

First Eigenvalue of Symmetric Minimal Surfaces in $\mathbb{S}^3$

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ABSTRACT. Let λ_1 be the first nontrivial eigenvalue of the Laplacian on a compact surface without boundary. We show that $\lambda_1 = 2$ on compact embedded minimal surfaces in $\mathbb{S}^3$ which are invariant under a finite group of reflections and whose fundamental piece is simply connected and has less than six edges. In particular $\lambda_1 = 2$ on compact embedded minimal surfaces in $\mathbb{S}^3$ that are constructed by Lawson and by Karcher-Pinkall-Sterling.

Given a minimal submanifold Σ of $\mathbb{R}^m$, it is well known that the Euclidean coordinates $x_1, \dots, x_m$ of $\mathbb{R}^m$ are harmonic on Σ . On the other hand, if Σ^n is an n -dimensional minimal submanifold of $\mathbb{S}^{m-1} \subset \mathbb{R}^m$, then $x_1, \dots, x_m$ are eigenfunctions of the Laplacian on Σ with eigenvalue n . Therefore the first nontrivial eigenvalue λ_1 of the Laplacian on Σ should be less than or equal to n . Indeed λ_1 is strictly less than 2 for a minimal torus with self intersection in $\mathbb{S}^3$ and for a Veronese surface in $\mathbb{S}^5$. However, in case Σ is a totally geodesic minimal submanifold $\mathbb{S}^n$ in $\mathbb{S}^{m-1}$, λ_1 equals n . Thus it is quite tempting to conjecture, as Yau did [9], the following:

The first eigenvalue λ_1 on an n -dimensional compact embedded minimal hypersurface in $\mathbb{S}^{n+1}$ is equal to n .

This conjecture is still open. The only partial result obtained so far is by Choi and Wang [2], which states $\lambda_1 \geq n/2$. In this paper we show that Yau's conjecture is true for all compact embedded minimal surfaces in $\mathbb{S}^3$ that are known to exist, as constructed by Lawson [7] and by Karcher-Pinkall-Sterling [6].

More generally, let $\Sigma \subset \mathbb{S}^3$ be a compact embedded minimal surface invariant under a group of reflections which tessellate $\mathbb{S}^3$ into tetrahedra. We prove that if the fundamental patch of Σ is simply connected and has less than six edges, then

$\lambda_1 = 2$. Also, let g be the genus of Σ and L the number of tetrahedra in the tessellation. If $g < 1 + L/4$, then $\lambda_1 = 2$.

In a sequel to this paper we also show that $\lambda_1 = 2$ on Kapouleas-Yang's minimal surfaces which were constructed recently [5].

1. TWO-PIECE PROPERTY

The first eigenvalue λ_1 of Δ on a compact Riemannian manifold M without boundary is defined by

$$\lambda_1 := \inf_{\varphi \in C^1(M), \int_M \varphi = 0} \frac{\int_M |\nabla \varphi|^2}{\int_M \varphi^2}.$$

If ψ is an eigenfunction with eigenvalue λ , then

$$\Delta \psi + \lambda \psi = 0.$$

If $\Delta \psi + \lambda_1 \psi = 0$, by Courant's nodal theorem [1] ψ has exactly two nodal domains (a nodal domain is a maximal connected domain of M on which ψ has a constant sign). Note, however, that an eigenfunction with two nodal domains is not necessarily the first eigenfunction.

Let Σ be an n -dimensional minimal submanifold of $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ and denote by $X = (x_1, \dots, x_m)$ the immersion of Σ in $\mathbb{R}^m$. The minimality of Σ in $\mathbb{S}^{m-1}$ implies that the cone $O \times \Sigma$ is minimal in $\mathbb{R}^m$. Therefore ΔX must be perpendicular to $\mathbb{S}^{m-1}$ and hence ΔX is parallel to X . Let $e_1, \dots, e_n$ be an orthonormal frame in a neighborhood of Σ such that $\nabla_{e_i} e_i = 0$ at a point p in the neighborhood.

Then at p

$$\begin{aligned} \langle \Delta X, X \rangle &= \sum_i \langle \nabla_{e_i} \nabla_{e_i} X, X \rangle \\ &= \sum_i (e_i \langle \nabla_{e_i} X, X \rangle - \langle \nabla_{e_i} X, \nabla_{e_i} X \rangle) \\ &= - \sum_i \langle e_i, e_i \rangle = -n. \end{aligned}$$

Hence

$$\Delta X + nX = 0.$$

Therefore the Euclidean coordinates $x_1, \dots, x_m$ of $\mathbb{R}^m$ are eigenfunctions of Δ with eigenvalue n on Σ^n .

If n is indeed the first eigenvalue of the Laplacian on Σ^n , then one can conclude, from Courant's nodal theorem applied to a linear function $a_1 x_1 + \dots + a_m x_m$ on Σ , that any great hypersphere in $\mathbb{S}^{m-1}$ will cut Σ into two connected

pieces. This two-piece property has been proved to be true by Ros for compact embedded minimal surfaces in $\mathbb{S}^3$ [8]. Ros's two-piece property hints that Yau's conjecture could be true in $\mathbb{S}^3$. In fact Ros's theorem can be easily extended to higher dimension as follows.

Given a unit vector $v \in \mathbb{S}^{n+1}$, define $S^n(v) = \{p \in \mathbb{S}^{n+1} \mid \langle v, p \rangle = 0\}$, $H_+(v) = \{p \in \mathbb{S}^{n+1} \mid \langle v, p \rangle > 0\}$ and $H_-(v) = \{p \in \mathbb{S}^{n+1} \mid \langle v, p \rangle < 0\}$.

Lemma 1.1. *If the boundary of a compact immersed orientable and stable minimal hypersurface Σ^n in $\mathbb{S}^{n+1}$ lies in a great sphere Π , then Σ is totally geodesic.*

Proof. By stability we have for any smooth function f on Σ vanishing on $\partial\Sigma$

$$(1.1) \quad \int_{\Sigma} (f\Delta f + nf^2 + |A|^2 f^2) \leq 0,$$

where A is the second fundamental form of Σ in $\mathbb{S}^{n+1}$. Choose v in such a way that $S^n(v)$ becomes the great sphere Π of the hypothesis. Let $f(p) = \langle v, p \rangle$ for $p \in \Sigma$. Then

$$f = 0 \text{ on } \partial\Sigma \quad \text{and} \quad \Delta f + nf = 0 \text{ on } \Sigma,$$

and it follows from (1.1) that

$$\int_{\Sigma} |A|^2 f^2 \leq 0.$$

If $A \neq 0$ in a neighborhood, then $f = 0$ in the same neighborhood. Either way Σ is totally geodesic. $\square$

Theorem 1.2. *Any great sphere $S^n(v)$ in $\mathbb{S}^{n+1}$ divides a compact embedded minimal hypersurface Σ of $\mathbb{S}^{n+1}$ into two connected pieces.*

Proof. Since the proof is trivial for totally geodesic Σ , let us assume that Σ is not a great sphere. Suppose $\Sigma \cap H_+(v)$ is not connected. Let Σ_1 be a connected component of $\Sigma \cap H_+(v)$ and let $\Sigma_2 = (\Sigma \cap H_+(v)) \setminus \Sigma_1$. Obviously $\Sigma_2 \neq \emptyset$. Denote the components of $\mathbb{S}^{n+1} \setminus \Sigma$ by U_1 and U_2 . Since U_1 is a mean convex domain and $\partial\Sigma_1$ is nullhomologous in $\bar{U}_1$, one can find an area minimizing hypersurface $\tilde{\Sigma}_1$ in $\bar{U}_1$ which is homologous to Σ_1 . Applying the same argument to U_2 , one obtains an area minimizing hypersurface $\tilde{\Sigma}_2$ in $\bar{U}_2$ which is homologous to Σ_2 . We claim that $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ is a great sphere. Since each of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ is stable and since $\partial\tilde{\Sigma}_1 = \partial\tilde{\Sigma}_2 \subset S^n(v)$, Lemma 1.1 tells us that both $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are totally geodesic. If $\partial\tilde{\Sigma}_1$ is not an $(n-1)$ -dimensional great sphere in $S^n(v)$, then $\tilde{\Sigma}_1, \tilde{\Sigma}_2 \subset S^n(v)$ and moreover, $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 = S^n(v)$. Suppose that $\partial\tilde{\Sigma}_1$ is a great sphere. Then $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are great hemispheres and in this case $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ may not be smooth along $\partial\tilde{\Sigma}_1$. Anyway we can rotate $\tilde{\Sigma}_1$ around $\partial\tilde{\Sigma}_1$ inside $\bar{U}_1$. Since $\partial\tilde{\Sigma}_1 \subset \partial U_1$, we can obtain a rotated copy of $\tilde{\Sigma}_1$ which touches Σ from inside $\bar{U}_1$. Then by

the boundary maximum principle Σ is totally geodesic, which is a contradiction. Hence $\partial\tilde{\Sigma}_1$ is not a great sphere and $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ is a great sphere.

Since $\partial\tilde{\Sigma}_1 \subset S^n(v) \cap (\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2)$, and $\partial\tilde{\Sigma}_1$ is not totally geodesic, $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_2$ coincides with $S^n(v)$. Now since $\tilde{\Sigma}_1 \subset \bar{U}_1$ and $\tilde{\Sigma}_2 \subset \bar{U}_2$, it follows that $\Sigma \cap S^n(v) = \partial\tilde{\Sigma}_1$. But this contradicts our original assumption because $\Sigma \cap S^n(v) = \partial\Sigma_1 \cup \partial\Sigma_2$ and $\partial\Sigma_2 \neq \emptyset$. Therefore $\Sigma \cap H_+(v)$ is connected. Similarly $\Sigma \cap H_-(v)$ is connected as well. $\square$

2. MINIMAL SURFACES IN $\mathbb{S}^3$

In 1970 Lawson [7] constructed a variety of compact minimal surfaces in $\mathbb{S}^3$. His construction starts from a geodesic polygon Γ consisting of four geodesic edges. At each vertex of Γ two adjacent geodesic edges are assumed to make an angle of $\pi/(k+1)$ and $\pi/(m+1)$ alternately, where k, m are positive integers. Morrey's solution to the Plateau problem for Γ is a disk-type minimal surface spanning Γ . Extending this surface by 180° rotations about its geodesic boundary arcs, one can eventually obtain Lawson's compact minimal surfaces in $\mathbb{S}^3$.

To be more precise, let C_1 and C_2 be the great circles in $\mathbb{S}^3 \subset \mathbb{R}^4$ defined by

$$C_1 = \{(x_1, x_2, 0, 0) \mid x_1^2 + x_2^2 = 1\} \quad \text{and} \quad C_2 = \{(0, 0, x_3, x_4) \mid x_3^2 + x_4^2 = 1\}.$$

Let k and m be positive integers and choose points $P_1, P_2 \in C_1$ and $Q_1, Q_2 \in C_2$ such that $\text{dist}(P_1, P_2) = \pi/(k+1)$ and $\text{dist}(Q_1, Q_2) = \pi/(m+1)$. For the geodesic polygon $\Gamma := P_1Q_1P_2Q_2$, the above construction gives rise to a minimal surface denoted $\xi_{m,k}$. Lawson showed that $\xi_{m,k}$ is a compact orientable surface of genus mk embedded in $\mathbb{S}^3$.

Adopting a different method which is dual to Lawson's construction, Karcher, Pinkall and Sterling constructed new compact embedded minimal surfaces in $\mathbb{S}^3$. They first find a disk-type minimal surface with boundary, called a patch, inside a tetrahedron T which orthogonally intersects the four totally geodesic faces of the tetrahedron. T is assumed to be a fundamental domain for a tessellation of $\mathbb{S}^3$. Repeatedly reflecting patches across the faces of the tetrahedra of the tessellation, they obtain a complete surface.

As a matter of fact, Lawson's surfaces $\xi_{m,k}$ can be constructed also in this way. Let S be Morrey's solution for the Jordan curve $\Gamma = P_1Q_1P_2Q_2$ as defined above. And let Π_1, Π_2 be the great spheres in $\mathbb{S}^3$ such that $C_i \subset \Pi_i$ and Γ is symmetric with respect to Π_i , $i = 1, 2$. We claim that S is also symmetric with respect to Π_i . Suppose S is not symmetric. Π_i cuts S into two parts S_a and S_b . Assume that $\text{area}(S_a) \leq \text{area}(S_b)$ and let $\tilde{S}_a$ be the mirror image of S_a across Π_i . If S is perpendicular to Π_i along a neighborhood U in $S \cap \Pi_i$, then $\tilde{S}_a$ locally lies on one side of S_b near an open subset of U , which contradicts the boundary point lemma [4]. Therefore $S_a \cup \tilde{S}_a$ is not smooth along $S \cap \Pi_i$. Hence by a small perturbation of $S_a \cup \tilde{S}_a$ along $S \cap \Pi_i$ one can construct a surface $\hat{S}$ with $\partial\hat{S} = \Gamma$ and $\text{area}(\hat{S}) < \text{area}(S)$. This contradiction proves the claim.

Π_1 and Π_2 cut S into four congruent pieces whose boundary contains the geodesic segment $\overline{P_1Q_1}$, $\overline{Q_1P_2}$, $\overline{P_2Q_2}$, and $\overline{Q_2P_1}$, respectively. Let $S_{1/4}$ denote the piece which contains $\overline{P_1Q_1}$ as a boundary curve. Let $\tilde{S}_{1/4}$ be the rotation of $S_{1/4}$ about $\overline{P_1Q_1}$ by 180° and define $S_{1/2} = S_{1/4} \cup \tilde{S}_{1/4}$. Then there exist two great spheres $\hat{\Pi}_1$ and $\hat{\Pi}_2$ such that $C_i \subset \hat{\Pi}_i$, $\angle(\Pi_1, \hat{\Pi}_1) = \pi/(m+1)$, $\angle(\Pi_2, \hat{\Pi}_2) = \pi/(k+1)$ and $S_{1/2}$ meets $\Pi_1, \Pi_2, \hat{\Pi}_1, \hat{\Pi}_2$ orthogonally. Let T be the tetrahedron with dihedral angles $\pi/2, \pi/2, \pi/2, \pi/2, \pi/(k+1), \pi/(m+1)$ which is surrounded by $\Pi_1, \Pi_2, \hat{\Pi}_1, \hat{\Pi}_2$ such that $S_{1/2} \subset T$ and $\partial S_{1/2} \subset \partial T$. T obviously determines a tessellation of $\mathbb{S}^3$ into cells which are congruent to T . Then the minimal surface obtained by repeated reflections across the faces of the tetrahedra in the tessellation is nothing but $\xi_{m,k}$.

3. A THEOREM ON THE EXISTING SURFACES

Every great sphere Π in $\mathbb{S}^{n+1}$ gives rise to an isometry on $\mathbb{S}^{n+1}$ which is the reflection across Π . In this section we will see how the symmetry of a minimal surface Σ influences the first eigenvalue and eigenfunction. From this we obtain the following.

Lemma 3.1. *Let G be a group of reflections in $\mathbb{S}^{n+1}$. Assume that a compact minimal hypersurface $\Sigma \subset \mathbb{S}^{n+1}$ is invariant under G . If the first eigenvalue of the Laplacian on Σ is less than n , then the first eigenfunction must be invariant under G .*

Proof. Let $\sigma \in G$ be the reflection across a great sphere Π in $\mathbb{S}^{n+1}$ and let φ be an eigenfunction on Σ corresponding to the first eigenvalue λ_1 . Note that $\varphi \circ \sigma$ is also an eigenfunction with eigenvalue λ_1 . Consider

$$\psi(x) := \varphi(x) - \varphi \circ \sigma(x).$$

If ψ is the null function, then φ is invariant under σ . If $\psi \neq 0$, then ψ itself is an eigenfunction with eigenvalue λ_1 . Furthermore, its nodal set, the zero set of ψ , contains $\Sigma \cap \Pi$ because for $p \in \Sigma \cap \Pi$,

$$\psi(p) = \varphi(p) - \varphi \circ \sigma(p) = \varphi(p) - \varphi(p) = 0.$$

But Courant's nodal theorem implies that ψ vanishes only on $\Sigma \cap \Pi$. Let D_1, D_2 be the components of $\Sigma \setminus \Pi$ such that ψ is positive on D_1 and negative on D_2 . One can find a linear function on $\mathbb{R}^{n+2}$ $\xi = a_1x_1 + \cdots + a_{n+2}x_{n+2}$ that vanishes on Π and is positive on D_1 . ξ is an eigenfunction on Σ with eigenvalue n . Since $\lambda_1 < n$, ξ is orthogonal to ψ on Σ . Since ψ and ξ have the same sign on $D_1 \cup D_2$, we have $\int_{\Sigma} \psi \xi > 0$, which contradicts the orthogonality of ψ and ξ . Therefore ψ must vanish on Σ . This completes the proof as σ is an arbitrary element of G . $\square$

Theorem 3.2. *Let Σ be a compact embedded minimal surface in $\mathbb{S}^3$ which is invariant under a group G of reflections. Suppose that the fundamental domain of G*

is a tetrahedron T . If the fundamental patch $P := \Sigma \cap T$ is simply connected and has four edges, then the first eigenvalue of the Laplacian on Σ equals 2.

Proof. Suppose $\lambda_1 < 2$. Let φ be an eigenfunction with eigenvalue λ_1 on Σ and $N \subset \Sigma$ the nodal set of φ . Lemma 3.1 tells us that φ is invariant under G . First we claim that N contains an interior point of P and $P \setminus N$ has at least two connected components. Suppose $\bar{P} \cap N \subset \partial P$. Then Courant's nodal theorem implies that $N = \Sigma \cap \mathbb{S}^2$ for some great sphere $\mathbb{S}^2$. By the orthogonality argument as used in the proof of Lemma 3.1, we get a contradiction. Hence $P \setminus N$ is not connected. Now one can find a face F of T and a component D of $P \setminus N$ such that ∂D is disjoint from F . Let Π be the great sphere containing F and let $\hat{D}$ be the mirror image of D across Π . Denote by D_1, D_2, D_3 the components of $\Sigma \setminus N$ containing $D, \hat{D}$ and intersecting Π , respectively. We claim that D_1, D_2, D_3 are all distinct. D_2 is the mirror image of D_1 and D_3 is nonempty and symmetric with respect to Π . Suppose on the contrary that D_1 and D_2 are identical. Let $\hat{T}$ be the mirror image of T across Π and set $T_2 = T \cup \hat{T}$. Choose $p \in D$ and let $\hat{p} \in \hat{D}$ be its mirror image. By the assumption, there is a curve γ connecting p to $\hat{p}$ which is disjoint from N . Let H be the normal closure of the subgroup of G which is generated by the reflections across the faces of T different from F . H is then a subgroup of G of index 2 whose fundamental domain is T_2 . Define $\tilde{\gamma} = \{\sigma(\gamma) \cap \tilde{T}_2 \mid \sigma \in H\}$. Then $\tilde{\gamma}$ is a curve in $\tilde{T}_2$ connecting p to $\hat{p}$ and disjoint from $N \cap T_2$. But this is impossible. Hence D_1 and D_2 are distinct components. The same argument can be used to conclude that D_1 and D_3 are also distinct. Therefore φ has at least three nodal domains, which contradicts Courant's nodal theorem. Thus $\lambda_1 = 2$. $\square$

Corollary 3.3. *The first eigenvalue of the Laplacian on Lawson's embedded minimal surfaces $\xi_{m,k}$ and Karcher-Pinkall-Sterling's minimal surfaces in $\mathbb{S}^3$ is equal to 2.*

Proof. As observed in the preceding section, both types of surfaces Σ originate from a simply connected fundamental patch S with four edges inside a tetrahedron T of a tessellation of $\mathbb{S}^3$. $\square$

4. BOUNDARY OF THE FUNDAMENTAL DOMAIN OF Σ

In this section we extend the results of the preceding section. Let T be an $(n+1)$ -simplex that is a fundamental domain for a finite group of reflections of $\mathbb{S}^{n+1}$. Given a compact embedded minimal hypersurface Σ^n in $\mathbb{S}^{n+1}$, the fundamental patch P of Σ is defined by $P = \Sigma \cap T$.

Lemma 4.1. *Let Σ^n be a compact embedded minimal hypersurface of $\mathbb{S}^{n+1}$ which is invariant under the group G of reflections in $\mathbb{S}^{n+1}$. Suppose that the fundamental patch P is homeomorphic to a ball. If $\lambda_1 < n$, then there are at least two disjoint nodal domains in P and the boundary of each domain must intersect every n -face of T .*

Proof. Suppose that λ_1 is less than n . Then by Lemma 3.1 the first eigenfunction φ is G -invariant and so is its nodal set N . Consider the nodal set restricted to $\bar{P}$. This set must be nonempty because otherwise φ would have only one nodal domain on P , which contradicts Courant's nodal theorem. Moreover, by the orthogonality argument as in the proof of Lemma 3.1, N contains an interior point of P and hence there are at least two nonempty nodal domains in P . Suppose that the boundary ∂D of one of the nodal domains of P is disjoint from an n -face F of T . Let Π be the great sphere containing F and $\hat{D}$ the mirror image of D across Π . Denote by D_1, D_2, D_3 the components of $\Sigma \setminus N$ containing $D, \hat{D}$, and intersecting Π , respectively. Then by reasoning as in the proof of Theorem 3.2, one can show that D_1, D_2, D_3 are distinct components, contradicting Courant's nodal theorem. $\square$

Theorem 3.2 generalizes as follows.

Theorem 4.2. *Let Σ be a compact embedded minimal surface in $\mathbb{S}^3$ invariant under a group of reflections that tessellate $\mathbb{S}^3$ into tetrahedra. If the fundamental patch P of Σ is simply connected and has less than six edges, then λ_1 is equal to 2.*

Proof. Suppose $\lambda_1 < 2$. We know from the proof of Lemma 4.1 that $N \cap P$, the nodal line in P , is nonempty. First let us suppose that $N \cap P$ is connected. Since P is simply connected, N cuts P into two sets, D_1 and D_2 . Suppose ∂P consists of edges $e_1, \dots, e_m$. Let $p, q \in \partial P$ be the end points of $N \cap P$. Then

- (i) p, q are interior points of some edges, or
- (ii) p is an interior point of an edge and q is a vertex of P , or
- (iii) both p and q are vertices.

In case of (i), assume $p \in e_1$ and $q \in e_k$. Then $e_1, \dots, e_k$ and $N \cap P$ bound D_1 . Since by Lemma 4.1 ∂D_1 intersects all four faces of T , we have $k \geq 4$. Similarly $e_k, e_{k+1}, \dots, e_m, e_1$ and $N \cap P$ bound D_2 and hence $m - k + 2 \geq 4$. Therefore $m \geq 6$. In case of (ii), assume $p \in e_1$ and $q \in \bar{e}_k \cap \bar{e}_{k+1}$. Then $e_1, \dots, e_k, N \cap P$ bound D_1 and $e_{k+1}, \dots, e_m, e_1, N \cap P$ bound D_2 . Hence $k \geq 4$ and $m - k + 1 \geq 4$, and therefore $m \geq 7$. In case of (iii), assume $p = \bar{e}_1 \cap \bar{e}_m$ and $q = \bar{e}_k \cap \bar{e}_{k+1}$. Then $k \geq 4$ and $m - k \geq 4$ and hence $m \geq 8$. In any case, we have $m \geq 6$. A similar proof holds even when $N \cap P$ is not connected. $\square$

5. FINITE GROUP OF REFLECTIONS

In this section we describe all the groups of reflections of $\mathbb{S}^3$ whose fundamental domain is a tetrahedron. The reason for this presentation is two-fold. On the first hand we will show that there exists a group of reflections for which the [6]'s construction would give a new embedded minimal surface but which was not presented in [6]. On the second hand, the description of the associated tetrahedra is needed in the last section in order to compute the genus of the G -invariant minimal surfaces.

The finite groups of reflections have been classified by Coxeter in [3]. We will consider groups generated by four reflections, i.e., the groups that induce a tessellation of $\mathbb{S}^3$ by tetrahedra. Indeed reflections are represented by vectorial hyperplanes of $\mathbb{R}^4$. Just as in dimension three, where planes of $\mathbb{R}^3$ define a division of the sphere into congruent triangles, hyperplanes define a division of the hypersphere $\mathbb{S}^3$ by congruent tetrahedra. Reciprocally, a tetrahedron T that tessellates $\mathbb{S}^3$ defines a group of reflections generated by the four reflections of the hyperplanes corresponding to the four faces of T . T is then the fundamental domain of G , hence G is characterized by the six dihedral angles of T : β_{ij} , $i, j = 1, \dots, 4$, where β_{ij} denotes the dihedral angle between faces F_i and F_j . The β_{ij} must be divisors of π , hence there exist integers b_k such that $\beta_{ij} = \pi/b_k$.

We will denote the reflection group G by $\langle b_1, b_2, b_3, b_4, b_5, b_6 \rangle$, the angles being classified according to the lexicographical order.

Furthermore, a sphere around any vertex of T is tessellated by congruent triangles which lie in each tetrahedron near the vertex. Necessary conditions on the existence of triangles tessellations on the sphere will then yield necessary conditions on the existence of tetrahedra tessellations of $\mathbb{S}^3$.

5.1. Reducible group of reflections If one face of T , say F_1 , is perpendicular to the other three, then the angles $\beta_{1,j} = \pi/2$, $j = 2, 3, 4$, and the vertex opposite to F_1 is at a constant distance $\pi/2$ to any point of F_1 . The tessellation of the hypersphere reduces in that case to the tessellation of the sphere Π containing F_1 by triangles congruent to F_1 and Π is globally invariant under G .

F_1 in turn is determined by three angles that correspond to the three remaining dihedral angles of T . They must be divisors of π , hence equal respectively to π/k_1 , π/k_2 , π/k_3 where k_i are integers. The fundamental domain of G is then a tetrahedron whose faces are pieces of spherical planes and whose six dihedral angles are

$$\beta_{1,2} = \frac{\pi}{2}, \beta_{1,3} = \frac{\pi}{2}, \beta_{1,4} = \frac{\pi}{2}, \beta_{2,3} = \frac{\pi}{k_1}, \beta_{3,4} = \frac{\pi}{k_2}, \beta_{4,2} = \frac{\pi}{k_3}.$$

Thus

$$G = \langle 2, 2, 2, k_1, k_2, k_3 \rangle.$$

In dimension two, conditions on the angles of congruent triangles to tessellate $\mathbb{S}^2$ are easily obtained by the Gauss-Bonnet theorem. Applied to the triangle $F_1 \subset \Pi_1 = \mathbb{S}^2$:

$$\int_{F_1} K + \sum_{i=1}^3 \delta_i = 2\pi$$

where K is the Gaussian curvature (equal to 1) and δ_i is the exterior angle of the triangle F_1 . Hence

$$(5.1) \quad \text{area}(F_1) = \pi \left(\sum_{i=1}^3 \frac{1}{k_i} - 1 \right).$$

In particular there is only a finite number of solutions that satisfy the Diophantine inequality

$$\sum_{i=1}^3 \frac{1}{k_i} - 1 > 0$$

with $k_i > 1$, $i = 1, 2, 3$. This yields the following four types of groups (with possible permutations on the integers k_i)

$$\begin{aligned} G_1 &= \langle 2, 2, 2, 2, 2, n \rangle, \quad n \geq 2; & G_2 &= \langle 2, 2, 2, 2, 3, 3 \rangle; \\ G_3 &= \langle 2, 2, 2, 2, 3, 4 \rangle; & G_4 &= \langle 2, 2, 2, 2, 3, 5 \rangle. \end{aligned}$$

All these groups are symmetry groups of regular polyhedra (except for the first one which is reducible to the symmetry group of the $2n$ -gon).

$\mathbb{S}^3$ is then tessellated by two mirror copies through Π of the corresponding regular polyhedra. Each polyhedron in turn is being divided by congruent copies of the tetrahedron T as follows.

Take the center P_0 of the polyhedron together with the center P_1 of a face of the polyhedron. Then draw a triangle on the face, whose vertices are P_1 , one vertex P_2 of the face, and a mid-point P_3 of one of the two edges of the edge adjacent to P_2 . The fundamental tetrahedron T of G is $P_0P_1P_2P_3$. Notice that any of these tetrahedra is quadri-rectangular.

It is convenient to use Schläfli's notation to represent polyhedra, and Coxeter's graphical representation of reflection groups [3].

Schläfli's notation goes as follows. The regular polygon of k edges is denoted by $[k]$; then a regular polyhedron whose bounding figure (i.e., bounding face) is a polygon $[k_1]$ and whose vertex figure (i.e., the set of vertices joined to a given vertex by an edge) is $[k_2]$ is denoted by $[k_1, k_2]$ (for example, the bounding figure and vertex figure of a cube are respectively a square and a triangle; hence Schläfli's symbol of the cube is $[4, 3]$). By induction, an n -dimensional polytope whose bounding figure is $[k_1, k_2, \dots, k_{n-1}]$ and whose vertex figure is $[k_2, \dots, k_n]$ is denoted by $[k_1, \dots, k_n]$.

Coxeter's notation goes as follows. Each dot represents a reflection plane. Two dots are joined by a link marked k if the dihedral angle of the two corresponding planes is π/k . By convention, the unmarked link corresponds to $k = 3$, and two dots are not connected by a direct link if the corresponding planes are perpendicular.

- (1) $\overset{m}{\bullet} \text{---} \bullet \overset{k}{\bullet}$ G_1 is the direct product of 2 dihedral groups, each being the symmetry group of a $[2m]$ and a $[2k]$. This is the group of invariance of Lawson's $\xi_{m-1,k-1}$ surfaces.
- (2) $\bullet \text{---} \bullet \text{---} \bullet$ $G_2 = S_3$ is the symmetry group of the regular tetrahedra $[3, 3]$.
- (3) $\bullet \text{---} \bullet \text{---} \overset{4}{\bullet}$ $G_3 = S_3 \times \mathbb{Z}_2^3$ is the symmetry group of the cube $[4, 3]$ (and its dual, the octahedron $[3, 4]$, or cocube).
- (4) $\bullet \text{---} \bullet \text{---} \overset{5}{\bullet}$ G_4 is the symmetry group of the icosahedron $[3, 5]$ (or its dual, the dodecahedron $[5, 3]$).

The number N of tetrahedron cells that tessellate $\mathbb{S}^3$ is given by equation (5.1). As N is twice the number of triangles that tessellate $\mathbb{S}^2$,

$$N = \frac{8}{\sum_{i=1}^3 \frac{1}{k_i} - 1}.$$

This gives the order of three of the four groups, namely

$$N_2 = 48, N_3 = 96, N_4 = 240.$$

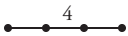
As for G_1 , notice that the first dihedral subgroup subdivides $\mathbb{S}^3$ into $N = 8m$ tetrahedra. Each tetrahedron is subdivided into $k/2$ tetrahedra by the action of the second dihedral subgroup. Hence

$$N_1 = 4mk.$$

This classifies the reducible groups of tessellations by tetrahedra.

5.2. Irreducible group of reflections We follow the classification given in [3]; the situation for dimension four is particularly rich since we obtain the largest possible number of groups, namely five. In each case the fundamental domain T is tri-rectangular and all groups except one are symmetry groups of regular hyperpolyhedra of $\mathbb{R}^4$. The cells of each hyperpolyhedron tessellate in turn the circumscribed hypersphere $\mathbb{S}^3$ by congruent regular polyhedra. Each polyhedron is in turn decomposed by congruent tetrahedra as it is described in the reducible case.

- (1) $\bullet \text{---} \bullet \text{---} \bullet$ $G_5 = \langle 3, 2, 2, 3, 2, 3 \rangle$. It is the symmetry group of the 4-dimensional tetrahedron $T_4 = [3, 3, 3]$. The projection of this regular polyhedron on the circumscribed hypersphere tessellates $\mathbb{S}^3$ by five T_4 , four lying at each vertex. As before, each tetrahedron is divided into 24 T 's totaling $N_5 = 120$ tetrahedra to cover $\mathbb{S}^3$.
- (2) $\bullet \text{---} \bullet \text{---} \overset{4}{\bullet}$ $G_6 = \langle 3, 2, 2, 3, 2, 4 \rangle$ is the symmetry group of the hyper-cube $C_4 = [4, 3, 3]$ or its dual, the cocube, or hyper-octahedra $C_4^* = [3, 3, 4]$; $\mathbb{S}^3$ is tessellated by the cells of C_4 , that comprise eight cubes, four cubes meeting at each of the 16 vertices of C_4 . Dually, $\mathbb{S}^3$ is tessellated by 16 tetrahedra; in both cases $N_6 = 384$.

- (3)  $G_7 = \langle 3, 2, 2, 4, 2, 3 \rangle$ is the symmetry group of the 24-cell $[3, 4, 3]$; $\mathbb{S}^3$ is tessellated by 24 octahedra, six occurring at each of a total of 24 vertices of the 24-cell. Hence $N_7 = 1152$.
- (4)  $G_8 = \langle 3, 2, 2, 3, 2, 5 \rangle$ is the symmetry group of the 120-cell $[3, 3, 5]$ or its dual, the 600-cell. $\mathbb{S}^3$ is tessellated by 120 dodecahedra or 600 octahedra. $N_8 = 14400$.
- (5)  Finally $G_9 = \langle 2, 2, 3, 2, 3, 3 \rangle$; there is a vertex whose 3 adjacent edges have a $\pi/2$ dihedral angle. It is remarkable that G_9 does not correspond to a tessellation of $\mathbb{S}^3$ by regular polyhedra. It does correspond, however, to a symmetry group of a semi-regular polyhedron. This fact is true for any reflection group: given a tessellation of $\mathbb{S}^3$ by tetrahedra, the centers of the in-sphere of each tetrahedron define the vertices of a polyhedra; the adjacent edges join any vertex to its reflected points by the faces of the T 's (four edges to each vertex of equal length) and the configuration at each vertex is identical. G_9 is the symmetry group of a C_4^* -like polyhedron with any other two non congruent tetrahedra. $N_9 = 192$.

[6]'s construction should apply for the tetrahedron $\langle 2, 2, 3, 2, 3, 3 \rangle$ to give a new embedded minimal surface of genus 9 in $\mathbb{S}^3$.

6. SYMMETRIC MINIMAL SURFACES OF LOW GENUS

In Section 4 we obtained a condition on the number of edges which ensures that λ_1 equals two. In this section, by contrast, this condition will be replaced by an upper bound on the genus of Σ . For this purpose we introduce a combinatorial argument which gives a relationship between the number of edges and the genus.

Let Σ be an embedded minimal surface invariant under a reflection group G . Denote by e the number of edges of P (that correspond to the intersection of P with the faces of T), by v the number of vertices of P (intersection of P with the edges of T), and by f the number of components of P . More precisely, denote by v_i the number of vertices belonging to the edges of T whose adjacent faces make an angle of π/k_i for an integer $k_i \geq 2$.

Let L be the number of cells that tessellate $\mathbb{S}^3$ by G . If we denote respectively by V, E, F the total number of vertices, edges and faces of Σ , and by V_i the total number of vertices on Σ corresponding to v_i in P , then

$$Lf = F, \quad Le = 2E, \quad Lv_i = 2k_i V_i, \quad i = 1, \dots, 6.$$

Let g be the genus of Σ . The Euler characteristic of the surface Σ is

$$\chi(\Sigma) = 2 - 2g = F - E + V = Lf - \frac{L}{2}e + \sum_{i=1}^6 v_i \frac{L}{2k_i}.$$

As Σ is connected, $f = 1$; as ∂P is polygonal, $e = v$ and hence

$$(6.1) \quad g = 1 + \frac{L}{2} \left(\frac{v}{2} - 1 - \sum_{i=1}^6 \frac{v_i}{2k_i} \right).$$

Example 6.1. In Karcher-Pinkall-Sterling's examples, the reflection group is one of the nine groups, and P is a simply connected domain with four edges, and three of the four vertices belong to a rectangular edge of T (i.e., their vertex angle is $\pi/2$). If the fourth angle equals π/k , Σ has genus

$$g = 1 + \frac{L}{8} \left(1 - \frac{2}{k} \right).$$

The summation term in (6.1) can be estimated in terms of v . Therefore (6.1) will turn an estimate of g into an estimate of v as follows.

Since $k_i = 2$, we have

$$\sum_{i=1}^6 \frac{v_i}{2k_i} \leq \frac{v}{4}.$$

So if $g < 1 + L/4$, then $v < 6$ and from Theorem 4.2, λ_1 equals two. Thus we have proved the following theorem.

Theorem 6.2. *Let $\Sigma \subset \mathbb{S}^3$ be a compact embedded minimal surface of genus g invariant under a reflection group G of order L that tessellates $\mathbb{S}^3$ into tetrahedra. If $g < 1 + L/4$, then $\lambda_1 = 2$.*

Example 6.3. Let G be the reflection group generated by four mutually orthogonal spheres. If P has at least four vertices, then

$$g = 1 + 8 \left(\frac{v}{4} - 1 \right) = 2v - 7$$

and λ_1 equals two for any Σ of genus less than five. The Clifford torus is an example of G -invariant minimal surfaces.

Acknowledgements Work of J. C. was supported in part by KOSEF R01-2005-000-10077-0 and KICOS-MOST 2006-08745.

Work of M. S. was supported in part by the Korean MOST and the French Ministère des Affaires Étrangères STAR-12976XL.

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KEY WORDS AND PHRASES: Laplacian, eigenvalue, minimal surface.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 49Q05, 35J05, 53A10.

Received: January 4th, 2007; revised: November 12th, 2008.

Article electronically published on February 23rd, 2009.