = 2k\omega_k + O(R\varepsilon^{k-2}),$$

while a longer computation shows that

$$\text{Volume}(p \times\!\!\!\times C) = 2R\omega_k + O(R\varepsilon^{k-1}),$$

so that for large R the k-dimensional Euclidean isoperimetric inequality

$$(\text{Volume}(C))^k \geqq k^k\omega_k(\text{Volume}(p \times\!\!\!\times C))^{k-1}$$

is certainly false. Thus there is no hope of extending Lemma 4 to submanifolds of dimension greater than two. On the other hand, the minimal submanifold Σ has

$$\text{Volume}(\Sigma) \leqq 2\omega_k + 2R\omega_{k-1}\varepsilon^{k-1},$$

as follows from the proof of Smale's theorem. For small ε, Σ itself therefore satisfies the k-dimensional Euclidean isoperimetric inequality

$$(\text{Volume}(\partial\Sigma))^k \geqq k^k\omega_k(\text{Volume}(\Sigma))^{k-1}.$$

That this inequality be valid for every k-dimensional minimal submanifold Σ of $\mathbf{R}^n$ remains a challenging conjecture; an eventual proof cannot be found through the straightforward intermediation of a cone $p \times\!\!\!\times \partial\Sigma$.

Using Proposition 1, Proposition 2, and Lemma 4, and the monotonicity of the quadratic function $4\pi A + A^2$ for positive area A, we may now prove our main result.

Theorem 1 *Let Σ^2 be an immersed compact minimal surface with boundary in hyperbolic space H^n. Assume there exists $p \in \Sigma$ such that $\partial\Sigma$ is radially connected from p. Then* Area(Σ) *and* Length$(\partial\Sigma)$ *satisfy the isoperimetric inequality*

$$4\pi A \leqq L^2 - A^2,$$

with equality if and only if Σ is a geodesic ball in a totally geodesic $H^2 \subset H^n$.

Remark 3 If $\partial\Sigma$ has two components, choose two points p_1 and p_2, one from each component. Then there exists a point q on Σ with $\operatorname{dist}(q, p_1) = \operatorname{dist}(q, p_2)$, which implies that $\partial\Sigma$ is radially connected from q. Consequently Σ satisfies the above isoperimetric inequality.

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