EXISTENCE AND GEOMETRIC STRUCTURE OF METRICS ON SURFACES WHICH EXTREMIZE EIGENVALUES

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Abstract. This is an exposition of the research area around our lecture at the 60th anniversary conference of IMPA which was held in October of 2012. It is a survey of results which have been obtained over many years concerning sharp upper bounds on the first eigenvalue of a surface, either with or without boundary, in terms or area or boundary length and the surface topology. It is mostly expository, but contains a new coarse upper bound for non-orientable surfaces with boundary. It also contains a classical reformulation of recent results in [FS2].

1. Introduction

In this paper we survey two extremal problems for eigenvalues on surfaces. The first problem is to determine a metric g on a given closed smooth surface M^2, normalized so that $A(g) = 1$, with the largest first eigenvalue. Another way to put the question is the problem of finding a sharp upper bound on λ_1 in terms of the area and topology of a surface M. This problem turns out to be intimately related to the study of minimal surfaces in the standard sphere $\mathbb{S}^n$ whose coordinate functions are first eigenfunctions for the induced metric.

The second problem that we survey is an eigenvalue problem for surfaces with non-empty boundary for which the maximization problem is also intimately related to a class of minimal surfaces called free boundary solutions in the standard ball B^n. This eigenvalue problem is the problem of maximizing the first Steklov eigenvalue (first nonzero eigenvalue of the Dirichlet-Neumann map) among all metrics on M with $L(\partial M) = 1$.

In Section 2 we discuss the connection between the eigenvalue problems and minimal immersions. These basically arise from the study of conformal maps from a surface into the sphere or the ball whose component functions are eigenfunctions. We also give some basic examples of metrics arising on minimal surfaces which turn out to be extremals for their topology.

In Section 3 we give a fairly detailed description of the current state of affairs for compact closed surfaces. This work extends over a period of more than forty years and many authors have made contributions. Even with the many successes, we understand very little about the problem for surfaces of negative Euler characteristic, so there is still much to be done.

In Section 4 we introduce the problem for surfaces with boundary. Although this problem historically begins earlier than the closed case, there are fewer results, and most of these have been obtained relatively recently. A large part of this section gives a detailed discussion of the work contained in [FS2]. There is one result in this section which does not seem to be

The author was partially supported by NSF grant DMS-1105323
Mathematics Subject Classification: 53A10, 58C40
Key Words: spectral theory, minimal surface, free boundary condition, Steklov eigenvalue.

in the literature and that is Theorem 4.4. It gives a coarse upper bound on the first Steklov eigenvalue for a non-orientable surface of genus γ with k boundary components. The result states that for any metric on such a surface we have the bound

$$\sigma_1 L \leq 24\pi(\gamma + 1),$$

so is independent of k.

In Section 5 we give a formulation of the results of Section 3 and Section 4 in terms of inhomogeneous membranes. In particular the statements of theorems of [FS2] for plane domains will be of interest to people working on spectral problems but not necessarily differential geometry.

2. Maps defined by eigenfunctions and minimal submanifolds

Suppose that (M^m, g) is a compact Riemannian manifold with or without boundary. A function u on M is an eigenfunction with eigenvalue λ if there is a positive constant λ such that u satisfies the equation

$$\Delta u + \lambda u = 0.$$

For now we will work locally and not worry about boundary conditions. Recall that the energy of a map $U : M \to N^n$ between Riemannian manifolds is the integral of its energy density $\|dU\|^2$ where the norm is taken with respect to the metrics on M and N, so in local coordinates we have

$$\|dU\|^2 = \sum_{i,j} g^{ij} \langle \partial_{x^i} U, \partial_{x^j} U \rangle.$$

Thus we have the energy $E(U)$ given by

$$E(U) = \int_M \|dU\|^2 \, d\mu$$

where μ is the volume measure of the metric g on M. A map is said to be *harmonic* if it is stationary for the energy functional; that is, for any smooth family U_t of maps with $U_0 = U$ and $U_t = U$ outside a compact set we have the derivative with respect to t at $t = 0$ vanishes, $\dot{E} = 0$. If N is isometrically embedded in $\mathbb{R}^p$, then this condition expresses the fact that the component of ΔU at $U(x)$ tangential to N vanishes for every $x \in M$. Thus the harmonic map equation is

$$\Delta U = (\Delta U)^{\perp} = \sum_{ij} g^{ij} A_{U(x)}(\partial_{x^i} U, \partial_{x^j} U)$$

where $A_P(\cdot, \cdot)$ is the normal vector valued second fundamental form at P of N in $\mathbb{R}^p$. In particular if N is the unit sphere in $\mathbb{R}^{n+1}$ the harmonic map equation is

$$\Delta U = -\|dU\|^2 U.$$

We have the following result for maps defined by eigenfunctions.

Proposition 2.1. *If $U : M \to \mathbb{S}^n$ is a map such that $U = (u_1, \ldots, u_{n+1})$ where each u_i is an eigenfunction with eigenvalue λ, then U is a harmonic map with constant energy density $\|dU\|^2 = \lambda$. If in addition U is a conformal map, then it follows that $U^*(g_0) = \frac{\lambda}{m} g$ where g_0 is the metric of $\mathbb{S}^n$. In this case the image $\Sigma = U(M)$ is a minimal submanifold of $\mathbb{S}^n$.*

Proof. We observe that for any point $x \in M$ we have $\Delta U(x) = -\lambda U(x)$, and therefore $\Delta U(x)$ is perpendicular to $\mathbb{S}^n$ at $U(x)$. It follows that U is harmonic and therefore satisfies the harmonic map equation. Thus we must have $\|dU\|^2 \equiv \lambda$.

If we now assume that U is conformal then we have $U^*(g_0) = \rho^2 g$ for some positive function ρ on M. We then see that the energy density of U is given by $\|dU\|^2 = m\rho^2$. If U is given by eigenfunctions we then have from the previous paragraph $\rho^2 = m^{-1}\|dU\|^2 = \frac{\lambda}{m}$ as claimed. Thus U becomes an isometric harmonic immersion for the metric $\frac{\lambda}{m}g$ and is therefore a minimal immersion. □

Remark 2.2. We note that the converse of Proposition 2.1 is also true in that if we are given a minimal immersion $U : M \to \mathbb{S}^n$, then if we take $g = U^*(g_0)$ so that U is isometric, then the components of U are eigenfunctions with eigenvalue m.

There are many examples of minimal submanifolds of spheres, but we describe four which will be of particular interest for us.

Example 2.3. (Standard Sphere) Let M^m be the standard sphere and consider the standard embedding as the unit sphere in $\mathbb{R}^{m+1}$. This is clearly given by eigenfunctions with eigenvalue m, the first nonzero eigenvalue of M.

Example 2.4. (Veronese Embedding of $\mathbb{RP}^2$) We think of $\mathbb{RP}^2$ as the quotient of $\mathbb{S}^2$ by the antipodal map $x \to -x$. Thus the eigenfunctions of $\mathbb{RP}^2$ are the even eigenfunctions of $\mathbb{S}^2$. We consider the map $U : \mathbb{S}^2 \to \mathbb{R}^5$ given by

$$U(x, y, z) = \sqrt{3}(xy, xz, yz, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2).$$

We see that the map is given by second eigenfunctions of $\mathbb{S}^2$ and therefore defines a map $U : \mathbb{RP}^2 \to \mathbb{R}^5$ by first eigenfunctions. It is easy to check that $|U|^2 \equiv 1$, and therefore U is a map to $\mathbb{S}^4$. By Proposition 2.1, U is harmonic, and it can either be checked directly or one can quote the general theorem that *any* harmonic map from $\mathbb{S}^2$ to a Riemannian manifold is conformal (vanishing Hopf differential) to see that U is conformal. Therefore U defines a minimal immersion by first eigenfunctions. The eigenvalue is $\lambda = 6$, and the induced metric $U^*(g_0) = 3g$ again from Proposition 2.1. Thus the area of $\Sigma = U(\mathbb{RP}^2) = 6\pi$, three times the area of the curvature 1 metric on $\mathbb{RP}^2$.

Example 2.5. (Clifford Torus) We consider the square torus $T_c = \mathbb{R}^2/\mathbb{Z}^2$ and we define a map $U : \mathbb{R}^2 \to \mathbb{R}^4 \approx \mathbb{C}^2$ by

$$U(x, y) = \frac{1}{\sqrt{2}}(e^{2\pi ix}, e^{2\pi iy}).$$

We see that U is given by eigenfunctions, is periodic under the integer lattice, has values in the unit sphere, and so defines a harmonic map $U : T_c \to \mathbb{S}^3$. Furthermore U is given by first eigenfunctions and it is easily checked that U is conformal. The induced metric $U^*(g_0)$ is equal to $2\pi^2$ times the flat metric on $\mathbb{R}^2/\mathbb{Z}^2$, and so the area of $\Sigma = U(T_c)$ is $2\pi^2$.

Example 2.6. (Equilateral Torus) The 60^0 equilateral torus is $T_e = \mathbb{R}^2/\Lambda$ where Λ is the lattice generated by the vectors $(1, 0)$ and $(1/2, \sqrt{3}/2)$. We define a map $U : \mathbb{R}^2 \to \mathbb{R}^6 \approx \mathbb{C}^3$ by

$$U(x, y) = \frac{1}{\sqrt{3}}(e^{4\pi iy/\sqrt{3}}, e^{2\pi i(x-y/\sqrt{3})}, e^{2\pi i(x+y/\sqrt{3})}).$$

It is easily checked that U is invariant under Λ and so defines a map $U : T_e \to \mathbb{S}^5$. The component functions are first eigenfunctions on T_e, and the map is conformal, so it defines a flat minimal immersion. The area of $\Sigma = U(T_e)$ is $\frac{4\pi^2}{\sqrt{3}}$ which is larger than $2\pi^2$, the area of the Clifford torus.

We now discuss some analogous ideas for manifolds with boundary. Let M^m be a Riemannian manifold with nonempty boundary ∂M. If we are given a smooth function u on ∂M we denote by $\hat{u}$ its harmonic extension to M; that is, we have $\Delta\hat{u} = 0$ in M and $\hat{u} = u$ on ∂M. We then consider the Dirichlet-Neumann map L given by

$$L(u) = \partial_\nu \hat{u}$$

where ν is the outward unit normal vector to ∂M. It turns out that L is a self-adjoint elliptic pseudo-differential operator of order 1, and therefore has discrete spectrum $0 = \sigma_0 < \sigma_1 \leq \sigma_2 < \dots$ with real eigenvalues which tend to infinity. We refer to the σ_i as Steklov eigenvalues and we observe that an eigenfunction u for an eigenvalue σ is a function which satisfies

$$\Delta u = 0 \text{ in } M,\ \partial_\nu u = \sigma u \text{ on } \partial M.$$

The eigenfunctions corresponding to the trivial eigenvalue $\sigma_0 = 0$ are the constant functions. We will consider proper maps $U : M \to B^n$ consisting of Steklov eigenfunctions.

We first introduce a variational problem for maps $U : \partial M \to N$ where we assume that N^n is a compact submanifold of $\mathbb{R}^p$. This problem is called the $\frac{1}{2}$-harmonic map problem (see Da Lio and Riviére [DR]). The functional is given by

$$E_{1/2}(U) = E(\hat{U})$$

where as above $\hat{U}$ is the harmonic extension of U. If U is a critical point of the functional it is called a $\frac{1}{2}$-harmonic map. This terminology is used because if M is the unit ball, then $E_{1/2}$ is the $H^{1/2}$ energy. We can derive the Euler-Lagrange equation by considering a variation η of U where $\eta(x)$ is a tangent vector to N at $U(x)$ for each $x \in \partial M$. We then have

$$\dot{E}_{1/2} = \int_M \langle \nabla\hat{\eta}, \nabla\hat{U}\rangle \, d\mu = \int_{\partial M} \langle \eta, \partial_\nu \hat{U}\rangle = 0.$$

Thus a $\frac{1}{2}$-harmonic map satisfies the equation

$$(\partial_\nu \hat{U}(x))^t = 0 \text{ for } x \in \partial M$$

where $(\cdot)^t$ denotes the component tangential to N at $U(x)$.

A related boundary condition is the free boundary condition for a minimal submanifold Σ^m of $\mathbb{R}^p$. This is the condition that $\partial\Sigma \subseteq N$, and the conormal vector η along $\partial\Sigma$ is orthogonal to N. This boundary condition arises when Σ is a critical point for the volume functional under deformations which preserve N, but are not necessarily the identity on N. The following proposition relates these concepts with Steklov eigenfunctions.

Proposition 2.7. *Suppose U is a proper isometric minimal immersion from M^m into B^n. The following are equivalent:*

(1) *The component functions of U are Steklov eigenfunctions with eigenvalue* 1.

(2) *The map $U : \partial M \to \mathbb{S}^{n-1}$ is a $\frac{1}{2}$-harmonic map with the map in the interior being its harmonic extension.*

(3) *The minimal submanifold $\Sigma = U(M)$ is a free boundary solution in the ball.*

Proof. Condition 1 implies that $\partial_\nu U = U$ at each point of ∂M, and therefore $U : \partial M \to \mathbb{S}^{n-1}$ is a $\frac{1}{2}$-harmonic map. Since U is harmonic by assumption, we have verified condition 2.

Assuming condition 2 we note that the conormal vector of Σ is given by $\eta = \partial_\nu U$, and therefore the $\frac{1}{2}$-harmonic condition implies that η is orthogonal to $\mathbb{S}^{n-1}$ at each point of $\partial\Sigma$. Therefore Σ is a free boundary solution in the ball as required by condition 3.

Finally, if condition 3 holds then at each point x of $\partial\Sigma$ we have the outward conormal vector η orthogonal to $\mathbb{S}^{n-1}$, and therefore we have $\eta(P) = P$ for all $P \in \partial\Sigma$. Since $\eta = \partial_\nu U$, this is the condition that $\partial_\nu U(x) = U(x)$ for each $x \in \partial M$. This completes the proof that condition 3 implies condition 1. □

In the case of surfaces, $m = 2$, the conformal invariance of the harmonic condition can be used to derive the following useful result.

Proposition 2.8. *Suppose that $U : M^2 \to B^n$ is a proper conformal harmonic map whose component functions are Steklov eigenfunctions with eigenvalue $\sigma > 0$. It then follows that $\Sigma = U(M)$ is a branched minimal surface which is a free boundary solution in the ball, and that the induced metric $U^*(\delta) = \rho^2 g$ where $\rho \equiv \sigma$ on ∂M.*

Proof. The assumption implies that U is a proper branched minimal immersion from M into B^n. The Steklov boundary condition then says $\partial_\nu U = \sigma U$ on $\partial\Sigma$ where ν is the unit normal with respect to g, the given metric on M. Now the unit conormal vector of $\partial\Sigma$ is $\eta = \|\partial_\nu U\|^{-1}\partial_\nu U$ and this is orthogonal to $\mathbb{S}^{n-1}$, so Σ is a free boundary minimal surface. Now the induced metric is given by $U^*(\delta) = \rho^2 g$ for some non-negative function ρ (positive with isolated zeroes). For any unit vector v on M we have $\rho = \|\partial_v U\|$, so taking the norm of the boundary condition we find that along ∂M we have $\rho = \|\partial_\nu U\| = \sigma$. This completes the proof. □

We now describe four examples of free boundary minimal surfaces which will be important for us. It seems that there are many fewer known examples of two dimensional free boundary solutions than there are minimal surfaces in spheres.

Example 2.9. (Flat disk) If we take B^m and take its standard embedding into B^n for $n \geq m$, then this embedding is by first Steklov eigenfunctions. There is a theorem of J. C. C. Nitsche [Ni] which says that in the case $m = 2$ and $n = 3$, this is the only free boundary minimal disk. The proof of this uses the Hopf differential, and may be viewed as the analogue of the result of F. Almgren that the only branched minimal immersion of S^2 into S^3 is the equatorial one. In the spherical case there are many non-equatorial minimal two-spheres in $\mathbb{S}^n$ for $n \geq 4$. We expect that there are also non-flat free boundary minimal immersions of the 2-disk into B^n for $n \geq 4$, but we do not know how to construct them.

Example 2.10. (Minimal cones) Given any minimal surface Σ^{m-1} in the sphere $\mathbb{S}^{n-1}$, the cone $C = C(\Sigma)$ defines an m dimensional free boundary minimal submanifold of B^n with an isolated singularity at the origin. This construction gives a large class of examples for

$m \geq 3$, but does not produce two dimensional examples. It also shows that the free boundary problem contains in a natural way the theory of minimal submanifolds of spheres. Note that the Steklov spectrum of a unit cone is the same as the spectrum of the link.

Example 2.11. (Critical catenoid) If we parametrize the catenoid by $F : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3$ given by

$$F(t, \theta) = (\cosh(t)\cos(\theta), \cosh(t)\sin(\theta), t),$$

we can see that this is a conformal harmonic map. If we restrict it to $[-T, T] \times \mathbb{S}^1$, then the conormal vector at $t = T$ is parallel to $\partial_t F(T, \theta)$ and there is a unique choice of T for which $\partial_t F(T, \theta)$ is parallel to $F(T, \theta)$. This T is determined by the condition at $\theta = 0$

$$(\sinh(T), 0, 1) = c(\cosh(T), 0, T)$$

for a constant c. This holds when T is the positive solution of $T = \cosh(T)/\sinh(T) = \coth(T)$. If we then write

$$U(t, \theta) = (\cosh^2(T) + T^2)^{-1/2} F(t, \theta),$$

then the map $U : [-T, T] \times \mathbb{S}^1 \to B^3$ is a free boundary solution which we call the *critical catenoid*. This seems to be the only known free boundary minimal annulus in B^3. It is easy to check that the component functions of U are first Steklov eigenfunctions.

Example 2.12. (Critical Möbius band) We think of the Möbius band M as $\mathbb{R} \times \mathbb{S}^1$ with the identification $(t, \theta) \approx (-t, \theta + \pi)$. There is a minimal embedding of M into $\mathbb{R}^4$ given by

$$F(t, \theta) = (2\sinh t \cos\theta, 2\sinh t \sin\theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta).$$

It can be directly checked that F is a conformal harmonic map. For a unique choice of T the restriction of F to $[-T, T] \times S^1$ defines an embedding into a ball by first Steklov eigenfunctions. This choice of T is determined by the condition that $\partial_t F(T, \theta)$ is parallel to $F(T, \theta)$. Putting $\theta = 0$, we see that the condition is

$$(2\cosh(T), 0, 2\sinh(2T), 0) = c(2\sinh(T), 0, \cosh(2T), 0)$$

for a constant c. This implies that $\cosh(T)/\sinh(T) = 2\sinh(2T)/\cosh(2T)$, so that T is the unique positive solution of $\coth t = 2\tanh 2t$. We may rescale the radius of the ball to 1 to get the *critical Möbius band*

$$U(t, \theta) = (4\sinh^2(T) + \cosh^2(2T))^{-1/2} F(t, \theta)$$

on the quotient of $[-T, T] \times \mathbb{S}^1$ by the identification. It can be seen that the component functions of U are first Steklov eigenfunctions. See Section 7 of [FS2] for a detailed discussion of the critical Möbius band.

3. The eigenvalue extremal problem for compact surfaces

The extremal problem for compact surfaces begins with the result of J. Hersch [H] in 1970 which considers arbitrary metrics on $\mathbb{S}^2$. Hersch was able to obtain a sharp upper bound on the first nonzero eigenvalue λ_1 in terms of only the area of the metric. The precise theorem is the following.

Theorem 3.1. *(J. Hersch* [H]*) Let g be any smooth metric on $\mathbb{S}^2$. We then have the bound $\lambda_1(g)A(g) \leq 8\pi$ with equality only for metrics of constant curvature.*

Remark 3.2. Hersch actually proved the stronger bound

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\geq 3\frac{A(g)}{8\pi}$$

with equality only for constant curvature metrics.

The first geometric ingredient of the Hersch proof is the uniformization theorem which implies that any metric on $\mathbb{S}^2$ is conformally diffeomorphic to the standard metric g_0 of curvature 1. The proof then uses the properties of the group of conformal transformations of $\mathbb{S}^2$ to balance this conformal diffeomorphism in order to obtain test functions in the Raleigh quotient which are orthogonal to the constant functions on $(\mathbb{S}^2, g)$.

In 1980 P. Yang and S. T. Yau [YY] obtained a coarse upper bound on $\lambda_1 A$ for smooth metrics on arbitrary oriented compact surfaces with a constant depending only on the genus of the surface. Let M be a smooth oriented compact surface with genus γ. We then have the bound.

Theorem 3.3. *(P. Yang, S. T. Yau* [YY]*) Let g be a smooth metric on a surface of genus γ. The following bound holds*

$$\lambda_1(g)A(g)\leq 8\pi\big[\frac{\gamma+3}{2}\big]$$

where $[\cdot]$ denotes the integer part of a number.

Remark 3.4. In the paper [YY] the bound given is weaker because the authors used a weaker result form the theory of Riemann surfaces. The term on the right is $8\pi D(\gamma)$ where $D(\gamma)$ is the least integer such that every Riemann surface of genus γ has a meromorphic function of degree D. The standard estimate, which is the one used in [YY], is $D(\gamma)\leq\gamma+1$. It was later observed that more refined results in complex analysis yield the result $D(\gamma)=[(\gamma+3)/2]$.

Based on this theorem we can then formulate the variational problem on a given smooth orientable surface. We define a number $\lambda^*(\gamma)$ by

$$\lambda^*(\gamma)=\sup\{\lambda_1(g) : g \text{ with } A(g)=1\}$$

where the supremum is taken over all smooth metrics on M. The Hersch theorem then may be restated as $\lambda^*(1)=8\pi$ and the supremum is achieved by the constant curvature metrics. The Yang-Yau theorem implies

$$\lambda^*(\gamma)\leq 8\pi\big[\frac{\gamma+3}{2}\big].$$

It is natural to ask if the variational problem can be solved for a smooth metric and if so what is the geometry of such a maximizing metric. More precisely the problem is to find a smooth metric g on a surface of genus γ with $A(g)=1$ which has the largest first eigenvalue among all smooth metrics of area 1. Having found such a metric, what can be said about its geometric properties?

For a non-orientable surface M we let γ denote the genus of its orientable double covering. There is also an upper bound on $\sigma_1 A$ for any metric on M depending only on γ although the bound is not so explicit. Such bounds on all eigenvalues follow from the work of N. Korevaar [Kor]. We can therefore pose our variational problem for non-orientable M as well. We denote by $\lambda^{\#}(\gamma)$ the supremum of λ_1 taken over all unit area metrics on M.

In 1982, P. Li and S. T. Yau [LY] developed the theory of conformal volume which considers conformal maps to higher dimensional spheres. Given a compact conformal surface M and a conformal map $\varphi : M \to \mathbb{S}^n$, the conformal volume with respect to φ is defined by

$$V_c(n, \varphi) = \sup\{A(F \circ \varphi)\}$$

where the supremum is taken over all conformal diffeomorphisms F of $\mathbb{S}^n$. The conformal volume of M in $\mathbb{S}^n$ is then defined by

$$V_c(n, M) = \inf_{\varphi} V_c(n, \varphi)$$

where the infimum is taken over all conformal maps from M into $\mathbb{S}^n$. There are two main theorems of [LY] which are important for our variational problem. The first is a generalization of [H] and [YY].

Theorem 3.5. *(Li-Yau* [LY]*) For any metric g in the given conformal class on M we have the bound*

$$\lambda_1(g)A(g) \leq 2V_c(n, M).$$

If equality holds then g is a constant multiple of the induced metric on a conformal minimal immersion $\varphi : M \to \mathbb{S}^n$.

The second main theorem is the result that minimal surfaces in $\mathbb{S}^n$ have the maximum area among their images under conformal diffeomorphisms of $\mathbb{S}^n$.

Theorem 3.6. *(Li-Yau* [LY]*) If $\varphi : M \to \mathbb{S}^n$ is a branched conformal minimal immersion, then we have*

$$A(\varphi) = V_c(n, \varphi).$$

An important consequence of these theorems is the following result which allows us to compute the conformal volume and also to give a sharp eigenvalue bound for metrics in a conformal class.

Corollary 3.7. *(Li-Yau* [LY]*) If $\varphi : M \to \mathbb{S}^n$ is a branched conformal immersion by first eigenfunctions for a conformal metric g, then $\lambda_1(g)A(g) = 2V_c(n, M)$ and g maximizes $\lambda_1 A$ over all metrics in its conformal class.*

Proof. We may scale the metric g so that $\lambda_1(g) = 2$, and then since the coordinate functions are eigenfunctions of the induced metric with eigenvalue 2, it follows that g is the induced metric defined by φ so in particular we have $A(g) = A(\varphi)$. By Theorem 3.5 and the definition of the conformal volume we have

$$\lambda_1(g)A(g) = 2A(g) \leq 2V_c(n, M) \leq 2V_c(n, \varphi).$$

Now by Thoerem 3.6 we have $V_c(n, \varphi) = A(\varphi) = A(g)$, so combining these we find that $\lambda_1(g)A(g) = 2V_c(n, M)$ as claimed.

The fact that g maximizes $\lambda_1 A$ in its conformal class now follows from the above together with Theorem 3.5. □

Now $\mathbb{RP}^2$ has only one conformal class of metrics by the uniformization theorem, so using the Veronese embedding into $\mathbb{S}^4$ we have the following sharp bound for metrics on $\mathbb{RP}^2$.

Theorem 3.8. *(Li-Yau* [LY]*) For any smooth metric g on $\mathbb{RP}^2$ we have the bound $\lambda_1(g)A(g) \leq 12\pi$. Equality holds for the constant curvature metrics on $\mathbb{RP}^2$.*

In 1985 S. Montiel and A. Ros [MR] obtained the following strong uniqueness property for minimal immersions by first eigenfunctions. As a consequence they were able to prove that the constant curvature metrics on $\mathbb{RP}^2$ uniquely maximize $\lambda_1 A$.

Theorem 3.9. *(Montiel-Ros* [MR]*) Given a compact conformal surface M, there is at most one conformal metric g which is induced on a conformal immersion into $\mathbb{S}^n$ by first eigenfunctions. In particular, for metrics g on $\mathbb{RP}^2$, we have $\lambda_1(g)A(g) \leq 12\pi$ with equality only for the constant curvature metrics.*

The second assertion of the theorem follows from Theorem 3.5 which shows that if such a metric g exists, then it arises (up to a constant factor) as the induced metric on a conformal immersion by first eigenfunctions. Since $\mathbb{RP}^2$ has only one conformal structure, the first assertion of the theorem implies that the minimal immersion is the Veronese surface and therefore g is a constant curvature metric.

In 1986 A. El Soufi and S. Ilias [EI1] generalized Theorem 3.6 to arbitrary dimension. In particular they showed by a direct argument (that avoids use of the Willmore functional and the Gauss-Bonnet Theorem) that conformal transformations of the sphere always reduce the volumes of minimal submanifolds and that non-isometric transformations do so strictly unless the minimal submanifold is an equatorial sphere.

All compact surfaces except $\mathbb{S}^2$ and $\mathbb{RP}^2$ have a noncompact family of distinct conformal structures, so it appears that the conformal volume ideas will be difficult to employ to solve our variational problem for other surfaces. In 1996, N. Nadirashvili [N2] took a direct approach to the construction of a maximizing metric on a torus. He was able to prove the following result. See also the more recent paper of N. Nadirashvili and Y. Sire [NS] where a refinement of this result is obtained.

Theorem 3.10. *(Nadirashvili* [N2]*) There exists a metric g on the torus which maximizes $\lambda_1 A$, and this metric is flat and is a multiple of the induced metric on the 60^0 equilateral torus T_e described in Section 2. It is the unique maximizing metric up to scaling.*

The main content of this theorem is the existence of a maximizing metric and the proof that it arises as the induced metric on a conformal immersion into a sphere by first eigenfunctions. Once we have such a metric and conformal immersion, it follows from Theorem 3.9 that the metric is flat. This is because any conformal torus has a transitive group of conformal transformations, but Theorem 3.9 implies that such a group acts isometrically on M. This is because if $f : M \to M$ is a conformal transformation, then $\varphi \circ f$ is a minimal immersion by first eigenfunctions from $(M, f^*(g))$ into the sphere and therefore by Theorem 3.9 we have $f^*(g) = g$. Therefore the metric g must be flat, and classification results (see [EI2]) imply that the only flat minimal tori in spheres which are immersed by first eigenfunctions are the Clifford torus T_c in $\mathbb{S}^3$ and the equilateral torus T_e in $\mathbb{S}^5$. Since the equilateral torus has larger area it must be the unique maximizer.

If we choose a metric g on T^2, then there is a unique flat metric g_0 with area 1 in the conformal class of g. Thus if we let μ be the volume measure for g we have $\mu = \rho^2\mu_0$ for a positive function ρ where $g = \rho^2 g_0$ and μ_0 is the volume measure of g_0. Thus the first eigenvalue of g can be written $\lambda_1(g_0, \mu)$ and the problem of constructing a maximizing metric is equivalent to maximizing over g_0 and μ. Controlling g_0 is equivalent to giving a lower bound on its injectivity radius. If the injectivity radius is small, then there is a short

simple closed geodesic. For T^2 the only possibility is that the surface with this geodesic removed is $\mathbb{S}^2$ with two disks removed. Thus one might expect that if g_0 has small injectivity radius then the eigenvalue is approximately bounded above by 8π (the maximum for unit area metrics on $\mathbb{S}^2$). That this is so is implied by the following theorem of A. Girouard [G] which is written in a slightly different language.

Theorem 3.11. *(Girouard* [G]*) Let $\lambda > 8\pi$ be chosen. There is a positive constant δ_0 depending only on λ such that if g_0 is a flat unit area metric and μ a smooth measure such that $\lambda_1(g_0, \mu) \geq \lambda$, then the injectivity radius of g_0 is greater than δ_0.*

In 2006, D. Jakobson, N. Nadirashvili, and I. Polterovich [JNP] constructed a metric on the Klein bottle for which there is conformal immersion by first eigenfunctions into $\mathbb{S}^4$. This metric was shown to be the unique such metric up to a scale on the Klein bottle by A. El Soufi, H. Giacomini, M. Jazar [EGJ]. Combining this with the existence of a smooth maximizing metric we have the following for the Klein bottle.

Theorem 3.12. *(*[JNP], [EGJ], [N2]*) If K is the Klein bottle, then there is a unique metric g of unit area which maximizes the first eigenvalue. The metric is $\mathbb{S}^1$ invariant, but not flat and a multiple of it arises as the induced metric on a conformal minimal immersion into $\mathbb{S}^4$.*

The existence question for the Klein bottle can be formulated as for the torus. We can consider g_0 to be a unit area flat metric and μ a smooth measure. For the Klein bottle the injectivity radius of g_0 can become small only if there is a short closed geodesic. There are two possibilities for such a closed geodesic C; either C divides K into two components or it doesn't. If C does not divide, then when C is removed the remaining surface is $\mathbb{S}^2$ with two disks removed as for the torus. However, if C does divide, then the removal of C results in two copies of the Möbius band ($\mathbb{RP}^2$ with a disk removed). Note that one can classify these possibilities by noting that the Euler characteristic of the surface minus the curve must be zero, which together with the number of boundary components, determines the possibilities. In order to rule out both of these degeneracies it is necessary to assume that $\lambda_1(g_0, \mu)$ is bounded strictly above 12π, the maximum for metrics on $\mathbb{RP}^2$ (see [G]).

For surfaces with negative Euler characteristic there is very little known about maximizing metrics. One can ask whether the coarse upper bound of Theorem 3.3 can ever be sharp. Notice that a conformal map φ from a surface to $\mathbb{S}^2$ may be considered as a degenerate minimal surface with the induced metric $g = \varphi^*(g_0)$. Now the components of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ are eigenfunctions of g with eigenvalue 2. If it were true that 2 is the *first* eigenvalue of g, and the degree D of φ is such that every conformal surface of genus γ has a conformal map to $\mathbb{S}^2$ of degree at most D, then we will have shown that g achieves the maximum and $\lambda^*(\gamma) = 8\pi D$. In general it is quite difficult to check that the components of φ are first eigenfunctions, but we mention here the paper of D. Jakobson, M. Levitin, N. Nadirashvili, N. Nigram, and I. Polterovich [JLNNP]. They conjecture that for a particular degree 2 conformal map φ from a special conformal genus 2 surface, called the Bolza surface, it is true that the components of φ are first eigenfunctions. Since every genus 2 conformal surface admits a degree 2 conformal map to $\mathbb{S}^2$ (hyperelliptic property) this would imply that the metric $g = \varphi^*(g_0)$ maximizes. There is as yet no rigorous proof of this conjecture, but numerical evidence is given in the paper.

Conjecture 3.13. *([JLNNP]) The value of $\lambda^*(2)$ is 16π and it is achieved by a metric of the form $g = \varphi^*(g_0)$ for a particular degree 2 conformal map from the Bolza surface to $\mathbb{S}^2$. In particular, there is an extremal metric with curvature 1 at all but a finite number of cone points.*

Another very interesting question is to understand the asymptotic behavior of $\lambda^*(\gamma)$ as γ tends to infinity, and also to understand the geometry of maximizing metrics for large genus. The following upper and lower bounds follow from Theorem 3.3 and known results concerning the lowest eigenvalue of hyperbolic metrics on arithmetic surfaces.

Theorem 3.14. *For any $\gamma > 1$ we have the bounds*

$$\frac{3\pi}{4}(\gamma - 1) \leq \lambda^*(\gamma) \leq 8\pi\big[\frac{\gamma + 3}{2}\big].$$

Proof. The upper bound is from Theorem 3.3. To get the lower bound we use a result of M-F. Vignéras [V] (see also [BBD] and [BM]) which asserts that there exists a compact hyperbolic Riemann surface of any genus greater than 1 with lowest eigenvalue at least $\frac{3}{16}$. For such a surface we then have by Gauss-Bonnet

$$\lambda_1 A \geq \frac{3}{16} 4\pi(\gamma - 1) = \frac{3\pi}{4}(\gamma - 1).$$

This gives the lower bound. □

Remark 3.15. A conjecture of A. Selberg would imply that for large γ the $\frac{3}{16}$ can be replaced by $\frac{1}{4}$. If this is true the lower bound would become $\pi(\gamma - 1)$ for large γ.

4. The eigenvalue extremal problem for surfaces with boundary

We now consider the case of metrics on a surface with boundary. We let M be a smooth surface of genus γ with k boundary components. Thus if M is orientable it is homeomorphic to a compact surface of genus γ with k disks removed. If M is not orientable we take the genus to be that of the orientable double covering, but we take k to be the number of boundary components of M; for example, the Möbius band has genus 0 and 1 boundary component. The history of our specific question begins in 1954 with R. Weinstock [W], although the eigenvalue problem studied in [W] goes back much further as can be guessed from the word 'classical' in the title of [W].

Theorem 4.1. *(Weinstock [W]) Let M be a simply connected plane domain. We then have the inequality $\sigma_1(M)L(\partial M) \leq 2\pi$ with equality achieved only by a round disk.*

In [W] it is also shown that if the arclength of ∂M is weighted by a positive function p, then the same inequality holds for the eigenvalue problem $\Delta u = 0$ on M with boundary condition $\nabla_\nu u = \sigma p u$ on ∂M. Furthermore, equality holds if and only if $p = 1$ and M is a round disk. This result is essentially the extension of the bound to general simply connected Riemannian surfaces.

To pose the problem for Riemannian surfaces it should be observed that the Steklov eigenvalues depend only on the boundary arclength measure and the conformal structure. We say that two surfaces (M_1, g_1) and (M_2, g_2) are *σ-isometric* if there is a conformal diffeomorphism $F : M_1 \to M_2$ which is an isometry on the boundary; that is, if we write $F^*(g_2) = \rho^2 g_1$, then

we have $\rho = 1$ on ∂M_1. It is clear that surfaces which are σ-isometric have the same Steklov eigenvalues and the eigenfunctions are related by composition with F.

This notion can be generalized slightly to allow a rescaling of the boundary metric. We say that (M_1, g_1) and (M_2, g_2) are *σ-homothetic* if there is a conformal diffeomorphism $F : M_1 \to M_2$ which is a scaling on the boundary; that is, $\rho = c$ for some positive constant c on ∂M_1. It is clear that if M_1 and M_2 are σ-homothetic, then for all i we have

$$\sigma_i(M_1)L(\partial M_1) = \sigma_i(M_2)L(\partial M_2),$$

and the eigenfunctions are related by composition with F. Thus the precise restatement of Weinstock's theorem for Riemannian surfaces is the following.

Theorem 4.2. *(Weinstock* [W]*) If M is a simply connected Riemannian surface, then we have*

$$\sigma_1(M)L(\partial M) \leq 2\pi$$

with equality if and only if M is σ-homothetic to the unit disk.

There have been efforts to generalize this sharp upper bound to other surfaces over the past several years. First there is a coarse upper bound for orientable surfaces which was obtained in [FS1] and by G. Kokarev [Kok]. The combination of these results implies the following.

Theorem 4.3. *(*[FS1], [Kok]*) For any orientable surface of genus γ with k boundary components we have*

$$\sigma_1 L \leq \min\{2\pi(\gamma + k),\ 8\pi\big[\frac{\gamma + 3}{2}\big]\}.$$

Using the coarse upper bound, we can again formulate a variational problem for metrics on an orientable surface with boundary. We let $\sigma^*(\gamma, k)$ denote the supremum of σ_1 taken over all smooth metrics on M with boundary length 1. We can ask if the supremum is achieved and if so what is the geometric structure of a maximizing metric. Theorem 4.2 can be restated $\sigma^*(0, 1) = 2\pi$ and equality is achieved precisely by the surfaces which are σ-homothetic to the unit disk.

Upper bounds on higher eigenvalues in the case of orientable surfaces were obtained recently by A. Girouard and I. Polterovich [GP]. While there should be coarse bounds on all eigenvalues for non-orientable surfaces, we observe that a bound on the first eigenvalue for such a surface can be obtained in terms of the value $\lambda^{\#}(0)$ for $\mathbb{RP}^2$ (which is equal to 12π).

Theorem 4.4. *Let M be a non-orientable surface of genus γ with k boundary components. Let g be any smooth metric on M. If $\gamma = 0$ we have $\sigma_1 L \leq \lambda^{\#}(0) = 12\pi$. If $\gamma \geq 1$ we have*

$$\sigma_1 L \leq 2(\gamma + 1)\lambda^{\#}(0) = 24\pi(\gamma + 1).$$

Proof. First consider the case $\gamma = 0$. In this case M is $\mathbb{RP}^2$ with k disks removed. We extend the metric g to a metric on $\mathbb{RP}^2$. By the uniformization theorem there is a conformal diffeomorphism $\psi : (\mathbb{RP}^2, g) \to (\mathbb{RP}^2, g_0)$ where g_0 is the standard metric. Let $\varphi : (\mathbb{RP}^2, g_0) \to \mathbb{S}^4$ be the Veronese embedding, and thus $\varphi \circ \psi$ is a conformal minimal embedding of $(\mathbb{RP}^2, g)$ into $\mathbb{S}^4$. Now find a conformal map F of $\mathbb{S}^4$ so that the map $\hat{\varphi} = F \circ \varphi \circ \psi$ is balanced in the sense that

$$\int_{\partial M} \hat{\varphi}\ ds = 0.$$

Using the variational characterization of σ_1 we then have for $i = 1, \ldots, 5$

$$\sigma_1 \int_{\partial M} \hat{\varphi}_i^2 \, ds \leq \int_M |\nabla \hat{\varphi}_i|^2 \, d\mu.$$

Summing on i gives $\sigma_1 L \leq E(\hat{\varphi}) = 2A(\hat{\varphi})$, and by Theorem 3.6 we have $\sigma_1 L \leq 2A(\varphi) = \lambda^{\#}(0)$.

For $\gamma \geq 1$, the manifold M is a compact non-orientable surface M_1 with k disks removed. We extend the metric g to a smooth metric on M_1. We let $\hat{M}_1$ denote the orientable double covering of M_1 so that M_1 is the quotient of $\hat{M}_1$ by an orientation reversing isometric involution f acting freely on $\hat{M}_1$. By a result of Li and Yau ([LY], page 272) there exists an orientation preserving conformal map $\psi : \hat{M}_1 \to \mathbb{S}^2$ which is equivariant with respect to the action of f and the antipodal map on $\mathbb{S}^2$. This map can be taken to have degree at most $2(\gamma + 1)$. Therefore we have defined a conformal map $\psi : M_1 \to \mathbb{RP}^2$. We may now repeat the argument in the first paragraph using the Veronese embedding and a conformal map of $\mathbb{S}^4$ to prove the bound $\sigma_1 L \leq 2A(\hat{\varphi})$ where as above $\hat{\varphi} = F \circ \varphi \circ \psi$. This implies by Theorem 3.6

$$\sigma_1 L \leq 2A(\varphi) \deg(\psi) = 12\pi \deg(\psi) \leq 24\pi(\gamma + 1).$$

□

For the non-orientable surface M of genus γ with k boundary components we may now define $\sigma^{\#}(\gamma, k)$ to be the supremum of $\sigma_1(g)$ taken over all metrics on M with $L(\partial M) = 1$.

In order to construct maximizing metrics we need some preliminary results. Given an eigenfunction u, the set $\Gamma = \{u = 0\}$ is called the nodal set. The open set $M \setminus \Gamma$ is a disjoint union of connected open sets on which u does not change sign. These connected components are called nodal domains. The Courant Nodal Domain Theorem (which works as well for Steklov eigenfunctions) implies that if u is an i-th eigenfunction then the number of nodal domains is at most $i + 1$. It follows from the maximum principle that each nodal domain must have boundary points of M in its closure. At an interior point P of Γ for which $\nabla u(P) \neq 0$ it follows from the implicit function that near P, Γ is a smooth curve, while at a point P where $\nabla u(P) = 0$ there is a holomorphic choice of coordinates in which $u(z) = \Re(z^l)$ for some integer $l \geq 2$. In particular the nodal set near P consists of l smooth arcs passing through P at equal angles π/l. At a point $P \in \Gamma \cap \partial M$ we can choose coordinates (x, y) centered at P for which ∂M near P consists of the x-axis. The following is the description of Γ near such a point.

Theorem 4.5. *(*[FS2]*, Theorem 2.2) There is a C^1 coordinate system centered and conformal at P such that in a neighborhood of P we have that ∂M is the x-axis and Γ is the zero set of $\Re(z^l)$ for some $l \geq 1$. In particular the intersection of Γ with each boundary component is finite (possibly empty) and the intersection of Γ with a neighborhood of a component of ∂M consists of an even number of arcs.*

If $l = 1$ this says that Γ near P is an arc which hits ∂M orthogonally at P. For $l \geq 2$, near P, Γ consists of l arcs which make angles π/l with each other and angle $\pi/(2l)$ with ∂M.

An application of the structure of the nodal set which will be important for us is a bound on the multiplicity of the first eigenvalue in terms of the topology of the surface (independent of the metric). Such results were first obtained for compact surfaces by S. Y. Cheng [C].

These bounds were improved by G. Besson [B] and by N. Nadirashvili [N1]. Using similar ideas of these authors together with the description of the nodal set given in Theorem 4.5, multiplicity bounds were obtained in [FS2] for Steklov eigenvalues. In the case of oriented surfaces the following result was obtained.

Theorem 4.6. *(*[FS2]*, Theorem 2.2) For any metric on an oriented surface of genus γ with k boundary components, the multiplicity of the i-th Steklov eigenvalue is at most $4\gamma + 2i + 1$.*

The recent papers by P. Jammes [J] and by M. Karpukhin, G. Kokarev, and I. Polterovich [KKP] give similar bounds. A bound for non-orientable surfaces was also obtained in [FS2]; however, this was improved in [KKP] to the following result. We state only a special case of their bounds.

Theorem 4.7. *(*[KKP]*) Let M be a non-orientable surface of genus γ with k boundary components. For any metric on M, the multiplicity m_i of the i-th Steklov eigenvalue satisfies the following bound*

$$m_i \leq \min\{2(1+\gamma) + 2i + 1, 1 + 2\gamma + 2k + i\}.$$

In particular, for the Möbius band ($\gamma = 0$, $k = 1$) the multiplicity of σ_1 is at most 4 and the bound is sharp since σ_1 for the critical Möbius band has multiplicity 4.

It turns out that variational aspects of free boundary solutions play an important role in the theory. Recall that the free boundary problem arises from the variational problem involving critical points for the volume functional among variations which allow the boundary to move along a specified free boundary submanifold. In the case of Σ^k in B^n, a free boundary solution is critical for variations which are tangential to $\mathbb{S}^{n-1}$ along $\partial\Sigma$. Because of the free boundary condition, any vector field which is normal to Σ has the property that it is tangential to $\mathbb{S}^{n-1}$ along $\partial\Sigma$. It seems intuitively reasonable that no free boundary solution could be a stable critical point for the volume. The second variation of the volume for a variation W which is normal to Σ is given by the quadratic form

$$S(W, W) = \int_\Sigma (|DW|^2 - |A^W|^2)\, d\mu - \int_{\partial\Sigma} |W|^2\, ds$$

where μ and ds are the induced volume and surface measures, D is the connection in the normal bundle of Σ, and A^W is the second fundamental form of Σ in the W direction; that is, $A^W(X, Y) = \nabla_X Y \cdot W$ for tangent vector fields X, Y. The Morse index will denote the maximal dimensional space of such W on which the index form is negative definite. Note that if we have a vector field V in $\mathbb{R}^n$ we can set $W = V^\perp$ to be its normal component and it defines a valid variational vector field for the free boundary problem. The following result shows that for a typical free boundary Σ there is a natural n-dimensional space on which the index form is negative definite.

Theorem 4.8. *(*[FS2]*, Theorem 3.1) If v is a constant vector in $\mathbb{R}^n$, then we have*

$$S(v^\perp, v^\perp) = -k \int_\Sigma |v^\perp|^2\, d\mu.$$

If Σ is not contained in a product $\Sigma_0 \times \mathbb{R}$ where Σ_0 is a free boundary solution, then the Morse index of Σ is at least n. In particular, if $k = 2$ and Σ is not a plane disk, its Morse index is at least n.

In order to construct metrics which maximize σ_1 given the normalization $L(\partial M) = 1$, we observe that we may conveniently separate the conformal part of a metric from the non-conformal part. We use a class of constant curvature metrics to mark the conformal class. Precisely we observe that any smooth metric on M is conformally equivalent to a metric g_0 which has constant curvature and for which each component of ∂M is a geodesic. We assume that M is not a disk since we have already handled that case, and we observe that we can double M to produce a closed surface $\hat{M}$ with twice the Euler characteristic. We also double the metric to produce a Lipschitz metric which is invariant by reflection across ∂M. Since the Euler characteristic of M is less than or equal to 0 we may apply the uniformization theorem to produce a metric g_0 on $\hat{M}$ which is either flat or hyperbolic and for which reflection across ∂M is an isometry. Thus for the metric g_0 the boundary curves are geodesics. The metric g_0 is flat if the Euler characteristic of M is 0, and this hold only for the annulus in the oriented case and the Möbius band in the non-orientable case. In these cases we make g_0 unique by normalizing $A(M, g_0) = 1$.

Thus given a metric g we determine g_0 as above, and then we write $g = \rho^2 g_0$ and we denote the boundary arclength measure of g by μ so that $\mu = \rho ds_0$. Our normalization of the boundary length then says that $\mu(\partial M) = 1$. Conversely, given g_0 and a smooth probability measure μ on ∂M we may write $\mu = \rho ds_0$ for a positive smooth function ρ. The data (g_0, μ) then determine an equivalence class of σ-isometric metrics gotten by extending ρ to the interior as a positive function and defining $g = \rho^2 g_0$. These all have the same Steklov spectrum, so we may denote σ_1 of any of these metrics by $\sigma_1(g_0, \mu)$.

The problem of constructing maximizing (g_0, μ) then has two parts. The first is to control the conformal class determined by the geometry of g_0, and the second is to control the measure for a maximizing sequence. It is well known that the metric g_0 is controlled up to diffeomorphism provided the injectivity radius is bounded from below. By this we mean that the set of metrics g_0 normalized as above with injectivity radius bounded from below by a positive constant is C^2 compact modulo the diffeomorphism group. Thus given a sequence $g_0^{(i)}$ with injectivity radius bounded below by $\delta_0 > 0$, there exist diffeomorphisms F_i of M such that the sequence $F_i^*(g_0^{(i)})$ has a subsequence which converges in C^2 norm to a metric g_0 with the same conditions.

There are two main results which give us control on g_0 when there is a measure μ with $\sigma_1(g_0, \mu)$ large enough. The first is the statement that adding a boundary component strictly increases σ^*; that is, $\sigma^*(\gamma, k+1) > \sigma^*(\gamma, k)$. The next result tells us that this is true provided $\sigma^*(\gamma, k)$ is achieved by a smooth metric.

Theorem 4.9. *(*[FS2]*, Proposition 4.2) Assume that $\sigma^*(\gamma, k)$ is achieved by a metric g. It then follows that $\sigma^*(\gamma, k+1) > \sigma^*(\gamma, k)$.*

Note that to prove this theorem it suffices to exhibit any metric on a surface of genus γ with $k+1$ boundary components satisfying $\sigma_1 L > \sigma^*(\gamma, k)$. For example, it follows from the construction of the critical catenoid that $\sigma^*(0, 2) > \sigma^*(0, 1) = 2\pi$. In general it is difficult to exhibit such a metric except as a perturbation of g, the maximal metric on the surface of genus γ with k boundary components. The idea of the proof is to start with a surface M of genus γ with k boundary components and to choose a point P interior to M, and to construct a surface M_ϵ of genus γ with $k+1$ boundary components by removing a disk of radius ϵ centered at P. Having done this there is freedom in choosing the measure μ_ϵ on the

new boundary component ∂D_ϵ which has been added. The arclength measure given by g does not appear to work in general. Instead we place a stronger weighting on this boundary component by taking μ_ϵ to be $(\delta\epsilon\log(1/\epsilon))^{-1}ds$ where δ is a number which goes to 0 with ϵ but at a much slower rate ($\delta = (\log(1/\epsilon))^{-1/2}$ turns out to work). The motivation for this choice of weighting is that it is the induced arclength measure for the conformal metric $(\delta r\log(1/r))^{-2}(dx^2+dy^2)$ where $r^2 = x^2+y^2$. This is a complete constant curvature metric on the punctured disk which is a cusp of curvature $-\delta^2$. By a careful analysis of a first eigenfunction u_ϵ for (g_0,μ_ϵ) it is shown in [FS2] that the average value of u_ϵ^2 on the new boundary component is small and this causes the boundary length to be increased by more than the eigenvalue is decreased, so the product strictly increases.

The second main result shows that the conformal structure can be controlled provided σ_1 is sufficiently large compared to the maximum for simpler surfaces.

Theorem 4.10. *(*[FS2]*, Theorem 4.4) Assume that g is a metric on a surface M of genus 0 with $k \geq 2$ boundary components and that $L(\partial M) = 1$ and $\sigma_1(g) \geq \sigma$ for a number $\sigma > \sigma^*(0,k-1)$. Let g_0 be the associated constant curvature metric. There is a number $\delta > 0$ depending only on σ such that the injectivity radius of g_0 is at least δ.*

The idea of the proof is to use the fact that if the injectivity radius goes to 0, then there is a short embedded closed geodesic. We then estimate the eigenvalue for the various conformal degenerations and measure degenerations which can occur. In each case it is shown that the degeneration can only occur if σ_1 becomes smaller than or equal to $\sigma^*(0,k-1)$ in the limit.

Under the assumption that we have control on the background metric g_0, we now must face the problem of constructing a measure μ which maximizes $\sigma_1(g_0,\mu)$ over all smooth (g_0,μ). The difficulty is that a limit of smooth measures may be very singular, but we can solve the problem if we allow arbitrary measures with an appropriate definition of σ_1. Note that for any probability measure on ∂M we can define a substitute for σ_1

$$\sigma_1(g_0,\mu) = \inf\{E(u):\ u \in H^1(M)\cap C^0(\bar{M}),\ \int_{\partial M} u\ d\mu = 0,\ \int_{\partial M} u^2\ d\mu = 1\}.$$

This definition clearly agrees with σ_1 for smooth measures μ, and because the domain of the test functions is fixed, it is easily checked to be upper-semicontinuous in μ with respect to weak* convergence. It is not true for general μ that one can construct an eigenspace, and the regularity of a maximizing measure μ is challenging. The paper of G. Kokarev [Kok] addresses such questions for compact surfaces without boundary. In fact, if we define

$$\sigma^{**}(\gamma,k) = \sup_{(g_0,\mu)} \sigma_1(g_0,\mu)$$

where the measures are allowed to be arbitrary, then we clearly have $\sigma^{**}(\gamma,k) \geq \sigma^*(\gamma,k)$. It is not clear that these values are the same.

The approach that is taken in [FS2] to the construction of maximizing measures instead involved solving a regularized problem and producing a maximizer as a weak* limit of maximizers of these regularized problems. We first have the following result which is the analogue for surfaces with boundary of a result of N. Nadirashvili [N2].

Theorem 4.11. *(*[FS2]*, Proposition 5.2) Assume that g is a smooth metric on a surface with boundary which maximizes $\sigma_1 L$ over all smooth metrics. It follows that the multiplicity of σ_1 is at least 2 and there are n independent eigenfunctions $u_1,\dots,u_n$ such that the map*

$U = (u_1, \dots, u_n)$ defines a proper conformal immersion to B^n. In particular, $\Sigma = U(M)$ is a free boundary minimal surface and the induced metric $U^*(\delta)$ is σ-homothetic to g.

We now describe the regularized problem. Given a constant curvature background metric g_0 and a measure μ on ∂M we consider the smoothed measure $\mu_\epsilon = \rho_\epsilon \, ds_0$ where

$$\rho_\epsilon(x) = \int_{\partial M} K_\epsilon(x, y) \, d\mu(y),$$

ϵ is a small positive number, ds_0 is the arclength measure defined by g_0, and $K_t(x, y)$ is the heat kernel on ∂M defined by g_0. Note that $K_t(x, y) = 0$ if x and y are in different connected components of ∂M. We can then find a maximizer for the problem

$$\sigma^*(\gamma, k, \epsilon) = \sup_{(g_0, \mu)} \sigma_1(g_0, \mu_\epsilon)$$

provided we are able to control the metrics g_0. This is straigthforward because the measure μ_ϵ is uniformly controlled in any C^p norm for a fixed positive ϵ. It is also easy to see that

$$\lim_{\epsilon \to 0} \sigma^*(\gamma, k, \epsilon) = \sigma^*(\gamma, k).$$

Thus if we let $(g_0^{(\epsilon)}, \nu^{(\epsilon)})$ denote a maximizer for $\sigma^*(\gamma, k, \epsilon)$, and we choose a sequence ϵ_i going to 0 such that the $\nu^{(i)}$ converge to ν in weak* topology and the $g_0^{(i)}$ converge to g_0 in C^2 norm, we have constructed a special maximizing sequence for the problem $\sigma^*(\gamma, k)$. The main theorem is then the following.

Theorem 4.12. *(*[FS2]*, Theorem 5.8) The measure ν is a smooth positive measure and $\sigma_1(g_0, \nu) = \sigma^*(\gamma, k)$. Therefore we have the conclusions of Theorem 4.11 for a metric g associated to (g_0, ν).*

The following result is proven in a similar way as Theorem 4.11 and it shows that the $(g_0^{(\epsilon)}, \nu^{(\epsilon)})$ have some useful structure.

Theorem 4.13. *(*[FS2]*, Proposition 5.6) There are independent first eigenfunctions for $(g_0^{(\epsilon)}, (\nu^{(\epsilon)})_\epsilon)$, denoted $u_1, \dots, u_n$, such that the map $U = (u_1, \dots, u_n)$ is a harmonic map into $\mathbb{R}^n$ with the following properties $(|u|^2)_\epsilon = 1$ and $\nabla_T (|u|^2)_\epsilon = 0$ for $\nu^{(\epsilon)}$-almost all points of ∂M where T is the unit tangent vector with respect to $g_0^{(\epsilon)}$.*

Furthermore, for any h of compact support, the function

$$F(t) = \int_M |\nabla_t u|^2 \, da_t - \sigma_1(g_0, \epsilon) \int_{\partial M} |u|^2 \, d(\nu)_{t,\epsilon}$$

has a critical point at $t = 0$ where we have $g_t = g_0^{(\epsilon)} + th$, and the quantities are computed with respect to g_t. Note that $(\nu)_{t,\epsilon}$ denotes the time ϵ solution of the heat equation with initial data $\nu^{(\epsilon)}$ for the canonical metric in the conformal class of g_t.

Note that the equation satisfied by an eigenfunction u_j of $(g_0^{(i)}, (\nu^{(i)})_{\epsilon_i})$ is

$$\Delta u_j = 0 \text{ in } M, \ \nabla_\eta u_j \, ds_0^{(i)} = \sigma_1 u_j (\nu^{(i)})_{\epsilon_i} \text{ on } \partial M \tag{4.1}$$

where Δ and η are with respect to $g_0^{(i)}$. We normalize $U = (u_1, \dots, u_n)$ so that $\int_{\partial M} |U|^2 \, d(\nu^{(i)})_{\epsilon_i} = 1$ so that we have $E(U) = \sigma^*(\gamma, k, \epsilon)$. It can also be shown ([FS2], Lemma 5.10) that $U = U^{(i)}$ is uniformly bounded independent of i. Thus the sequence $U^{(i)}$ is uniformly bounded in

$H^1(M, \mathbb{R}^n)$ and so by extracting a subsequence we may assume the $U^{(i)}$ converge H^1-weakly to $U \in H^1(M, \mathbb{R}^n)$. By weak* compactness and the boundedness of the $U^{(i)}$ we may also assume that the vector-valued signed measures $U^{(i)}(\nu^{(i)})_{\epsilon_i}$ converge to a signed measure which may be written $\hat{U}\nu$ for an $L^\infty(\partial M, \nu, \mathbb{R}^n)$ function $\hat{U}$. We may write (4.1) in weak form as

$$\int_M \langle \nabla\varphi, \nabla U^{(i)} \rangle \, da_0^{(i)} = \sigma_1^{(i)} \int_{\partial M} \langle \varphi, U^{(i)} \rangle \, d(\nu^{(i)})_{\epsilon_i}$$

for all smooth $\mathbb{R}^n$ valued test functions φ. We can then conclude that the limiting function U satisfies

$$\int_M \langle \nabla\varphi, \nabla U \rangle \, da_0 = \sigma^*(\gamma, k) \int_{\partial M} \langle \varphi, \hat{U} \rangle \, d\nu \tag{4.2}$$

for all φ. Clearly the limit U is smooth and harmonic in the interior of M, and it follows (see [FS2], Lemma 5.11) from Theorem 4.13 that the limit U is also conformal.

To get the nontriviality of U we first show that $|\hat{U}|$ is bounded from below by a positive constant. This is done in Proposition 5.13 of [FS2] and it uses the fact that the number of sign changes of any eigenfunction on ∂M can be bounded in terms of the topology of M. From (4.2) we then see that U is not a constant map and therefore defines a nontrivial minimal surface in $\mathbb{R}^n$. From this it follows that the support of the limiting measure ν is all of ∂M.

Properties of the nodal set then can be used (see [FS2], Proposition 5.13) to show that on open intervals of ∂M on which $|U^{(i)}|$ is bounded from below by a positive number, the angle $U^{(i)}/|U^{(i)}|$ is uniformly equicontinuous. From this and the conformality it can be shown that on such intervals, the limit U is smooth (see [FS2], Lemma 5.12) and $|U|$ is constant. The regularity of the measure ν follows directly from the smoothness of U at ∂M. This completes a rough outline of the proof of Theorem 4.12.

A consequence of Theorem 4.10 and Theorem 4.12 is that for any oriented surface of genus 0 with k boundary components there exists a smooth maximizing metric. It is a difficult problem to determine these metrics and associated free boundary surfaces explicitly, but we do know from the multiplicty bounds that they all must have multiplicity at most 3. It is a consequence of the fact that a proper conformal map to $B^2 = D$ cannot be achieved by first eigenfunctions (see [FS1]) that for all k the multiplicity is 3, and we have constructed a free boundary surface in B^3 which has maximal eigenvalue.

In the case $k = 2$ we show that the solution is the critical catenoid. This follows from the general uniqueness theorem for the critical catenoid.

Theorem 4.14. *(*[FS2]*, Theorem 6.6) If Σ is a free boundary minimal surface in B^n which is homeomorphic to the annulus and such that the coordinate functions are first eigenfunctions, then $n = 3$ and Σ is congruent to the critical catenoid.*

In combination with Theorem 4.10 and Theorem 4.12 we have then shown.

Theorem 4.15. *(*[FS2]*, Theorem 6.7) For any smooth metric g on the annulus we have $\sigma_1 L$ bounded above by $(\sigma_1 L)_{cc} \approx 4\pi/1.2$ with equality only if g is σ-homothetic to the critical catenoid.*

There are similar theorems for the critical Möbius band.

Theorem 4.16. *(*[FS2]*, Theorem 7.4) Assume that Σ is a free boundary minimal Möbius band in B^n such that the coordinate functions are first eigenfunctions. Then $n = 4$ and Σ is the critical Möbius band.*

Theorem 4.17. *(*[FS2]*, Theorem 7.5) For any metric on the Möbius band M we have*

$$\sigma_1 L \leq (\sigma_1 L)_{cmb} = 2\pi\sqrt{3}$$

with equality if and only if M is σ-homothetic to the critical Möbius band.

The idea of the proof of Theorem 4.14 is that any conformal annulus has an $\mathbb{S}^1$ group of conformal diffeomorphisms generated by the rotation vector field $\frac{\partial}{\partial\theta}$ where we may think of the conformal structure as $[-T, T] \times \mathbb{S}^1$. If $\varphi : [-T, T] \times \mathbb{S}^1 \to B^n$ is a conformal parametrization of Σ, then the vector field $X = \frac{\partial\varphi}{\partial\theta}$ is a conformal vector field along Σ. We want to show that X is a Killing vector field so that the metric on Σ is $\mathbb{S}^1$-invariant. The result can then be shown by ODE methods.

In order to show that X is a Killing vector field it suffices to show that X has components which are first Steklov eigenfunctions; that is,

$$\frac{\partial X}{\partial t} = X\rho$$

where $\rho = |\partial_t\varphi| = |\partial_\theta\varphi|$ is the conformal factor which gives the induced metric. Since $\partial_t\varphi = \varphi\rho$ on the boundary, it follows that

$$\frac{\partial X}{\partial t} \cdot \frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial t\partial\theta} \cdot \frac{\partial\varphi}{\partial t} = \rho\frac{\partial\rho}{\partial\theta} = 0.$$

Thus ρ is constant on ∂M, and the induced metric is σ-homothetic to a flat metric on the annulus. That Σ is the critical catenoid then follows from an ODE argument of [FS1].

The proof that X is a first eigenfunction uses variational ideas. The quadratic form

$$Q(V, V) = \int_M |\nabla V|^2 \, da - \int_{\partial M} |V|^2 \, ds$$

for vector fields which are tangent to $\mathbb{S}^{n-1}$ along ∂M is the second variation of $1/2E(\cdot)$ for such variations. It is easily checked that X is a null vector field for this form. This would imply that X is a first eigenfunction if we knew that $\int_{\partial M} X \, ds = 0$. In order to show this we consider adding a carefully chosen vector field V to X with the property that $\int_{\partial M}(X + V) \, ds = 0$ and $Q(X + V, X + V) \leq 0$. Since X is null for Q, this is the same as requiring that $Q(V, V) \leq 0$.

While we can't easily construct vector fields for which Q is nonpositive, Theorem 4.8 tells us that we have at least an n dimensional space of vector fields on which S is negative definite. In addition to this space we also have a null vector field (Jacobi field) which vanishes on $\partial\Sigma$ given by $W(x) = x^\perp$ for $x \in \Sigma$. In [FS2] the relationship between S and Q is carefully analyzed. If a vector field V is *conformal* in the sense that a variation of Σ in the direction of V preserves the conformal structure up to first order, then it is true that $Q(V, V) = S(V, V)$. In the $n+1$ dimensional space of vector fields for which S is nonpositive we show that there is at least an n dimensional subspace of vector fields which have this conformal property. The proof of this involves analysis of a boundary value problem for the Cauchy-Riemann equation on M. Eventually we succeed in finding a vector field V such that $\int_{\partial M}(X + V) \, ds = 0$ and

$Q(V,V) \leq 0$. Because of the specific nature of this V it is possible to show that in fact V must have been 0, and therefore $\int_{\partial M} X\ ds = 0$. This completes the proof.

The proof of uniqueness for the critical Möbius band is similar except that it uses the same ideas in higher codimension. For the critical catenoid it was sufficient to work in B^3 because of the multiplicity bounds.

For oriented surfaces of genus 0 with $k \geq 3$ boundary components we cannot explicitly determine the surfaces, but we can give sharp asymptotic conditions on these surfaces and on the $\sigma^*(0,k)$. We let $\Sigma_k \subseteq B^3$ denote a free boundary surface which achieves $\sigma^*(0,k)$. We know that Σ_1 is the flat disk and Σ_2 is the critical catenoid. We have the following asymptotic description of the Σ_k.

Theorem 4.18. *(*[FS2]*, Theorem 8.2) After a suitable rotation of each Σ_k, the sequence Σ_k converges in C^3 norm on compact subsets of B^3 to the disk $\{z = 0\}$ taken with multiplicity two. Furthermore we have $\lim_{k\to\infty} A(\Sigma_k) = 2\pi$ and $\lim_{k\to\infty} L(\partial\Sigma_k) = 4\pi$.*

This implies the following bounds and asymptotic description of the $\sigma^*(0,k)$.

Theorem 4.19. *(*[FS2]*, Theorem 8.3) The sequence $\sigma^*(0,k)$ is strictly increasing in k and converges to 4π as k tends to infinity. For each k a maximizing metric is achieved by a free boundary minimal surface Σ_k in B^3 of area less than 2π. The limit of these minimal surfaces as k tends to infinity is a double disk.*

The proof of these asymptotic theorems uses minimal surface theory combined with geometric interpretations of the conditions on nodal sets which follow from the fact that the coordinate functions are first Steklov eigenfunctions. In particular this condition implies that any plane through origin intersects Σ_k transversally in its interior. This implies that the support function $x\cdot\nu$, where ν is the unit normal vector of Σ_k, does not change sign in the interior of Σ_k. It then follows that each Σ_k does not contain the origin, is star-shaped with respect to the origin (each ray intersects Σ_k at most one time), and is stable with respect to area for variations of compact support in Σ_k. The stability gives us curvature estimates in the interior and allows us to prove strong compactness results. Another ingredient is a version of boundary monotonicity of area which is used to avoid area loss near the boundary.

5. Interpretation in terms of inhomogeneous mass distributions

The first eigenvalue problem we discussed in this paper for a compact surface may be written

$$\Delta_0 u + \lambda u \rho^2 = 0$$

where $g = \rho^2 g_0$ and g_0 is any chosen background metric in the conformal class. We note that this is also the eigenvalue problem for a membrane with mass distribution ρ^2 on the background geometry determined by g_0. It is natural to take g_0 to be a constant curvature metric. The extremal question is then to determine the mass distribution with a given total mass with the largest first fundamental frequency. Indeed this is the way that J. Hersch [H] stated his theorem. The maximizing question on the space of all metrics is then the problem of determining the background constant curvature geometry g_0 and the mass distribution with largest first fundamental frequency.

The Steklov eigenvalue problem, on the other hand, may be thought of in the following way. Again let's take a background metric g_0 on a given complete surface $\hat{M}$, and let's choose

a domain M in $\hat{M}$. The eigenvalue problem is then

$$\Delta_0 u = 0 \text{ in } M, \quad \nabla_\eta u = \sigma u \rho \text{ on } \partial M$$

where ρ is a given positive function on ∂M. This may also be thought of as the eigenvalue problem associated with vibrations of the inhomogeneous membrane with mass distribution given by the singular measure ρds_0 with support on ∂M. From this point of view one can pose the question of determining the density function ρ which maximizes the first fundamental frequency among all mass distributions with the same total mass which are supported on boundaries of domains in $\hat{M}$ with the topology of M. The global maximization problem would ask to determine the background metric g_0, the domain M, and the density function ρ with maximal first eigenvalue.

We take $\hat{M} = \mathbb{R}^2$ with the euclidean metric, Ω a domain which is homeomorphic to an annulus, and ρ a positive density function on $\partial\Omega$. There is a unique circular annulus $D_R \setminus D_1$ which is conformally equivalent to Ω, and given the inverse of such a holomorphic map $w = f(z)$ where $z \in D_R \setminus D_1$, we can transform the measure $\rho(w)|dw|$ on $\partial\Omega$ to $\hat{\rho}(z)|dz|$ on $\partial(D_R \setminus D_1)$ where $\hat{\rho}(z) = \rho(w(z))|dw/dz|$. The pairs (Ω, ρ) and $(D_R \setminus D_1, \hat{\rho})$ are then σ-isometric and so have the same Steklov spectrum. Thus there is no loss of generality to assume our annular domain is $D_R \setminus D_1$. We can now restate Theorem 4.15 as follows.

Theorem 5.1. *Suppose we are given the domain $\Omega = D_R \setminus D_1$ and a positive density function ρ on $\partial\Omega$ with the property that $\int_{\partial\Omega} \rho \, ds = 1$. The lowest Steklov eigenvalue then satisfies $\sigma_1(\Omega, \rho) \leq (\sigma_1 L)_{cc} \approx 4\pi/1.2$. Equality holds if and only if $T \equiv 1/2 \log(R)$ is the unique positive number with $T = \coth(T)$, and $\rho = (4\pi)^{-1}$ on ∂D_1 and $\rho = (4\pi R)^{-1}$ on ∂D_R.*

Proof. The inequality follows directly from Theorem 4.15. To analyze the case of equality we must show that the only circular annulus with boundary density function of total mass 1 which is σ-homothetic to the critical catenoid is the one described in the statement of the theorem. Assume we have such an annulus $(D_R \setminus D_1, \rho)$ which is σ-homothetic to the critical catenoid. The conformal structure is uniquely determined by R and therefore R must be as described. Now if φ is the σ-homothety, we have $\varphi : D_R \setminus D_1 \to D_R \setminus D_1$ is a holomorphic map, and it is therefore a rotation $\varphi(z) = e^{i\alpha} z$ for some α. If $\hat{\rho}$ is the density function coming from the critical catenoid, it is clear that $\hat{\rho}$ is constant on each boundary component and since the boundary components have the same length we have $\hat{\rho}(1) = R\hat{\rho}(R)$. In order that φ be a σ-homothety it must therefore be true that ρ is also constant on each boundary component and satisfies $\rho(R)/\rho(1) = \hat{\rho}(R)/\hat{\rho}(1) = 1/R$. Since ρ also satisfies the mass 1 condition, it is true that $\rho(1) = (4\pi)^{-1}$ and $\rho(R) = (4\pi R)^{-1}$. □

Similarly Theorem 4.18 has the following consequence for plane domains.

Theorem 5.2. *Let $k \geq 3$ and let Ω be a plane domain with k boundary components. Let ρ be a positive density function on $\partial\Omega$ satisfying $\int_{\partial\Omega} \rho \, ds = 1$. We then have the inequality*

$$\sigma_1(\Omega, \rho) \leq \sigma^*(0, k) < 4\pi.$$

Equality is attained by a unique equivalence class of σ-isometric pairs (Ω, ρ), and the sequence $\sigma^*(0, k)$ is strictly increasing in k with $\lim_{k\to\infty} \sigma^*(0, k) = 4\pi$.

Describing the geometry of the maximizing domains in terms of euclidean geometry seems difficult, and we do not know a good description.

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