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On the existence of higher dimensional Enneper's surface

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Enneper's surface and the catenoid are the simplest minimal surfaces in $\mathbf{R}^3$ that are complete, orientable and nonplanar. This is because a complete orientable minimal surface has the total curvature of $-4k\pi$ for some nonnegative integer k, while $k = 1$ for Enneper's surface and the catenoid. Enneper's surface has one end and is a minimal immersion of $\mathbf{R}^2$ in $\mathbf{R}^3$, whereas the catenoid has two ends and is a surface of revolution.

Not only is $\mathbf{R}^3$ but also in $\mathbf{R}^n$, $n \geq 4$, the catenoid has been known to exist. It is a minimal hypersurface which is rotationally symmetric. The higher dimensional catenoid has been the only example that is a higher dimensional analogue of a 2-dimensional minimal surface. In this paper, however, we prove that there also exists an n-dimensional Enneper's surface Σ^n in $\mathbf{R}^{n+1}$ for $n = 3, 4, 5, 6$, which is a minimal immersion of $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$.

For two-dimensional minimal surfaces in $\mathbf{R}^3$ there is the Weierstrass representation. This representation makes it easy to write down an enormous number of complete minimal surfaces in $\mathbf{R}^3$. Moreover, one can construct arbitrarily many minimal submanifolds of even codimension in $\mathbf{R}^{2n}$, as every complex submanifold of $\mathbf{R}^{2n}$ is minimal. But in higher dimension one does not even have a good way to construct examples of complete immersed minimal hypersurfaces. Among a few known examples are the higher dimensional catenoids, area minimizing cones and graphs constructed by Bombieri-De Giorgi-Giusti [BDG], minimal hypersurfaces in $\mathbf{R}^4$ and $\mathbf{R}^6$ passing through the Clifford tori in S^3 and S^5 [B], minimal hypersurfaces as leaves of a foliation arising from isoparametric hypersurfaces [FK], and F-invariant minimal hypersurfaces [W].

All the examples above have been found by solving ordinary differential equations which were induced from the partial differential equation of minimal hypersurfaces by exploiting certain symmetry conditions. Higher dimensional Enneper's surface Σ, by contrast, is constructed by solving the partial differential equation directly as follows. First construct a compact minimal hypersurface by finding Jenkins-Serrin's solution [JS] to the Dirichlet problem for the minimal

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surface equation with suitably prescribed boundary data. Second obtain a compact Enneper type surface by reflecting the minimal hypersurface across the totally geodesic part of its boundary. Third blow up the compact Enneper type surface by an appropriately chosen scale to obtain a complete minimal immersion of $\mathbf{R}^n$ in $\mathbf{R}^{n+1}$. In this process we have used the curvature estimates of [SSY] and [SS], and for this reason we have the dimension restriction that $n = 3, 4, 5, 6$.

Our higher dimensional Enneper's surface Σ satisfies some properties which are analogous to those of classical Enneper's surface. Namely, Σ^n contains n mutually orthogonal $(n-1)$-planes. Asymptotically, i.e., viewed from infinity, Σ^n looks like an n-plane with multiplicity $2^n - 1$. On the other hand, a high dimensional analogue of the total curvature for Σ^n $\int_\Sigma |A|^n$, A being the second fundamental form, becomes infinite. Moreover, the Gauss map is not well defined at the point at infinity of Σ. Several interesting features of higher dimensional Enneper's surface are remarked in Section 6.

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1. Definitions and notations

(1) Let $O = (0, \ldots, 0)$, $p_1 = (1, 0, \ldots, 0)$, $p_2 = (0, 1, 0, \ldots, 0), \ldots, p_n = (0, \ldots, 0, 1, 0) \in \mathbf{R}^{n+1}$. Define T to be the regular $(n-1)$-simplex with $p_1, \ldots, p_n$ as its vertices. Let $p_\varepsilon = (-1, \ldots, -1, \varepsilon) \in \mathbf{R}^{n+1}$, $0 < \varepsilon < 1$, and define $\Gamma_\varepsilon = (O \mathbin{⋘} \partial T) \cup (p_\varepsilon \mathbin{⋘} \partial T) \subset \mathbf{R}^{n+1}$. Here $p \mathbin{⋘} S$ denotes the cone from p to S, the union of all line segments from p to the points of S.

(2) Define C as the n-dimensional catenoid which is rotationally symmetric about the x_{n+1}-axis. C satisfies the equation $x_{n+1} = f(r)$, $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, where

$$f(r) = \int_1^r [t^{2(n-1)} - 1]^{-1/2}\, dt.$$

(3) For each $r > 0$ we define

$$\mu_r : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}, \qquad \mu_r(x) = rx,$$

and for each $q \in \mathbf{R}^{n+1}$ define

$$\tau_q : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}, \qquad \tau_q(x) = x - q.$$

(4) Let Λ_i, $1 \le i \le n+1$, be the hyperplane $\{(x_1, \ldots, x_{n+1}) : x_i = 0\}$ and let $\Lambda_{n+2} = \{(x_1, \ldots, x_{n+1}) : x_1 + \cdots + x_n = 0\}$, $\Lambda_0 = \{(x_1, \ldots, x_n, 0) : x_1 \cdots x_n = 0\}$,

$\Lambda_{-1} = \{(x_1, \ldots, x_n, 0): \Pi_{1 \le i < j \le n}(x_i^2 + x_j^2) = 0\}$, i.e., Λ_{-1} is the $(n-2)$-skeleton of Λ_0, $\Lambda^+ = \{(x_1, \ldots, x_{n+1}): x_1 \cdots x_{n+1} > 0\}$, $\Lambda^* = \{(x_1, \ldots, x_{n+1}): x_i \ge 0, i = 1, \ldots, n+1\}$, $\Lambda^\sharp_{n+1} = \Lambda_{n+1} \cap \Lambda^*$, $\Lambda^\flat_{n+1} =$ the closure of $(\Lambda_{n+1} \sim \Lambda^*)$, $\Lambda^\varepsilon_{n+1} = \{(x_1, \ldots, x_{n+1}): 0 \le x_{n+1} \le \varepsilon\}$, $\Lambda^+_{n+1} = \{(x_1, \ldots, x_{n+1}): x_{n+1} \ge 0\}$. Let $\Lambda_{i,\theta}$ be the hyperplane which passes through the origin, is disjoint from the interior of Λ^*, is perpendicular to Λ_{n+1}, and makes an angle of θ with Λ_i and an angle of φ, $(n-1)\cos^2\varphi + \cos^2\theta = 1$, with every $\Lambda_j, j \ne i, n+1$, and let $\Lambda_{n+1,\theta}$ be the hyperplane in $\mathbf{R}^{n+1}$ which contains the $(n-1)$-plane $\Lambda_{n+1} \cap \Lambda_{n+2}$ and makes an angle of $0 \le \theta < \pi$ with Λ_{n+1}. Let ℓ be the straight line $\{(x_1, \ldots, x_n, 0): x_1 = \cdots = x_n\}$.

(5) Define π_1, π_2 as the projections from $\mathbf{R}^{n+1}$ onto $\Lambda_{n+1}, \Lambda_{n+2}$, respectively. Define ρ_i, $1 \le i \le n$, as the rotation by 180° about the $(n-1)$-plane $\Lambda_{n+1} \cap \Lambda_i$ and ρ_{ij}, $1 \le i \ne j \le n$, as the rotation by 90° about the $(n-1)$-plane $\Lambda_i \cap \Lambda_j$ taking the positive x_i-axis to the positive x_j-axis. ξ_{n+1} is the reflection with respect to the hyperplane Λ_{n+1}.

(6) For $1 \le i \ne j \le n$, let $\phi_{ij} = \xi_{n+1} \circ \rho_{ij}$ and let ξ_{ij} be the reflection with respect to the hyperplane $x_i = x_j$. Define G to be the subgroup of $O(n+1)$ generated by $\{\phi_{ij}, \xi_{ij}\}_{1 \le i \ne j \le n}$.

(7) Let $B_r(q)$ be the ball of radius r with center at q and $\mathring{B}_r(q)$ its interior. Z_r is the cylinder defined by $Z_r = \{(x_1, \ldots, x_{n+1}): x_1^2 + \cdots + x_n^2 \le r^2\}$.

2. Compact Enneper type surface

The first step towards the proof of the existence of higher dimensional Enneper's surface is to construct a compact minimal hypersurface which resembles the fundamental region of 2-dimensional Enneper's surface (Lemma 1). Then a compact Enneper type surface is obtained from this fundamental piece by 180° rotations (Lemma 2).

LEMMA 1. *For each $\varepsilon > 0$ there exists a unique n-dimensional compact minimal hypersurface Σ_ε in $\mathbf{R}^{n+1}$ bounded by Γ_ε. Σ_ε is area minimizing and stable.*

Proof. The projection π_2 maps Γ_ε one-to-one onto $\pi_2(\Gamma_\varepsilon)$ which is the boundary of a convex domain in Λ_{n+2}. By [GT, Theorem 16.8] the Dirichlet problem for the minimal surface equation is uniquely solvable. It is well known that a minimal graph over a convex domain is area minimizing. Hence it is stable.

LEMMA 2. *2^n congruent copies of Σ_ε can be pieced together to form a compact smooth minimal hypersurface $\tilde{\Sigma}_\varepsilon$ which is invariant under the group G.*

Proof. Assume that for α, $\beta \in G$,

$$\alpha(\partial\Sigma_\varepsilon \cap \Lambda_{n+1}) = \beta(\partial\Sigma_\varepsilon \cap \Lambda_{n+1}). \tag{1}$$

Note that $\phi_{ij}(\Lambda^+) = \Lambda^+$ and $\xi_{ij}(\Lambda^+) = \Lambda^+$. Hence

$$\beta^{-1}\alpha(\Lambda^+) = \Lambda^+. \tag{2}$$

It follows from (1) and (2) that

$$\beta^{-1}\alpha(\Lambda^*) = \Lambda^* \quad \text{and} \quad \alpha(\partial\Sigma_\varepsilon) = \beta(\partial\Sigma_\varepsilon).$$

Hence $\beta^{-1}\alpha$ is the product of some ξ_{ij}'s. From the invariance of Γ_ε under ξ_{ij} and the uniqueness of Σ_ε spanning Γ_ε, one obtains $\xi_{ij}(\Sigma_\varepsilon) = \Sigma_\varepsilon$. Therefore

$$\alpha(\Sigma_\varepsilon) = \beta(\Sigma_\varepsilon) \tag{3}$$

Define

$$\tilde{\Sigma}_\varepsilon = \bigcup_{\alpha \in G} \alpha(\Sigma_\varepsilon).$$

Clearly $\tilde{\Sigma}_\varepsilon$ is invariant under G. That (1) implies (3) shows that $\tilde{\Sigma}_\varepsilon$ consists of 2^n (= the number of the components of Λ^+) copies of Σ_ε. Note now that

$$\rho_i(\Sigma_\varepsilon) = \phi_{ij}(\Sigma_\varepsilon) \qquad \text{for every } 1 \leq i \neq j \leq n.$$

Then a standard theory of the elliptic partial differential equations states that $\Sigma_\varepsilon \cup \phi_{ij}(\Sigma_\varepsilon)$ is an analytic extension of Σ_ε across $\partial\Sigma_\varepsilon \cap \Lambda_{n+1} \cap \Lambda_i$. Furthermore it follows that $\tilde{\Sigma}_\varepsilon$ is an analytic extension of Σ_ε across $\partial\Sigma_\varepsilon \cap \Lambda_{n+1}$.

3. Curvature estimates

Extending a compact Enneper type surface to a complete hypersurface requires detailed estimates on the curvature of the surface. A lower bound of the curvature is obtained by the maximum principle (Lemma 4) and an upper bound is derived from stability (Lemma 5).

LEMMA 3. *Let $\gamma(s) = (x(s), y(s))$, $0 \leq s \leq a$, be a C^2 curve in $\mathbf{R}^2$ parametrized by the arclength s satisfying $\gamma(0) = (0, 0)$, $\gamma'(0) = (1, 0)$, $\gamma'(a) = (0, 1)$ and*

$0 \le x(s) \le b$. Then there exists $0 \le s_0 \le a$ such that the curvature of γ at $\gamma(s_0)$ is not less than $1/b$.

Proof. Let ζ_c be the quarter circle defined by $\zeta_c(t) = (b \sin t, c - b \cos t)$, $0 \le t \le \pi/2$. If $\bar{c} = \sup\{c < b : \zeta_c \cap \gamma = \phi\}$, then γ lies on one side of $\zeta_{\bar{c}}$ touching $\zeta_{\bar{c}}$ at a point $\gamma(s_0)$, $0 \le s_0 \le a$. Hence the curvature of γ at $\gamma(s_0)$ is larger than or equal to that of $\zeta_{\bar{c}}$ which is $1/b$.

LEMMA 4. *For each $\varepsilon > 0$ there exist $q_\varepsilon \in \tilde{\Sigma}_\varepsilon$ and $a(\varepsilon) > 0$ such that*

$$\operatorname{dist}(q_\varepsilon, \Lambda_0) \le a(\varepsilon) + \varepsilon, \qquad |A|(q_\varepsilon) \ge \frac{1}{na(\varepsilon)}, \qquad \lim_{\varepsilon \to 0} a(\varepsilon) = 0, \tag{4}$$

where $|A|$ is the length of the second fundamental form of $\tilde{\Sigma}_\varepsilon$.

Proof. Let $a(\varepsilon) > 0$ be the smallest number such that for any $r > a(\varepsilon)$, the catenoid $\tau_{q(r)}\mu_r(C)$, $q(r) = (r, \ldots, r, 0) \in \ell$, is disjoint from $\Gamma_\varepsilon \sim \Lambda_{n+1}$. Then one can easily see that $a(\varepsilon)$ converges to 0 as ε goes to 0. For any $q(r)$ with $r > a(\varepsilon)$, $\tau_{q(r)}\mu_{a(\varepsilon)}(C)$ does not intersect Γ_ε. Also, for sufficiently large $b > 0$, $\tau_{q(b)}\mu_{a(\varepsilon)}(C)$ cannot intersect Σ_ε. It follows from the maximum principle that

$$\tau_{q(r)}\mu_{a(\varepsilon)}(C) \cap \Sigma_\varepsilon = \phi \qquad \text{for } a(\varepsilon) < r \le b.$$

Hence

$$\tau_{q(a(\varepsilon))}\mu_{a(\varepsilon)}(C) \cap (\Sigma_\varepsilon \sim \partial\Sigma_\varepsilon) = \phi.$$

Let

$$\bar{\ell} = \{q \in \Sigma_\varepsilon : \pi_1(q) \in \ell\}.$$

Since the plane curve $\bar{\ell}$ is invariant under the reflections ξ_{ij}, $\bar{\ell}$ is a principal curve in Σ_ε, that is, every tangent vector of $\bar{\ell}$ points along a principal direction of Σ_ε. Therefore

$$|A|(q) \ge \kappa(q), \qquad \text{the curvature of } \bar{\ell} \text{ at } q \in \bar{\ell}. \tag{5}$$

The tangent cone of Σ_ε at the origin O is $\Lambda_{n+1} \cap \Lambda^*$. Hence a tangent vector of $\bar{\ell}$ at O points along $\ell \subset \Lambda_{n+1}$. Moreover $\bar{\ell}$ is tangent to $p_\varepsilon \;\not\!\times\; T$ at p_ε. Hence the angle between two tangent vectors of $\bar{\ell}$ at O and at p_ε is larger than 90°. Thus there exists $q \in \bar{\ell}$ at which a tangent vector of $\bar{\ell}$ is perpendicular to ℓ. Since $\bar{\ell} \sim \{p_\varepsilon\}$

is disjoint from $\tau_{q(a(\varepsilon))}\mu_{a(\varepsilon)}(C)$ one can apply Lemma 3 and conclude that there exists a point $q_\varepsilon \in \bar{\ell}$ at which

$$\kappa(q_\varepsilon) \geq \frac{1}{(\sqrt{n}-1)a(\varepsilon)}. \tag{6}$$

Combining (5) and (6), we get (4b). Finally we can compute

$$\operatorname{dist}(q_\varepsilon, \Lambda_0) \leq \left[\frac{1}{n}(\sqrt{n}-1)^2 a(\varepsilon)^2 + \varepsilon^2\right]^{1/2},$$

which gives (4a).

DEFINITION. Fix $0 < d < 1$ in such a way that for any ε

$$\operatorname{dist}(Z_d, \partial\tilde{\Sigma}_\varepsilon) \geq d. \tag{7}$$

Define $\tilde{\Sigma}_{\varepsilon,c} = \{q \in \tilde{\Sigma}_\varepsilon \cap Z_d : \operatorname{dist}(q, \Lambda_0) \leq c\}$.

LEMMA 5. *If $n \leq 6$ and $\tilde{\Sigma}_{\varepsilon,c}$ is stable, then there exists $b > 0$ depending only on the dimension n such that for any interior point q of $\tilde{\Sigma}_{\varepsilon,c}$*

$$|A|(q) \leq \frac{b}{\operatorname{dist}(q, \partial\tilde{\Sigma}_{\varepsilon,c})}. \tag{8}$$

Proof. Let ω_{n+1} be the volume of a unit ball in $\mathbf{R}^{n+1}$. By Lemma 1, $\alpha(\Sigma_\varepsilon) \cap \tilde{\Sigma}_{\varepsilon,c}$ is area minimizing for any $\alpha \in G$. So it is easy to show that if $B_r(q)$ is disjoint from $\partial\tilde{\Sigma}_{\varepsilon,c}$ then

$$Vol(\tilde{\Sigma}_{\varepsilon,c} \cap \alpha(\Sigma_\varepsilon) \cap B_r(q)) \leq Vol(\partial B_r(q)) = (n+1)\omega_{n+1} r^n, \qquad \alpha \in G.$$

Summing up for all distinct $\alpha(\Sigma_\varepsilon)$ gives

$$r^{-n} Vol(\tilde{\Sigma}_{\varepsilon,c} \cap B_r(q)) \leq 2^n (n+1)\omega_{n+1}.$$

Thus (8) follows from [SSY, Theorem 3] for $n \leq 5$ and [SS, Theorem 3] for $n = 6$.

4. Blowing up

We are now in a position to blow up a compact Enneper type surface to obtain a higher dimensional Enneper's surface. But in this process correct scaling is needed (Lemma 6). Blowing up by correct scaling gives us a complete analytic hypersurface (Lemma 8). It may happen that this hypersurface becomes the hyperplane. However, an eigenvalue estimate rules out this possibility (Lemma 7).

LEMMA 6. *Suppose $n \leq 6$. For each ε, let*

$$c(\varepsilon) = \max\{c : \tilde{\Sigma}_{\varepsilon,c} \text{ is stable}\}.$$

Then

$$\lim_{\varepsilon \to 0} c(\varepsilon) = 0. \tag{9}$$

Proof. $\tilde{\Sigma}_{\varepsilon,c}$ is stable if and only if

$$\int_{\tilde{\Sigma}_{\varepsilon,c}} |\nabla f|^2 - |A|^2 f^2 \geq 0$$

for any smooth function f with compact support in $\tilde{\Sigma}_{\varepsilon,c}$. Hence $\tilde{\Sigma}_{\varepsilon,c}$ is stable for sufficiently small $c > 0$. So $c(\varepsilon) > 0$. Suppose there exist $\delta > 0$ and a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ converging to 0 such that $c(\varepsilon_i) > \delta$ for all $i = 1, 2, 3, \ldots$. Then (4a) and (4c) of Lemma 4 imply that q_{ε_i} lies in $\tilde{\Sigma}_{\varepsilon_i, c(\varepsilon_i)}$ for sufficiently large i. And then from (4b), (4a), (8) we see that

$$\frac{1}{na(\varepsilon_i)} \leq |A|(q_{\varepsilon_i}) \leq \frac{b}{\delta - a(\varepsilon_i) - \varepsilon_i},$$

which contradicts (4c). Therefore we get (9).

LEMMA 7. *For $i = 1, \ldots, n$, let $\Lambda^n_{r,i} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : |x_i| \leq r\}$ and $\Lambda^n_r = \bigcup_{1 \leq i \leq n} \Lambda^n_{r,i}$. Then on a domain $D \subset \Lambda^n_r$ the first nonzero eigenvalue $\lambda_1(D)$ of the Laplacian satisfies*

$$\lambda_1(D) \geq \frac{1}{4n^2 r^2}.$$

Proof. Define the projections $\pi_i^n: \mathbf{R}^n \to \mathbf{R}^n$ by $\pi_i^n(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$. Then for any $D' \subset\subset D$ and any i we have

$$Vol(\partial D') \geq 2 Vol(\pi_i^n(\partial D')).$$

However,

$$Vol(D') \leq \sum_{1 \leq i \leq n} Vol(D' \cap A_{r,i}^n) \leq 2r \sum_{1 \leq i \leq n} Vol(\pi_i^n(\partial D')).$$

Hence from Cheeger's estimate [C] we see that

$$\lambda_1(D) \geq \frac{1}{4}\left[\inf_{D' \subset\subset D} \frac{Vol(\partial D')}{Vol(D')}\right]^2$$

$$\geq \frac{1}{4}\left[\inf_{D' \subset\subset D} \frac{\frac{2}{n} \sum_{1 \leq i \leq n} Vol(\pi_i^n(\partial D'))}{2r \sum_{1 \leq i \leq n} Vol(\pi_i^n(\partial D'))}\right]^2 = \frac{1}{4n^2 r^2}.$$

LEMMA 8. *As $\varepsilon \to 0$, $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon)$ converges to a complete minimal hypersurface Σ in $\mathbf{R}^{n+1}$, $n = 3, 4, 5, 6$. Σ is distinct from the hyperplane.*

Proof. Since Σ_ε is area minimizing, one can apply the same argument as in the proof of Lemma 5 to show that

$$|A|(q) \leq \frac{b}{\operatorname{dist}(q, \partial\alpha(\Sigma_\varepsilon))}, \qquad q \in \alpha(\Sigma_\varepsilon), \quad \alpha \in G. \tag{10}$$

Take $q \in \tilde{\Sigma}_\varepsilon \cap Z_d$. Then

$$\operatorname{dist}(q, \partial\tilde{\Sigma}_\varepsilon) \geq d, \tag{11}$$

and q must belong to $\alpha(\Sigma_\varepsilon)$ for some $\alpha \in G$. Observe that $\partial\alpha(\Sigma_\varepsilon) \subset \Lambda_0 \cup \partial\tilde{\Sigma}_\varepsilon$. If $\operatorname{dist}(q, \Lambda_0) \leq c(\varepsilon)/2$, then by Lemma 5

$$|A|(q) \leq \frac{2b}{c(\varepsilon)}. \tag{12}$$

If $\operatorname{dist}(q, \Lambda_0) > c(\varepsilon)/2$, then (10) and (11) imply that

$$|A|(q) \leq \frac{b}{\min\{c(\varepsilon)/2, d\}}. \tag{13}$$

So it follows from (12) and (13) that for sufficiently small ε

$$\sup|A| \leq \frac{2b}{c(\varepsilon)} \qquad \text{on } \tilde{\Sigma}_\varepsilon \cap Z_d.$$

Hence on $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$ we have

$$\sup|A| \leq 2b.$$

Therefore $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$ converges as $\varepsilon \to 0$ to an analytic minimal hypersurface Σ in the C^2 topology. By (7) we see that the boundary of $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_\varepsilon \cap Z_d)$ lies in $\partial Z_{d/c(\varepsilon)}$, which disappears as $\varepsilon \to 0$. Thus Σ is complete.

We now show that Σ cannot be the hyperplane. Since $\tilde{\Sigma}_{\varepsilon,c(\varepsilon)}$ is stable and any subset of $\tilde{\Sigma}_\varepsilon$ properly containing $\tilde{\Sigma}_{\varepsilon,c(\varepsilon)}$ is unstable, the Jacobi operator $\Delta + |A|^2$ on $\tilde{\Sigma}_{\varepsilon,c(\varepsilon)}$ has an eigenfunction f_ε with the eigenvalue zero which is positive in the interior and zero on the boundary of $\tilde{\Sigma}_{\varepsilon,c(\varepsilon)}$. Consequently $\bar{f}_\varepsilon = f_\varepsilon \circ \mu^{-1}_{1/c(\varepsilon)}$ is an eigenfunction of the Jacobi operator on $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon,c(\varepsilon)})$. Let

$$\Sigma_J = \lim_{\varepsilon \to 0} \mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon,c(\varepsilon)}) = \{q \in \Sigma : \operatorname{dist}(q, \Lambda_0) \leq 1\}.$$

Suppose that Σ is the hyperplane. Σ must then coincide with Λ_{n+1}. Viewing Λ_{n+1} as $\mathbf{R}^n$, we see that $\Sigma_J = \Lambda_1^n$, as defined in the preceding lemma. $\mu_{1/c(\varepsilon)}(\tilde{\Sigma}_{\varepsilon,c(\varepsilon)})$ is close to $\Lambda(\varepsilon) = \Lambda_1^n \cap Z_{d/c(\varepsilon)}$ in the C^2 topology. Hence one can push $\bar{f}_\varepsilon$ forward to obtain a smooth function $\tilde{f}_\varepsilon$ on $\Lambda(\varepsilon)$ that vanishes on the boundary of $\Lambda(\varepsilon)$ and satisfies

$$\Delta\tilde{f}_\varepsilon + q\tilde{f}_\varepsilon = 0 \qquad \text{on } \Lambda(\varepsilon)$$

for a smooth function q with $|q| \leq b(\varepsilon)$, where $b(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then

$$\lambda_1(\Lambda(\varepsilon)) \leq \frac{\int_{\Lambda(\varepsilon)} |\nabla\tilde{f}_\varepsilon|^2}{\int_{\Lambda(\varepsilon)} \tilde{f}_\varepsilon^2} = \frac{\int_{\Lambda(\varepsilon)} q\tilde{f}_\varepsilon^2}{\int_{\Lambda(\varepsilon)} \tilde{f}_\varepsilon^2} \leq b(\varepsilon),$$

which contradicts the preceding lemma. Therefore Σ is not the hyperplane.

5. Existence theorem

In conclusion we prove the following theorem on the existence of higher dimensional Enneper's surface.

THEOREM. *In $\mathbf{R}^{n+1}$, $n = 3, 4, 5, 6$, there exists a complete minimal hypersurface Σ^n called higher dimensional Enneper's surface with the following properties.*

(i) *Σ is a minimal immersion of $\mathbf{R}^n$ into $\mathbf{R}^{n+1}$.*

(ii) *Asymptotically Σ is the hyperplane with multiplicity $2^n - 1$.*

(iii) *Σ contains Λ_0, the union of n mutually orthogonal $(n-1)$-planes.*

(iv) *Σ is invariant under G.*

(v) *$\int_\Sigma |A|^n = \infty$, $|A|$ being the length of the second fundamental form of Σ.*

(vi) *The Gauss map for Σ is not well defined at the point at infinity of Σ.*

(vii) *Σ consists of 2^n congruent embedded pieces. The union of two adjacent pieces is stable. More precisely, if $\hat{\Sigma}$ is one of the pieces with $\partial\hat{\Sigma} \subset \Lambda^*$, then $\hat{\Sigma} \cup \rho_i(\hat{\Sigma})$ is a stable subset of Σ.*

Proof. (i) From the construction of Σ_ε in Lemma 1 it is clear that the interior of Σ_ε is diffeomorphic to the interior of $O \ggg T$. Let $\hat{\Sigma} = \lim_{\varepsilon\to 0}\mu_{1/c(\varepsilon)}(\Sigma_\varepsilon)$. Then one can see that $\hat{\Sigma}$ is embedded and diffeomorphic to $\Lambda^\sharp_{n+1} (= \lim_{\varepsilon\to 0}\mu_{1/c(\varepsilon)}(O \ggg T))$. Let ψ be a diffeomorphism of $\Lambda^\sharp_{n+1}$ onto $\hat{\Sigma}$. Note that

$$\Sigma = \bigcup_{\alpha\in G} \alpha(\hat{\Sigma}).$$

Define $\tilde{\psi}\colon \Lambda_{n+1} \to \mathbf{R}^{n+1}$ by

$$\tilde{\psi}(x) = \alpha(\psi(y)) \qquad \text{if } x = \alpha(y), \quad x \in \Lambda_{n+1}, \quad y \in \Lambda^\sharp_{n+1}.$$

Then one easily verifies that $\tilde{\psi}$ is an immersion of $\mathbf{R}^n$ onto $\Sigma \subset \mathbf{R}^{n+1}$.

(ii) Since Σ_ε is area minimizing, we have for $r \le d$

$$Vol(\Sigma_\varepsilon \cap B_r(O)) \le Vol(\Sigma_\varepsilon \cap Z_r) \le Vol(\Lambda^\flat_{n+1} \cap Z_r) + Vol(\partial Z_r \cap \Lambda^\varepsilon_{n+1}).$$

Hence

$$\frac{1}{\omega_n r^n} Vol(\Sigma_\varepsilon \cap B_r(O)) \le \frac{2^n - 1}{2^n} + \frac{n\varepsilon}{r}.$$

By the monotonicity of the volume ratio,

$$\frac{1}{\omega_n r^n} Vol(\Sigma_\varepsilon \cap B_r(O)) \leq \frac{2^n - 1}{2^n} + \frac{n\varepsilon}{d}, \qquad 0 < r \leq d.$$

Because of the invariance of the volume ratio under scaling, we see that as $\varepsilon \to 0$

$$\frac{1}{\omega_n r^n} Vol(\hat{\Sigma} \cap B_r(O)) \leq \frac{2^n - 1}{2^n}, \qquad 0 < r < \infty. \tag{14}$$

Now define *the tangent cone* T_∞ *of* $\hat{\Sigma}$ *at infinity* by the current limit

$$T_\infty = \lim_{\varepsilon \to 0} \mu_\varepsilon(\hat{\Sigma}).$$

Then (14) gives

$$\Theta^n(T_\infty, O) \leq \frac{2^n - 1}{2^n}. \tag{15}$$

spt T_∞ lies in Λ^+_{n+1} because $\hat{\Sigma} \subset \Lambda^+_{n+1}$. Also

$$\text{spt}(\partial T_\infty) = \partial \Lambda^\sharp_{n+1}.$$

If spt $T_\infty \cap \Lambda^\flat_{n+1} \sim \partial \Lambda^\flat_{n+1} \neq \phi$, then the maximum principle implies that spt $T_\infty \supset \Lambda^\flat_{n+1}$. It follows from (15) that spt $T_\infty = \Lambda^\flat_{n+1}$. So let us suppose that spt $T_\infty \neq \Lambda^\flat_{n+1}$. Then either

$$\text{spt } T_\infty \cap \Lambda^\flat_{n+1} \sim \partial \Lambda^\flat_{n+1} = \phi \quad \text{and} \quad \text{spt } T_\infty \sim \Lambda_{n+1} \neq \phi, \tag{16}$$

or

$$\text{spt } T_\infty = \Lambda^\sharp_{n+1}. \tag{17}$$

In case of (16), there exists $0 < \theta < \pi$ such that spt T_∞ is tangent to $\Lambda_{n+1,\theta}$ and lies on one side of $\Lambda_{n+1,\theta}$. By the maximum principle one gets $\Lambda_{n+1,\theta} \cap \Lambda^+_{n+1} \subset$ spt T_∞, and so spt $T_\infty \cap \Lambda^\flat_{n+1} \supset \Lambda_{n+1} \cap \Lambda_{n+2}$, which is a contradiction. In case of (17), assume that $\hat{\Sigma} \cap \Lambda_{n+1,\theta} \sim \{O\} \neq \phi$ for some $0 < \theta < \pi$. (17) requires $\hat{\Sigma} \cap \Lambda_{n+1,\theta}$ to be compact. Then one can find a hyperplane Λ parallel to $\Lambda_{n+1,\theta}$ such that $\hat{\Sigma}$ is tangent to Λ at an interior point of $\hat{\Sigma}$ and lies on one side of Λ. This is impossible by the maximum principle. So $\hat{\Sigma} = \Lambda^\sharp_{n+1}$. But then $\Sigma = \Lambda_{n+1}$, which contradicts

Lemma 8. Therefore spt $T_\infty = \Lambda^\flat_{n+1}$ and it follows that the tangent cone of Σ at infinity is Λ_{n+1} with multiplicity $2^n - 1$.

(iii), (iv) These are obvious by Lemma 1 and Lemma 2.

(v) M. Anderson [A, Theorem 5.2] showed that if a complete n-dimensional minimally immersed submanifold $S \subset \mathbf{R}^m$ has one end and satisfies $\int_S |A|^n < \infty$, then S is an n-plane. So our claim follows.

(vi) Since $\Sigma \supset \Lambda_0$ and the $(n-1)$-planes of Λ_0 intersect each other along Λ_{-1}, one can see that any vector v normal to Σ at $q \in \Lambda_{-1}$ must be normal to every $(n-1)$-plane of Λ_0 passing through q. It follows that $v = (0, \ldots, 0, a)$, $a \neq 0$. Hence the Gauss map for Σ maps Λ_{-1} to the north pole of S^n. Now let $\hat{\Sigma}$ be the embedded surface as defined in the proof of part (i) above and let $\hat{\ell} \subset \hat{\Sigma}$ be the plane curve which is invariant under the reflections ξ_{ij}, $1 \leq i \neq j \leq n$. At the origin $\hat{\ell}$ is tangent to the horizontal hyperplane Λ_{n+1}. But as $\hat{\ell}$ goes toward the point at infinity, it is flipped over by 180° and becomes parallel to Λ_{n+1}. So the Gauss map maps $\hat{\ell}$ onto a great semicircle connecting the north pole to the south pole in S^n. Therefore the Gauss map cannot take on a single value at the point at infinity.

(vii) It follows from Lemma 2 that 2^n congruent copies of $\hat{\Sigma}$ comprise Σ. Since $\partial\Sigma_\varepsilon$ can be projected one-to-one into $\Lambda_{i,\theta}$ for $0 < \theta < \pi/2$, Σ_ε is a graph over $\Lambda_{i,\theta}$. Therefore, as a limiting case, the interior of Σ_ε is a graph over $\Lambda_{i,0} = \Lambda_i$ although Σ_ε itself is not. Similarly one can show that the interior of $\rho_i(\Sigma_\varepsilon)$ is a graph over Λ_i. Note that Σ_ε and $\rho_i(\Sigma_\varepsilon)$ lie in the opposite sides of Λ_{n+1} and that $(\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)) \cap \Lambda_{n+1} \subset \Lambda_0$. Hence the interior of $\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)$ is also a graph over Λ_i. Therefore $\Sigma_\varepsilon \cup \rho_i(\Sigma_\varepsilon)$ is stable by [Ch, Corollary 3] and so is $\hat{\Sigma} \cup \rho_i(\hat{\Sigma})$.

6. Concluding remarks

(1) When $n = 2$, the same construction as described above gives rise to classical Enneper's surface. This can be verified by observing that Σ^2 has one end and has the total curvature of -4π [O].

(2) We have seen that there exists n-dimensional Enneper's surface in $\mathbf{R}^{n+1}$ for $2 \leq n \leq 6$. On the other hand, our method of construction breaks down for $n \geq 7$ because the curvature estimates (8) and (10) are no longer valid. This is in sharp contrast with the following famous results: J. Simons [Si] has proved that there exists no n-dimensional entire nonlinear minimal graph in $\mathbf{R}^{n+1}$ when $2 \leq n \leq 7$; Bombieri-De Giorgi-Giusti [BDG] have shown that there exist n-dimensional entire nonlinear minimal graphs in $\mathbf{R}^{n+1}$ if $n \geq 8$. Let us explain in a naive and heuristic way why these dichotomies occur: The curvature estimates imply that low dimensional stable minimal submanifolds are *rigid*; Invalidity of the curvature estimates in high dimensions indicates that high dimensional stable minimal submanifolds are *flexible*.

Roughly speaking, one can obtain $\hat{\Sigma}$ from $\Lambda_{n+1}^{\#}=\{(x_1,\ldots,x_n,0):x_i\geq 0,$ $i=1,\ldots,n\}$ by fixing the boundary of $\Lambda_{n+1}^{\#}$ and bending the interior of $\Lambda_{n+1}^{\#}$ by 180°. When the dimension is low, the minimal submanifold $\hat{\Sigma}$ is so rigid that $\hat{\Sigma}$ can withstand the extreme bending and thereby giving rise to the higher dimensional Enneper's surface. However, when the dimension is high, $\hat{\Sigma}$ is so flexible that the 180° bending tears down and flattens $\hat{\Sigma}$ and then $\hat{\Sigma}$ becomes $\Lambda_{n+1}^{?}$. As for minimal graphs, one should note that graphs are obtained from the horizontal hyperplane by mild bendings of at most 90°. But low dimensional complete stable minimal submanifolds are too rigid to allow mild bendings and therefore hyperplanes are the only entire minimal graphs. Moreover, high dimensional stable minimal submanifolds are flexible enough to allow mild bendings to persist, thereby allowing nonlinear minimal graphs to exist.

In view of these interpretations let us make a guess as to n-dimensional Enneper's surface in $\mathbf{R}^{n+1}$ for $n\geq 7$. If such Enneper's surface is to exist, its fundamental piece should be constructed by bending $\Lambda_{n+1}^{\#}$ by less than 180°, and hence the support of its tangent cone at infinity should be distinct from Λ_{n+1}.

(3) L. Simon [S] proved that the curvature estimate of Lemma 5 also holds for $n=7$ in the nonparametric case. However, with the assumption that $\tilde{\Sigma}_{\varepsilon,c}$ is a local graph instead of being a stable hypersurface, we had difficulty ruling out the possibility that Σ^7 becomes the hyperplane or $\Sigma^6\times\mathbf{R}^1$ in the proof of Lemma 8.

(4) As for part (v) of the theorem, we recall that $\int_\Sigma K=-4\pi$ for the two-dimensional Enneper's surface $\Sigma\subset\mathbf{R}^3$. In fact, the total curvature of Σ is concentrated near the origin since $|K|$ takes on the maximum at the origin. On the other hand, for higher dimensional Enneper's surface we can argue that $\int_\Sigma|A|^n$ is concentrated near Λ_{-1} as follows. By part (vii) of the theorem, $\hat{\Sigma}\cup\rho_i(\hat{\Sigma})$ and $\hat{\Sigma}\cup\rho_j(\hat{\Sigma})$, $i\neq j$, are stable. Hence for $q\in\hat{\Sigma}$ one gets the curvature estimate

$$\begin{aligned}|A|(q)&\leq\min\{b/\operatorname{dist}(q,\partial(\hat{\Sigma}\cup\rho_i(\hat{\Sigma}))),\, b/\operatorname{dist}(q,\partial(\hat{\Sigma}\cup\rho_j(\hat{\Sigma})))\}\\&\leq\sqrt{2}b/\operatorname{dist}(q,\partial(\hat{\Sigma}\cup\rho_i(\hat{\Sigma}))\cap\partial(\hat{\Sigma}\cup\rho_j(\hat{\Sigma})))\\&=\sqrt{2}b/\operatorname{dist}(q,\Lambda_{-1}).\end{aligned}$$

This estimate indicates that $|A|(q)$ becomes large as q approaches Λ_{-1}, which also suggests that $\int_\Sigma|A|^n$ becomes infinite since Λ_{-1} has infinite $(n-2)$-dimensional volume unless $n=2$.

(5) From part (vii) of the theorem we see that the higher dimensional Enneper's surface Σ consists of 2^{n-1} disjoint stable subsets. In light of [Ch, Theorem 1] it is tempting to conjecture regarding the Morse index of Σ that

$$\operatorname{index}(\Sigma)=2^{n-1}-1.$$

(6) It is still interesting to show that index(Σ) is finite. This together with part (v) of the theorem would surprisingly contrast with Fischer-Colbrie's theorem that a complete minimal surface in $\mathbf{R}^3$ has finite total curvature if and only if it has finite index [F].

(7) The higher dimensional catenoid C lies between two parallel hyperplanes. In the proof of Lemma 4, C was used as a barrier in applying the maximum principle to Σ_ε. For this reason it seems quite probable that higher dimensional Enneper's surface might also lie between two parallel hyperplanes.

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