CHARACTERIZING THE SPHERE IN TERMS OF VOLUME, AREA, EIGENVALUE AND ENERGY

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The sphere is probably the most beautiful geometric object in mathematics. As such, $S^{n-1} \subset \mathbf{R}^n$ can be characterized in many ways. Let us mention some of them:
1) Among all closed hypersurfaces in $\mathbf{R}^n$ of fixed volume, the sphere is the unique one that encloses the largest volume. (Steiner, 1842)
2) If $S \subset \mathbf{R}^3$ is a convex surface of area A, with mean curvature H and enclosed volume V, then $A^2/3V \geq \int_S H \geq 2\sqrt{\pi A}$, where equality holds if and only if S is a sphere. (Minkowski)
3) If S is an embedded hypersurface in $\mathbf{R}^n$ with enclosed volume V and mean curvature $H \neq 0$, then $\int_S 1/H \geq nV$, and equality holds if and only if S is a sphere. (Ros, 1987)
4) Every compact embedded hypersurface of constant mean curvature in $\mathbf{R}^n$ is a sphere. (Alexandrov, 1956)
5) Every constant mean curvature immersion of S^2 in $\mathbf{R}^3$ is the standard sphere. (Hopf, 1951)
6) If S is a compact embedded surface of constant Gaussian curvature in $\mathbf{R}^3$, then S is a shpere. (Liebmann, 1899)
7) If S is a compact embedded hypersurface of constant scalar curvature in $\mathbf{R}^n$, then S is a sphere. (Ros, 1987).
8) The ball is the only equilibrium figure for a self-gravitating liquid at rest. (Carleman, 1919)
9) Among all domains $D \subset \mathbf{R}^n$ with given volume, the ball provides the smallest value of the first nonzero eigenvalue λ_1 of the Laplacian for functions vanishing on ∂D. (Krahn, 1926)
10) Among all compact embedded hypersurfaces $S \subset \mathbf{R}^n$ enclosing fixed volume, the sphere is the unique one that maximizes $\lambda_1(S)/Volume(S)^2$, where $\lambda_1(S)$ is the first nonzero eigenvalue of the Laplacian on S. (Chavel, 1978)
11) Let S be a two-dimensional Riemannian manifold homeomorphic to S^2. If $\lambda_1, \lambda_2, \lambda_3$ are the first three nontrivial eigenvalues, then $1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 \geq 3Area(S)/8\pi$, and equality holds if and only if S is the standard sphere. (Hersch, 1970)
12) Among all compact Riemannian manifolds M^n with Ricci curvature satisfying $Ric(v, v) \geq (n-1)|v|^2$ for all $v \in TM$, only the standard sphere has the maximum volume. (Bishop, 1964)
13) Among all compact Riemannian manifolds M^n with Ricci curvature satisfying $Ric(v, v) \geq (n-1)|v|^2$ for all $v \in TM$, the minimum value of $\lambda_1(M)$ is attained by the standard sphere only. (Obata, 1962)

In this note we will review the three characterizing properties of the sphere, following [B], [C], [Ch]: the isoperimetric inequality, the first eigenvalue of the Laplacian, and the isoenergy inequality of a

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harmonic map. Actually these geometric inequalities will be studied in more general setting: in a simply connected nonpositively curved Riemannian manifold M^n rather than in $\mathbf{R}^n$.

1. The isoperimetric inequality

The circle is uniquely characterized by the property that among all simple closed plane curves of given length L, the circle of circumference L encloses maximum area. This property is most succinctly expressed in the isoperimetric inequality

$$4\pi A \leq L^2, \tag{1}$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle. Similarly the isoperimetric problem in $\mathbf{R}^n$ is to minimize the surface area among all domains having given volume, or equivalently, maximize the volume among all domains whose boundary surfaces have fixed volume. The solution in both cases is that the unique extremal is the domain bounded by a sphere. Steiner proved the corresponding isoperimetric inequality

$$n^n \omega_n Volume(D)^{n-1} \ \leq \ Volume(\partial D)^n, \tag{2}$$

where ω_n is the volume of a unit ball in $\mathbf{R}^n$ and equality holds if and only if D is a ball.

Having settled the isoperimetric problem in $\mathbf{R}^n$, it is natural to consider the isoperimetric inequality on a curved space. Does (1) hold for surfaces in $\mathbf{R}^3$ with nonpositive Gaussian curvature? Does (2) hold for domains in a nonpositively curved n-dimensional Riemannian manifold M^n? The curvature assumption is necessary because one can easily find a positively curved space in which neither (1) nor (2) holds. In 1933 Beckenbach and Radó [BR] showed that the necessary and sufficient condition for $4\pi A \leq L^2$ to hlod for all simply connected domains on a surface S is that S be nonpositively curved. Recently Kleiner [K] and Croke [Cr] obtained (2) for domains in simply connected M^3 and M^4, respectively.

Kleiner's and Croke's results are rather complicated for this note. So in this section let us study just the two-dimensional case.

Theorem 1. *If D is a simply connected domain in a nonpositively curved surface with $A = Area(D)$, $L = Length(\partial D)$, then $4\pi A \leq L^2$.*

The condition $\pi_1(D) = 0$ is necessary because a long cylinder may have arbitrarily large area with fixed boundary length. Here we will give two proofs of Theorem 1. One is a geometric proof given in [BZ], and the other is more analytic in nature.

Geometric Proof (Outline). Consider the following special case. Let D be a domain with connected smooth boundary all of whose parallel curves $l_t = \{x \in D : dist(x, \partial D) = t\}$, except the furthest one l_r, are smooth simple closed curves. Denote by $A(t)$ the area of the set $D_t = \{x \in D : dist(x, \partial D) < t\}$. Under these assumptions $A'(t) = l(t)$, where $l(t)$ is the length of l_t. But the first variation formula for $l(t)$ says

$$\frac{dl}{dt} \ = \ -\int_{l_t} k ds$$

where k is the geodesic curvature of l_t with respect to the inward normal to l_t. Assume that $A''(t) = l'(t)$

also exists and is continuous in [0,r). By the Gauss-Bonnet formula for D_t, we have

$$A''(t) \;=\; l'(t) \;=\; -\int_{D_t} KdA \;-\; \int_{\partial D} kds.$$

Since $K \leq 0$, the Gauss-Bonnet formula for D implies

$$A''(t) \;\leq\; -\int_{D} KdA \;-\; \int_{\partial D} kds \;=\; -\,2\pi. \tag{3}$$

Multiplying (3) by $2A'(t) \geq 0$ and integrating from 0 to $r = \sup\{t : l_t \neq \phi\}$, we get

$$A'(r)^2 - A'(0)^2 \;\leq\; -\,4\pi[A(r) - A(0)].$$

Since $A'(0) = L,\;\; A(0) = 0,\;\; A(r) = A$, this yields

$$L^2 - 4\pi A \;\geq\; A'(r)^2 \;\geq\; 0,$$

which is the desired inequality.

As a rule, the assumptions on the structure of the parallel curves l_t and the differentiability of $A(t)$ do not hold. Nevertheless it is possible to obtain a rigorous proof along these lines. Such a proof is presented in [BZ, pp.20-27]. In order to overcome the technical difficulties, the argument will be carried out for polyhedra. The general case follows by passing to the limit.

Analytic Proof. Let x, y be isothermal coordinates on D. Then the metric and the Gaussian curvature K of D can be written as

$$ds^2 \;=\; e^{2\lambda}(dx^2 + dy^2), \quad K = \;-\,e^{-2\lambda}\triangle\lambda. \tag{4}$$

By the curvature assumption we have

$$\triangle\lambda \;\geq\; 0. \tag{5}$$

Let h be the solution of the Dirichlet problem

$$\triangle h = 0, \quad h = \lambda \; on \; \partial D, \tag{6}$$

and let $\tilde{ds^2}$ be the new metric on D defined by

$$\tilde{ds^2} \;=\; e^{2h}(dx^2 + dy^2).$$

Then, with respect to $\tilde{ds^2}$, we define $\tilde{A} = Area(D),\;\; \tilde{L} = Length(\partial D)$. Note that the boundary condition (6) implies $\tilde{L} = L$. From (5), (6) and the maximum principle we get $\tilde{A} \geq A$. But we see from (4) and (6) that $\tilde{ds^2}$ is a flat metric. Hence

$$4\pi A \;\leq\; 4\pi\tilde{A} \;\leq\; \tilde{L}^2 \;=\; L^2.$$

2. The first eigenvalue of the Laplacian

Let us start this section with a well-known result of partial differential equations:

Theorem A. *If*

$$\inf_{f\in\mathcal{F}} \frac{\int_D |\nabla f|^2}{\int_D f^2} \;=\; \lambda_1, \tag{7}$$

where $\mathcal{F}$ is the set of piecewise smooth functions in a domain $D \subset \mathbf{R}^2$ vanishing on the boundary, then λ_1 is the smallest eigenvalue of the equation

$$\triangle f + \lambda f \;=\; 0 \tag{8}$$

for solutions having zero boundary values.

The quotient on the left of (7) is called the *Rayleigh quotient*. Unlike the isoperimetric inequalities (1) and (2), it is not dimensionally invariant. If one applies a homothetic map to the domain D, multiplying distances by a factor h, then the left hand side of (7) is divided by h^2. Thus, to understand the dependence of λ_1 on the domain D, it is sufficient to normalize by fixing for example the area of D. One then has the result

Theorem B. *Among all domains $D \subset \mathbf{R}^2$ having fixed area, the left side of* (7) *attains a minimum if and only if D is a circular disk.*

The interest in this theorem derives from the physical interpretation of the quantity λ_1. (8) arises from separating space and time variables in the wave equation. If a homogeneously stretched membrane has the shape of the domain D, and is attached at the boundary, then solutions of (8) with zero boundary values represent the amplitude of vibrations of the membrane with frequency $\sqrt{\lambda}$. The eigenvalues λ_n are thus the squares of the frequencies of free vibration of the membrane, and the quantity λ_1 given by (7) corresponds to the lowest frequency, or the *fundamental tone* of the membrane.

The first statement of Theorem B is due to Rayleigh in his fundamental treatise *The theory of sound* (1877). He writes "If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (or the principal tone) is the gravest possible, and this form can be no other than the circle." By way of evidence, he offers a variational argument showing that the first variation of λ_1 is positive under variations that start with a circular domain and vary it keeping the area constant. He also lists a number of special domains for which λ_1 can be computed, such as various rectangles, triangles, and circular sectors, and it seems apparent that the more the domains deviate from circularity, the higher the value of λ_1.

The first actual proofs of theorem B were given by Faber (1923) and Krahn (1925). Its proof makes use of the isoperimetric inequality $4\pi A \leq L^2$ together with the technique of symmetrization. An extension of Theorem B to a higher dimensional euclidean space is also possible. The Rayleigh quotient $\inf_{f\in\mathcal{F}} \int_D |\nabla f|^2 / \int_D f^2 = \lambda_1$ again gives the lowest eigenvalue of the problem $\triangle f + \lambda f = 0$ in $D \subset \mathbf{R}^n$, $f = 0$ on ∂D. Krahn showed by means of the n-dimensional isoperimetric inequality (2), that among all domains D with given volume, the ball provides the smallest value of λ_1.

While Krahn's result gives a lower bound for $\lambda_1(D)$, Chavel's theorem [C] gives an upper bound of $\lambda_1(\partial D)$. In this section we discuss Chavel's result for a domain case:

Theorem 2 [C]. *Let D be a domain with connected boundary in an n-dimensional complete simply connected nonpositively curved Riemannian manifold M. Let V denote the volume of D, A the volume of*

∂D, and λ_1 the first nonzero eigenvalue of the Laplacian on ∂D. Then

$$\lambda_1 \;\leq\; \frac{n-1}{n^2}\frac{A^2}{V^2}, \tag{9}$$

where equality holds if and only if ∂D is a sphere in $\mathbf{R}^n$.

For the proof of Theorem 2 we need a suitable coordinate system on M. Let $p \in M$, $\{e_1, ..., e_n\}$ be an orthonormal basis of the tangent space of M at p, T_pM, and $y : M \to \mathbf{R}^n$ the Riemannian normal coordinates on M determined by $(p; e_1, ..., e_n)$. Our assumptions concerning M imply by the Cartan-Hadamard theorem that y is indeed defined on all of M and is a diffeomorphism of M onto $\mathbf{R}^n$. It is standard that geodesics emanating from p map onto rays emanating from the origin of $\mathbf{R}^n$.

Lemma 1. *We may choose p, $\{e_1, ..., e_n\}$ so that the respective coordinate functions $y^i : M \to \mathbf{R}$, $i = 1, 2, ..., n$, of $y : M \to \mathbf{R}^n$ satisfy*

$$\int_{\partial D} y^i \;=\; 0. \tag{10}$$

Proof. Parallel translate the frame $\{e_1, ..., e_n\}$ along every geodesic emanating from p and thereby obtain a differentiable orthonormal frame field $\{E_1, ..., E_n\}$ on M. Let $y_q : M \to \mathbf{R}^n$ denote the Riemann normal coordinates of M determined by $\{E_1, ..., E_n\}$ at q, and let $(y_q)^i, i = 1, ..., n$ be the coordinate functions of y_q. Then

$$Y(q) \;=\; \sum_{i=1}^{n} \left[\int_{\partial D} (y_q)^i\right] E_i(q)$$

is a continuous vector field on M. If we restrict Y to a geodesic ball B containing D then the convexity of B implies that on the boundary of B, Y points into B. (You can easily show this fact by taking B and $Y(q)$ to T_qM under exp_q^{-1} and by using the fact that T_qM is euclidean.) The Brouwer fixed point theorem then implies that Y has zero on B.

So we may assume that p, $\{e_1, ..., e_n\}$ actually satisfies (10). Let X be the vector field on M given by

$$X \;=\; \sum_{i=1}^{n} y^i \frac{\partial}{\partial y^i}$$

where $\{\frac{\partial}{\partial y^i} : i = 1, ...n\}$ is the natural basis of tangent spaces associated with the coordinate chart y. Of course if M is $\mathbf{R}^n$ with its usual flat metric then X is naturally identified with the position vector. Next set $\xi = X|_D$, i.e. the restriction of X to D. Then standard arguments using the Rauch comparison theorem [HS, p.721] imply

$$n \;\leq\; \text{div}\xi. \tag{11}$$

Thus (11) implies

$$\begin{aligned} nV \;&\leq\; \int_{\partial D} < \xi, \nu > \;\leq\; \int_{\partial D} |\xi| \;\leq\; A^{1/2}\left[\int_{\partial D} |\xi|^2\right]^{1/2} \\ &=\; A^{1/2}\left[\int_{\partial D} \sum_i (y^i)^2\right]^{1/2} \;\leq\; (A/\lambda_1)^{1/2}\left[\int_{\partial D} \sum_i |\text{grad}_{\partial D} y^i|^2\right]^{1/2}. \end{aligned}$$

The expression $\mathrm{grad}_{\partial D} y^i$ denotes the gradient of $(y^i|\partial D)$ in ∂D. The last inequality is, combined with (10), Rayleigh's characterization of the first nonzero eigenvalue λ_1 of the Laplacian on ∂D.

It remains to verify the estimate

$$\sum_i |\mathrm{grad}_{\partial D} y^i|^2 \ \leq \ n-1$$

on all of ∂D. Let $q \in \partial D$, $u : G \to \mathbf{R}^{n-1}$ be a coordinate chart on ∂D about q, $\{\frac{\partial}{\partial u^\alpha} : \alpha = 1, ..., n-1\}$ the natural basis of tangent spaces to ∂D at points of G such that $\{\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^{n-1}}\}$ is orthonormal at q. Note now that the exponential map $\exp : T_p M \to M$ preserves length in the radial direction and does *not decrease* length in any orthogonal direction. Thus at q

$$\begin{aligned} \sum_{i=1}^{n} |\mathrm{grad}_{\partial D} y^i|^2 \ &= \ \sum_{i=1}^{n}\sum_{\alpha=1}^{n-1} \left(\frac{\partial y^i}{\partial u^\alpha}\right)^2 \ = \ \sum_{\alpha=1}^{n-1} \left|(d\exp^{-1})_q\left(\frac{\partial}{\partial u^\alpha}\right)\right|^2 \\ &\leq \ \sum_{\alpha=1}^{n-1} \left|\frac{\partial}{\partial u^\alpha}\right|^2 \ = \ n-1, \end{aligned}$$

and (9) is proved. If we have equality in (9) then $|\xi|$ is constant on ∂D from which one concludes that ∂D is a geodesic sphere bounding the geodesic ball D. Furthermore $\mathrm{div}\xi = n$ on all of D. But this in turn implies by [BC, pp.253-257] that D is isometric to a ball in $\mathbf{R}^n$.

3. The isoenergy inequality of a harmonic map

Let us first introduce harmonic maps. Let (M^n, g),(N^k, h) be Riemannian manifolds and let u be a C^1 map from M^n to N^k. For metrics g and h, we use the following notations:

$$g \ = \ \sum g_{\alpha\beta}(x) dx^\alpha dx^\beta, \quad h \ = \ \sum h_{ij}(u(x)) du^i du^j.$$

The pull-back of h, $u^*(h)$ is a symmetric quadratic form such that

$$u^*(h) \ = \ \sum_{\alpha,\beta}\left(\sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}\right) dx^\alpha dx^\beta.$$

By finding orthonormal vector fields and their dual 1-forms $\omega^1, ..., \omega^n$, we can diagonalize $u^*(h)$ such that $u^*(h) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega^\alpha)^2$. Then the symmetric functions of $\lambda_1, ..., \lambda_n$ become invariants. The most interesting one of them is the trace:

$$|du|^2 := \mathrm{Tr}_g(u^* h) \ = \ \sum_{\alpha=1}^{n} \lambda_\alpha.$$

$|du|^2$ is called the *energy density* of u and its coordinate representation is given by

$$|du|^2 \ = \ \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}.$$

Note that the energy density of u is independent of the choice of a coordinate system on M^n.

We define the *energy functional* $E(u)$ by

$$E(u) \ = \ \int_M |du|^2$$

Then the critical points of E in the space of maps are called *harmonic maps.*

Example 1) If $N^k = \mathbf{R}^k$ and $h_{ij} = \delta_{ij}$, then

$$E(u) \;=\; \sum_{i=1}^{k} \int_M |\nabla u^i|^2.$$

Hence u is harmonic if and only if $\triangle_M u^i = 0, i = 1, ..., k$, that is, the component functions are harmonic.

Example 2) If M=[0,1] (in other words, if u is a curve in N), then the energy of u is $E(u) = \int_o^1 ||du/dt||^2 dt$ and the critical points of E are geodesics with constant speed parametrization. Note that another energy functional $\bar{E}(u) = \int_0^1 ||du/dt||dt$ gives us the same critical points, geodesics. However, these geodesics need not have constant speed parametrization because $\bar{E}(u)$ is independent of parametrization.

Suppose that N^k is isometrically embedded in $\mathbf{R}^m$. We look at a bounded map $u : M \to N$ whose first derivatives are in L^2; such a map is thought of as a map $u = (u^1, \cdot\cdot\cdot, u^m) : M \to \mathbf{R}^m$ having image almost everywhere in N. Then

$$E(u) \;=\; \sum_{i=1}^{m} \int_M |\nabla u^i|^2.$$

A nice way of getting harmonic maps is to look for extremals of E with $u(x) \in N$ almost everywhere. In this way we get the *harmonic map equation*

$$\triangle u^i - g^{\alpha\beta} A^i_{u(x)} \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta} \right) \;=\; 0, \quad i = 1, ..., m, \tag{12}$$

where A is the second fundamental form of N defined by $A_u(X, Y) = (D_X Y)^\perp$. Since the second term of (12) is perpendicular to $T_u N$, we can say that $u \in C^2$ is harmonic if and only if

$$\triangle_M \; u \;\perp\; T_u N. \tag{13}$$

In local coordinates on N (12) becomes

$$\triangle u^i + g^{\alpha\beta} \Gamma^i_{jl}(u(x)) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^l}{\partial x^\beta} \;=\; 0, \quad i = 1, ..., m,$$

where Γ^i_{jk} is the Christoffel symbols of N.

A harmonic map u is stationary if its energy is critical with respect to variations of the type $u \circ F_t$, where $F_t : M \to M$ is a smooth path of diffeomorphisms of M fixing the boundary. It can be shown that a C^0 harmonic map is stationary.

Now let us discuss the isoenergy inequality for harmonic maps. Roughly speaking, isoenergy inequality is a version of isoperimetric inequality for the energy of a harmonic map. Consider a smooth harmonic map u from a closed unit ball $\bar{B} \subset \mathbf{R}^n$ to $\mathbf{R}^m, n \geq 2$. Define $E(u)$ and $E(u|_{\partial B})$ to be the energy of the map u and the energy of the restriction of u to ∂B, respectively. Then is there any relationship between $E(u)$ and $E(u|_{\partial B})$ that resembles the isoperimetric inequality? We will answer this question affirmatively; we obtain a relationship in a sharp form, called the *isoenergy inequality*, for a nonpositively curved target manifold N as well as for $\mathbf{R}^m$.

Theorem 3. *Suppose that u is a smooth harmonic map from $\bar{B} \subset \mathbf{R}^n, n \geq 2$, into $\mathbf{R}^m$. Then we have the isoenergy inequality*

$$(n-1)E(u) \ \leq \ E(u|_{\partial B}),$$

where equality holds if and only if u is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Proof. Stationary harmonic maps satisfy the monotonicity property for the scale invariant energy in balls. Let $B_\rho = \{x \in \mathbf{R}^n : |x| < \rho\}$ and $B = B_1$. Suppose $u : B_{1+\epsilon} \to N$, $\epsilon > 0$, is a stationary harmonic map. Then the monotonicity formula [Sc] says

$$\rho^{2-n} \int_{B_\rho} |\nabla u|^2 - \sigma^{2-n} \int_{B_\sigma} |\nabla u|^2 \ = \ 2 \int_{B_\rho - B_\sigma} r^{2-n} \left| \frac{\partial u}{\partial r} \right|^2, \quad r = |x|,$$

for $0 < \sigma < \rho < 1+\epsilon$. Noting that $\int_{\partial B_\rho} f = \frac{d}{d\rho} \int_{B_\rho} f$ for almost all ρ, differentiate the formula with respect to ρ and set $\rho = 1$. Then we get an equivalent form of the monotonicity formula:

$$(n-2) \int_B |\nabla u|^2 \ = \ \int_{\partial B} \left(|\nabla u|^2 - 2 \left| \frac{\partial u}{\partial r} \right|^2 \right). \tag{14}$$

Let $\overline{\nabla} u^i$ denote the gradient of u^i on ∂B. Observe that

$$E(u|_{\partial B}) \ = \ \int_{\partial B} \sum_i |\overline{\nabla} u^i|^2 \ = \ \int_{\partial B} \left(|\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right).$$

It follows from (14) that

$$(n-2)E(u) \ = \ E(u|_{\partial B}) - \int_{\partial B} \left| \frac{\partial u}{\partial r} \right|^2, \tag{15}$$

which gives

$$(n-2)E(u) \ \leq \ E(u|_{\partial B}),$$

where equality can be attained if $N = S^{n-1} \subset \mathbf{R}^n$ and $u(x) = x/|x|$. Now (13) implies

$$\triangle u^i \ = \ 0, \quad i = 1, \cdots, m.$$

Hence

$$\begin{aligned} E(u) \ &= \ \frac{1}{2} \int_B \triangle \sum_i (u^i)^2 \ = \ \int_{\partial B} \sum_i u^i \frac{\partial u^i}{\partial r} \\ &\leq \ \left[\int_{\partial B} \sum_i (u^i)^2 \right]^{1/2} \left[\int_{\partial B} \left| \frac{\partial u}{\partial r} \right|^2 \right]^{1/2} \ = \ (*). \end{aligned}$$

Here, without loss of generality, let us assume

$$\int_{\partial B} u^i \ = \ 0, \quad i = 1, \cdots, m.$$

Using (15) and the fact that $n-1$ is the first eigenvalue of the Laplacian on ∂B, one sees that

$$(*) \ \leq \ \left[\frac{1}{n-1} \int_{\partial B} \sum_i |\overline{\nabla} u^i|^2 \right]^{1/2} \left[E(u|_{\partial B}) - (n-2)E(u) \right]^{1/2}.$$

Hence by combining the inequalities above one gets

$$E(u)^2 \;\le\; \frac{1}{n-1} E(u|_{\partial B})\left[E(u|_{\partial B}) - (n-2)E(u)\right],$$

which gives the desired isoenergy inequality. Moreover equality holds if and only if u^i is a constant multiple of $\partial u^i/\partial r$ and

$$\triangle_{\partial B}\, u^i + (n-1)u^i \;=\; 0, \quad i = 1, \cdots, m,$$

which holds if and only if u is a linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Theorem 4. *If u is a smooth harmonic map from $\bar{B} \subset \mathbf{R}^n, n \ge 2$, to N of nonpositive curvature, then*

$$(n-1)E(u) \;\le\; E(u|_{\partial B}).$$

Proof. The Bochner formula [EL] says that if $u : M^n \to N^k$ is harmonic then

$$\begin{aligned} \frac{1}{2}\triangle|\nabla u|^2 \;=\; ||\nabla' du||^2 \;&-\; \sum_{\alpha,\beta} R_N(u_*e_\alpha, u_*e_\beta, u_*e_\alpha, u_*e_\beta) \\ &+\; \sum_i Ric_M(u^*\theta^i, u^*\theta^i) \end{aligned} \tag{16}$$

where ∇' is the pullback connection from TN, $e_1, \cdots, e_n$ is an orthonormal basis for TM and $\theta^1, \cdots, \theta^k$ is orthonormal for T^*N. Hence for $M = \bar{B}$ and N nonpositively curved, $|\nabla u|^2$ is subharmonic. Since the mean value of a subharmonic function on a sphere of radius r centered at the origin is monotonically nondecreasing as a function of r, one can deduce that

$$\frac{E(u)}{\omega_n} \;\le\; \frac{1}{n\omega_n}\int_{\partial B} |\nabla u|^2 \;=\; \frac{1}{n\omega_n}\left[(n-2)E(u) + 2\int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2\right],$$

where equality follows from (14). So

$$E(u) \;\le\; \int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2. \tag{17}$$

Then adding (15) to (17) gives the isoenergy inequality.

Remark 1. In case u is a harmonic map from a ball B_ρ of radius ρ into N of nonpositive curvature, we obviously have

$$(n-1)E(u) \;\le\; \rho E(u|_{\partial B_\rho}).$$

4. An eigenvalue estimate

Given an $(n-1)-$dimensional minimal submanifold Σ in $S^k \subset \mathbf{R}^{k+1}$, $O ⋇ \Sigma$ is the cone from the origin O of $\mathbf{R}^{k+1}$ over Σ, that is, the union of the unit line segments from O to the points of Σ. It is well known that $O ⋇ \Sigma$ is an n-dimensional minimal submanifold of $\mathbf{R}^{k+1}$. In this section we want to consider the isoenergy inequality of a harmonic map u from $O ⋇ \Sigma$ into $\mathbf{R}^m$. In the proof of the isoenergy inequality of Theorem 3 we used the fact that $n-1$ is the first eigenvalue of the Laplacian on S^{n-1}. However, we do not know the exact value of the first eigenvalue $\lambda_1(\Sigma)$ of the minimal submanifold $\Sigma \subset S^k$. Therefore, instead of deriving an isoenergy inequality, we obtain an upper bound of the first eigenvalue in terms of the energy of the harmonic map u and the energy of $u|_\Sigma$. To do this, we need the following monotonicity on $O \circledast \Sigma$.

Lemma 2. *Let Σ be an $(n-1)$-dimensional submanifold of $S^k \subset \mathbf{R}^{k+1}$. If u is a harmonic map from $O \circledast \Sigma$ into $\mathbf{R}^m$ which is smooth up to the boundary Σ, then*

$$(n-2)\int_{O \circledast \Sigma} |\nabla u|^2 \;=\; \int_\Sigma \left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r}\right|^2 \right).$$

Proof. Let $\Sigma_\rho = \{x \in O \circledast \Sigma : |x| < \rho\}$. Note that the quantity

$$\Theta(\rho) \;=\; \rho^{2-n}\int_{\Sigma_\rho} |\nabla u|^2 \tag{18}$$

is invariant under scaling. More precisely, if we denote $u_\rho(x) = u(\rho x)$, then

$$\Theta(\rho) \;=\; \int_{\Sigma_1} |\nabla u_\rho|^2.$$

So at $\rho = 1$ we see that

$$\frac{d}{d\rho}\Theta(\rho) \;=\; 2\int_{\Sigma_1} \nabla u \cdot \nabla \frac{du_\rho}{d\rho} \;=\; 2\int_\Sigma (x \cdot \nabla u)\cdot \frac{du_\rho}{d\rho} - 2\int_{\Sigma_1} \Delta u \cdot \frac{du_\rho}{d\rho}.$$

Since $x \cdot \nabla u = \frac{du_\rho}{d\rho} = \frac{\partial u}{\partial r}$ on Σ and $\Delta u = 0$ on Σ_1 we get

$$\left[\frac{d}{d\rho}\Theta(\rho)\right]_{\rho=1} \;=\; 2\int_\Sigma \left|\frac{\partial u}{\partial r}\right|^2. \tag{19}$$

Thus (18) and (19) complete the proof.

Theorem 5. *Let Σ be an $(n-1)$-dimensional submanifold of $S^k \subset \mathbf{R}^{k+1}$ and let $u : O \circledast \Sigma \to \mathbf{R}^m$ be a harmonic map which is smooth up to the boundary Σ. Then*

$$\lambda_1(\Sigma) \;\le\; \frac{E(u|_\Sigma)}{E(u)}\left(\frac{E(u|_\Sigma)}{E(u)} - n + 2\right). \tag{20}$$

Proof. Basically we follow the proof of Theorem 3 and use Lemma 2. So

$$\begin{aligned} E(u) \;&=\; \frac{1}{2}\int_{O \circledast \Sigma} \triangle \sum_i (u^i)^2 \;=\; \int_\Sigma \sum_i u^i \frac{\partial u^i}{\partial r} \\ &\le\; \left[\int_\Sigma \sum_i (u^i)^2\right]^{1/2} \left[\int_\Sigma \left|\frac{\partial u}{\partial r}\right|^2\right]^{1/2} \\ &\le\; \left[\frac{1}{\lambda_1(\Sigma)}\int_\Sigma \sum_i |\overline{\nabla} u^i|^2\right]^{1/2} [E(u|_\Sigma) - (n-2)E(u)]^{1/2}. \end{aligned}$$

Hence

$$\lambda_1(\Sigma)E(u)^2 + (n-2)E(u)E(u|_\Sigma) - E(u|_\Sigma)^2 \;\le\; 0,$$

which gives (20).

Corollary. *Let Σ be an embedded minimal hypersurface in S^n. If $u : O ⋇ \Sigma \to \mathbf{R}^m$ is harmonic and smooth up to the boundary, then*

$$(n-1)E(u) \ \le \ (-n+2+\sqrt{n^2-2n+2})E(u|_\Sigma).$$

Proof. Combine (20) with Choi-Wang's estimate [CW]:

$$\frac{n-1}{2} \ \le \ \lambda_1(\Sigma).$$

Remark 2. In a sense (20) is similar to Chavel's estimate (9):

$$\lambda_1(S) \ \le \ \frac{(n-1)}{n^2}\frac{A^2}{V^2}.$$

Chavel's estimate, when applied to $S = O ⋇ \Sigma \subset M = \mathbf{R}^{n+1}$ with Σ minimal in $S^n \subset \mathbf{R}^{n+1}$, implies $\lambda_1(\Sigma) \le n-1$, which is nothing new. Also our estimate (20), when applied to $u =$ identity, draws the same conclusion because $(n-1)E(\mathrm{id}) = E(\mathrm{id}|_\Sigma)$. But should there exist a harmonic map $u : O ⋇ \Sigma \to \mathbf{R}^m$ satisfying

$$(n-1)E(u) \ > \ E(u|_\Sigma), \tag{21}$$

then one can conclude from (20) that

$$\lambda_1(\Sigma) \ < \ n-1, \tag{22}$$

which has not been disproved yet. In fact, since $Ric_{O ⋇ \Sigma}$ is nonpositive, one can deduce from the Bochner formula (16) that $|\nabla u|^2$ is strictly superharmonic provided

$$||\nabla' du||^2 + \sum_i Ric_{O ⋇ \Sigma}(u^*\theta^i, u^*\theta^i) \ < \ 0.$$

Then the argument of the proof of Theorem 4 implies (21).

Remark 4. Let Σ be an $(n-1)$-dimensional embedded minimal hypersurface of S^n. A map $u : \Sigma \to S^n \subset \mathbf{R}^{n+1}$ is said to be *balanced* if $\int_\Sigma u$ equals the zero vector in $\mathbf{R}^{n+1}$. *Does there exist a balanced energy minimizing map of Σ into S^n which is different from the identity? If it exists, is its energy smaller than that of the identity?* If this is the case, then one gets (22) since

$$\begin{aligned} \lambda_1(\Sigma) Volume(\Sigma) \ &= \ \lambda_1(\Sigma)\int_\Sigma |u|^2 \ \le \ \int_\Sigma |\nabla u|^2 \\ &< \ E(\mathrm{id}) \ = \ (n-1)Volume(\Sigma). \end{aligned}$$

For a minimal surface of codimension ≥ 2, that case really occurs. Let $\psi : S^2(\sqrt{3}) \to S^4$ be the two-to-one isometric minimal immersion whose image is the Veronese surface V diffeomorphic to the projective plane. Let $\tilde{V}$ be the double covering of V and define $u : \tilde{V} \to S^4$ by $u = \frac{1}{\sqrt{3}}\psi^{-1}$. Then u is a balanced harmonic map satisfying

$$E(u) \ = \ \frac{1}{3}E(\mathrm{id}).$$

Indeed $\lambda_1(\tilde{V}) = 2/3$.

References

[BR] E. F. Beckenbach and T. Radó, *Subharmonic functions and surfaces of negative curvature*, Trans. Amer. Math. Soc, 35(1933), 662–674.

[BC] R. L. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

[B] G. Bol, *Isoperimetrische Ungleichung für Bereiche auf Flächen*, Jber. Deutsch. Math.-Verein. 51(1941), 219–257.

[BZ] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der mathematischen Wissenschaften 285, Springer, New York, 1988.

[C] I. Chavel, *On A. Hurwitz' method in isoperimetric inequalities*, Proc. Amer. Math. Soc. 71(1978), 275–279.

[Ch] J. Choe, *The isoenergy inequality for a harmonic map*, preprint.

[CW] H.I. Choi and A.-N. Wang, *A first eigenvalue estimate for minimal hypersurfaces*, J. Diff. Geom. 18(1983), 559–562.

[Cr] C. Croke, *A sharp four dimensional isoperimetric inequality*, Comment. Math. Helv. 59(1984), 187–192.

[EL] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10(1978), 1–68.

[ES] J. Eells and J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86(1964), 109–160.

[H] R. Hamilton, *Harmonic maps of manifolds with boundary*, Lecture Notes 471, Springer, 1975.

[HS] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. 29(1974), 715–727.

[K] B. Kleiner, *An isoperimetric comparison theorem*, Invent. math. 108(1992), 37–47.

[Sc] R. Schoen, *Analytic aspects of the harmonic map problem*, Math. Sci. Res. Inst. Publ. Vol. 2, Springer, Berlin, 1984, 321–358.

[S] J. Steiner, *Sur le maximum et le minimum des figures dans le plan sur la sphère et dans l'espace en général*, J. Reine Angew. Math. 24(1842), 93–152.

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