The History of the Plateau Problem Jaigyoung Choe, Postech

Introduction and Historical Remarks

Surfaces of vanishing mean curvature are called minimal surfaces. Minimal surfaces can be physically realized as soap films. If we dip a thin wire frame consisting of one closed curve in a soap solution, and then skillfully remove it, we obtain one soap film bounded by the wire. The shape of this film approximates with great accuracy that of a minimal surface. While the influence of gravity can be neglected because of the vanishingly small mass, the surface tension in the soap film causes the film to take a form in which its surface area is an absolute, or at least a relative, minimum: in any small deformation, which can, for example, be produced by carefully blowing against the soap film, the surface area increases.

Since the fundamental research by the Belgian physicist J. Plateau (1801-1883), whose extensive studies of the phenomenon of surface tension and experiments with soap films and soap bubbles became famous in the middle of the nineteenth century, it has become customary to call the problem of determining a minimal surface of the type of the disk, bounded by a prescribed Jordan curve, the *Plateau problem*. (After 1843, owing to blindness Plateau was no longer able to admire these films himself—in 1829, he had once looked directly into the sun for more than 25 seconds.)

Indeed, Plateau not only described how to realize the minimal surfaces defined by mathematical equations; he also clearly stated what the results of geometry as well as his own experiments appeared to confirm, namely that for any given contour there is always a minimal surface bounded in its entirety by this contour. Naturally, experimental evidence can never replace a mathematical existence proof. But this existence proof confronts mathematicians with extraordinarily difficult problems.

The beauty and charm of these often bizarre surfaces produced by experiments have been frequently described and praised, as for example in G. Van der Mensbrugghe's orbituary of Plateau:

" " 'There is not, to our knowledge, an example where observation has supported theory with more delightful forms. What could be more beautiful, to the eyes of a mathematician, than these weightless shapes of the most brilliant colors, endowed, despite their extreme fragility, with an astonishing persistence ?' Apart from Plateau's works a description of experiments with soap films and discussions of the underlying mathematical, physical, and experimental questions can also be found in the books of Boys [B], and Hildebrandt and Tromba [HT]. Anyone who considers working in the subject of minimal surfaces should have carried out or at least seen the typical experiments.

When mathematicians first began to work on the Plateau problem, they tried to find an explicit representation for the desired minimal surface or at least a process which would make this explicit representation possible. Considering the bizarre forms which Jordan curves can take, it is clear in hindsight that such a venture could never succeed. Before real progress was feasible, the question of existence of a solution first had to be separated clearly from the problem of determining the solution surface explicitly, and then the existence proof had to be specialized by searching not just for any minimal surface bounded by a prescribed Jordan curve Γ, but for a minimal surface whose area is an absolute minimum. It was, in fact, not until 1930 that the Plateau problem was treated in satisfactory generality. Until the last third of the nineteenth century, the Plateau problem remained completely unsolved for any nonplanar contour. The first successful solution of the Plateau problem in a concrete case (and in explicit form) was derived by H. A. Schwarz in 1865. The bounding contour considered by him was the skew quadrilateral Γ consisting of four of the edges of a regular tetrahedron with side 1. Unknown to Schwarz, Riemann had attacked the same problem at about the same time, possibly even earlier. Weierstrass and Darboux devoted special efforts to solving the Plateau problem for a Schwarz chain boundary. In 1866, Weierstrass wrote that because of insurmountable difficulties, he restricted himself to investigate more closely the case where the boundary is composed of straight line segments instead of a Jordan curve. In 1914, Darboux still remarked:

" " 'Thus far, mathematical analysis has not been able to envisage any general method which would permit us to begin the study of this beautiful question.'

In his remarkable paper of 1928, Garnier was able to force the solution of Plateau's problem by determining the functions in the Weierstrass representation formula. His work extended the program outlined by Weierstrass and Darboux and was based on the investigation of G. D. Birkhoff. Utilizing a limiting process, Garnier showed that his proof also applied to Jordan curves Γ consisting of a finite number of unknotted arcs with bounded curvature. Garnier's accomplishment was outstanding even though it was eventually somewhat overshadowed by the subsequent proofs by Radó and Douglas.

Although Garnier had found a solution to the Plateau problem, his method could hardly have been perceived as definitive. For complicated, possibly knotted, boundary curves, his procedure (not to mention the methods of nineteenth century mathematicians) was either unusable or at least very cumbersome and barely generalizable. However, in the late twenties of this century, Douglas [D] and Rado [R1] have, quite independently of each other, been successful in developing new methods for solving the Plateau problem. Their methods were simpler, in a sense quite far ranging, and were completely satisfactory for their time. In the ensuing years, Douglas conceived and treated appreciable generalizations of the original problem. (Douglas's work has recently been revisited, and his methods criticized, by Tromba, Jost and others.) Since then, of course, there have been many extensions as well as new conjectures concerning the Plateau problem. Even today, important questions are still open and the most general version of the question, which by now has little in common with the original Plateau problem, is still being honed and perfected.

While Radó used conformal mappings of polyhedra and a limit theorem for solutions to certain approximating problems to obtain the desired minimal surface solution, Douglas applied the direct method of the calculus of variations. For his contributions to the Plateau problem, Douglas was awarded one of the two inaugural Fields medals at the International Congress of Mathematicians in 1936. (Ahlfors won the other.) In order to solve the Plateau problem, Douglas defined a certain functional A and showed that the solution to the variational problem $A = \min\,!$ was the desired (generalized) minimal surface. At the time, many mathematicians were so astonished by the simplicity of Douglas's method that at first they could not quite believe it actually worked. In 1933, McShane [Mc] improved and completed some ideas of Lebesgue and obtained a third solution to the Plateau problem.

Douglas's functional is closely related to the more easily handled Dirichlet integral. This integral, together with a lemma of Lebesgue and a variational theorem due to Radó, is the basis of an even more transparent solution of the Plateau problem obtained nearly simultaneously by Courant and Tonelli in 1936. Morrey [M] later solved the Plateau problem in its original and generalized forms not only in Euclidean space, but also in more general Riemannian spaces. In the years 1948-1954, Besicovitch developed a new existence proof for a surface of the type of the disk of smallest area bounded by an arbitrary closed curve, assuming that the closed curve actually bounds some surface of finite area. In Section I, we shall briefly describe the Douglas-Courant-Tonelli method, basically following the arguments of Lawson's book [L].

A special case of the general problem had already been solved, called the nonparametric Plateau problem, where the boundary curve Γ has a bijective projection onto a (convex) curve in the xy-plane, and where the desired minimal surface solution can be represented nonparametrically as $z = z(x, y)$. Many mathematicians contributed to this effort. They used not only the theory of partial differential equations, but also the direct methods of the calculus of variations in ever increasing generality and rigor. See Bernstein [Be] and Rado [R2]. In 1968, this nonparametric Plateau problem, which is precisely the first boundary value problem for the minimal surface equation, was also considerably generalized by Jenkins and Serrin [JS]. Section II concerns the nonparametric Plateau problem.

In 1956, Fleming constructed a Jordan curve which bounds no surface of smallest area with finite topological type. The infimum of the areas of all surfaces with fixed Euler characteristic bounded by this curve actually decreases as the Euler characteristic is allowed to decrease.

Such a situation cannot be adequately treated with classical methods which only allow comparisons between surfaces of the same topological type. This caused an appreciable generalization of the fundamental concepts. The notion of parametric surfaces has been replaced by that of generalized surfaces, or that of certain point sets — the so-called integral currents, varifolds, etc. In this connection, it was necessary to create the concept of *area* for these new structures. Also, the question of when to regard such a structure as being *bounded* by a given point set caused particular difficulties. Instead of two-dimensional surfaces in three-dimensional space, k-dimensional structures in n-dimensional space were investigated as well.

A few years ago, pioneering investigations led to the proof of existence of a structure of minimum area with a prescribed boundary, to proofs of the analyticity of the solution structure, its analyticity except at the boundary, or its analyticity except at certain exceptional sets (see Federer and Fleming [FF], Reifenberg [Re], Almgren [A], and Federer [F]). It was also shown that the solution behaves, at least locally, almost everywhere (in the two-dimensional case, everywhere), like a differential geometric hypersurface, and statements concerning the boundary behavior of the solution structure were obtained. Fomenko, Morrey, and Simons considered the problem on Riemannian manifolds rather than in Euclidean space.

There are several problems about which little is currently known. These include giving conditions guaranteeing *solution structures* of finite topological type, characterizing situations where the singularities which have to be allowed in higher dimensions do not occur, and the conditions under which a solution is unique or, if not, the number of solutions to the Plateau problem can be estimated.

The method of generalized surfaces will be discussed briefly in Section III.

I. The solution of Douglas-Rado

The first difficulty in trying to attack the Plateau problem is to find a workable formulation of the question. Roughly speaking, it is: "Given a Jordan curve Γ in R^n, find a surface in R^n of least area having Γ as boundary." However, a little thought on this problem (and perhaps some experimentation with soap films) quickly shows that the topological type of Γ may be quite complicated. Moreover, minimal surfaces of various topological type with Γ as boundary may exist (while only one may represent an *absolute* minimum in area.) In view of Fleming's example, where the surface of absolutely least area is infinitely connected, we shall restrict our attention to trying to find a surface Σ of least area and of prescribed topological type, the simplest being that of the disk.

We want to consider our "surfaces" here to be mappings of two-manifolds into R^n. Then our problem can be formulated as follows. Let $\Gamma \subset \mathrm{R}^n$ $(n \geq 2)$ be a Jordan curve, i.e., a subset homeomorphic to the circle, and set $\Delta = \{(x,y) \in \mathrm{R}^2 : x^2 + y^2 \leq 1\}$. A mapping $\psi : \Delta \to \mathrm{R}^n$ is called *piecewise* C^1 if it is continuous and if, except along $\partial\Delta$ and along a finite number of regular C^1 arcs and points in Δ^0, ψ is of class C^1. A continuous map $b : \partial\Delta \to \Gamma$ is called *monotone* if for each $p \in \Gamma$ the set $b^{-1}(p)$ is connected. We now define the class of competing surfaces

$$X_\Gamma = \{\psi : \Delta \to \mathrm{R}^n : \psi \text{ is piecewise } C^1 \text{ and}$$
$$\psi|\partial\Delta \text{ is a monotone parameterization of } \Gamma\}.$$

We then define the area function $A : X_\Gamma \to \mathrm{R}^+ \cup \{\infty\}$ by the following (generally improper) integral.

$$A(\psi) = \iint_\Delta |\psi_x \wedge \psi_y| dxdy$$

where $|\psi_x \wedge \psi_y|^2 = |\psi_x|^2|\psi_y|^2 - \langle\psi_x, \psi_y\rangle^2$. The precise statement of our problem is now *to find a* $\psi \in X_\Gamma$ *such that* $A(\psi) = \mathcal{A}_\Gamma$ *where*

$$\mathcal{A}_\Gamma = \inf_{\psi \in X_\Gamma} A(\psi).$$

Note that we have one minor complication. For this problem to be interesting we should know that $\mathcal{A}_\Gamma < \infty$. If Γ is rectifiable this is true, but in general it will have to be assumed since there exist curves Γ with $\mathcal{A}_\Gamma = \infty$.

The question now is how to solve the problem we have posed. Geometric intuition would suggest that we take a sequence of surfaces $\{\psi_n\}$ such that $A(\psi_n) \to \mathcal{A}_\Gamma$, and try to show that subsequence must converge to a solution. However, we must deal in this case not only with the geometric images of the surfaces but with the way they are parameterized. The area integral $A(\psi_n)$ is clearly invariant under even piecewise C^1 reparameterizations. Thus if we reparameterize ψ_n by a very wild homeomorphism on Δ for each n, then it may happen that no subsequence of $\{\psi_n\}$ could converge even to a continuous mapping.

The lesson here is that we must somehow control the parameterizations of our minimizing sequences of surfaces. In the one-dimensional case, i.e., curves in a Riemannian manifold, this is done by minimizing the *energy integral.* Here minimizing subsequences of curves tend to geodesics which not only minimize the length integral but which are forced to be parameterized by a multiple of arc length. Physically, one could think of a rubber band lying on a surface with its endpoints fixed. This band not only minimizes its length but also minimizes its potential energy by stretching itself uniformly along the surface. The analogous situation holds for surfaces. The corresponding energy integral is the so-called *Dirichlet integral*

$$D(\psi) = \iint_\Delta (|\psi_x|^2 + |\psi_y|^2) dxdy.$$

As we shall now see, the functions in X_Γ which minimize the Dirichlet integral not only minimize area but must have natural parameterizations, namely, conformal ones, which correspond to a tight, or least energy, spreading of the surface over the geometric configuration of least area.

To begin we note that for any two vectors $v, w \in \mathrm{R}^n$ we have

$$|v \wedge w|^2 = |v|^2|w|^2 - \langle v, w\rangle^2 \leq |v|^2|w|^2 \leq \frac{1}{4}(|v|^2 + |w|^2)^2$$

where equality holds if and only if $|v| = |w|$ and $\langle v, w\rangle = 0$. It follows immediately that for any $\psi \in X_\Gamma$,

$$A(\psi) \leq \frac{1}{2}D(\psi)$$

where equality holds if and only if

$$|\psi_x| = |\psi_y| \quad \text{and} \quad \langle \psi_x, \psi_y\rangle = 0$$

almost everywhere in Δ. Any mapping ψ with this property is called *almost conformal*. Whereever $|\psi_x| > 0$, such a map is *conformal*, or angle preserving, and induces a metric on Δ of the form

$$ds^2 = \lambda(dx^2 + dy^2)$$

where $\lambda = |\psi_x| = |\psi_y|$. Under this condition the parameters (x, y) are called *isothermal coordinates* for the surface. A theorem of fundamental importance for us is the following.

Theorem (The existence of isothermal coordinates) Let $\psi : \Delta \to \mathrm{R}^n$ be a continuous map such that $\psi|\Delta^0$ is an immersion of class C^k, $1 \leq k \leq \infty$ (or real analytic). Then there exists a homeomorphism $d : \Delta \to \Delta$ where $d|\Delta^0$ is of class C^k (or real analytic) such that the reparameterized mapping $\psi' = \psi \circ d$ is conformal.

We now consider a Jordan curve $\Gamma \subset \mathrm{R}^n$ and define

$$d_\Gamma = \inf_{\psi \in X_\Gamma} D(\psi).$$

Given a sequence of immersed surfaces $\{\psi_n\}$ with $A(\psi_n) \to \mathcal{A}_\Gamma$, the above theorem enables us to reparameterize each ψ_n into a conformal mapping ψ_n'. Since $A(\psi_n) = A(\psi_n') = \frac{1}{2}D(\psi_n')$, we have

$$\mathcal{A}_\Gamma = \frac{1}{2}d_\Gamma.$$

Therefore we have the following

Corollary Let $\Gamma \subset \mathrm{R}^n$ be a Jordan curve with $\mathcal{A}_\Gamma < \infty$. Then for any $\psi \in X_\Gamma$,

$$D(\psi) = d_\Gamma \text{ if and only if } A(\psi) = \mathcal{A}_\Gamma \text{ and } \psi \text{ is almost conformal.}$$

Hence, to solve the Plateau problem it is sufficient to find a function $\psi \in X_\Gamma$ which minimizes the Dirichlet integral. However, to do this we can use the following standard fact in harmonic function theory:

Dirichlet's Principle Let $b : \partial\Delta \to \mathrm{R}^n$ be a continuous map and define

$$X_b = \{\psi : \Delta \to \mathrm{R}^n : \psi \text{ is piecewise } C^1 \text{ and } \psi|\partial\Delta = b\}.$$

Assume that the number

$$d_b = \inf_{\psi \in X_b} D(\psi)$$

is finite. Then there exists a unique function $\psi_b \in X_b$ such that $D(\psi_b) = d_b$. The function ψ_b is harmonic in Δ^0 and represents the solution to the boundary value problem $\Delta\psi = 0$, $\psi|\partial\Delta = b$.

Our requirements for solving the Plateau problem are now vastly simplified. We want to minimize the integral $D(\psi)$ over the class X_Γ. However, for each *fixed parameterization* $b : \partial\Delta \to \Gamma$ we know that there exists a unique function $\psi_b \in X_b \subset X_\Gamma$ such that $D(\psi_b) = d_b$. However, for *different parameterizations* b of Γ we will, in general, have *different values* of d_b. Hence it remains only for us to find a parameterization b of Γ such that $d_b = d_\Gamma$.

To find such a minimal parameterization we shall choose a sequence $\{b_n\}$ such that $\lim d_{b_n} = d_\Gamma$ and show that there exists a uniformly convergent subsequence. To do this we will need to normalize the mappings b_n. This normalization will correspond to normalizing the maps ψ_{b_n}. For this we first note the following.

Lemma The Dirichlet integral $D(\psi)$ is invariant under conformal transformations of the disk.

Recall that by a conformal transformation of Δ it is possible to map any given three points of $\partial\Delta$ to any other three distinct points of $\partial\Delta$. Moreover, having prescribed the images of three such points the conformal transformation is uniquely determined. We now normalize our surfaces as follows. We choose three distinct points $p_1, p_2, p_3 \in \Gamma$ and three distinct points $z_1, z_2, z_3 \in \partial\Delta$, and we define

$$X'_\Gamma = \{\psi \in X_\Gamma : \psi(z_k) = p_k \text{ for } k = 1, 2, 3\}.$$

By the above lemma we have that $\inf\{D(\psi) : \psi \in X'_\Gamma\} = d_\Gamma$, and thus we may solve the Plateau problem by minimizing in this somewhat smaller class. Note here that the number *three* plays a crucial role. Prescribing the images of two points is not sufficient: As there exist conformal mappings of Δ which, fixing z_1 and z_2, map arbitrarily small neighborhoods of z_1 onto the complement in Δ of arbitrarily small neighborhoods of z_2, we may not obtain a convergent subsequence of the minimizing sequence of the Dirichlet integral. Moreover, for the class X'_Γ we have the following important fact.

Proposition Let M be a constant $> d_\Gamma$. Then the family of functions

$$\mathcal{F} = \{\psi|\partial\Delta : \psi \in X'_\Gamma \quad \text{and} \quad D(\psi) \leq M\}$$

is equicontinuous on $\partial\Delta$. Thus, by Arzela's theorem $\mathcal{F}$ is compact in the topology of uniform convergence.

The key idea in the proof of this proposition is that the finiteness of the Dirichlet integral of ψ gives the boundedness of the arc lengths of the images under ψ of curves in Δ. To see this we define C_r to be the intersection of Δ with the circle of radius r about the point $z \in \mathrm{R}^2$, and we denote by s the arc length parameter of C_r. Let $\ell(C_r)$ be the length of the curve $\psi(C_r)$. It follows from the Schwarz inequality that

$$\ell(C_r)^2 = (\int_{C_r} |\psi_s| ds)^2 \leq 2\pi r \int_{C_r} |\psi_s|^2 ds.$$

For $0 < \delta < 1$, consider the integral

$$I \equiv \int_\delta^{\sqrt{\delta}} \int_{C_r} |\psi_s|^2 ds dr \leq D(\psi) \leq M,$$

and express I as

$$I = \int_\delta^{\sqrt{\delta}} p(r) \frac{1}{r} dr \quad \text{with} \quad p(r) = r \int_{C_r} |\psi_s|^2 ds.$$

Then by the Mean Value Theorem (for the measure $d(\log r)$) we have that there exists a number ρ, with $\delta \leq \rho \leq \sqrt{\delta}$, such that

$$I = p(\rho) \int_\delta^{\sqrt{\delta}} d(\log r) = p(\rho) \frac{1}{2} \log(\frac{1}{\delta}).$$

Therefore $p(\rho) \leq \frac{2M}{\log \frac{1}{\delta}}$.

Thus for each $0 < \delta < 1$, there exists a number ρ with $\delta \leq \rho \leq \sqrt{\delta}$ such that

$$\ell(C_\rho)^2 \leq \frac{4\pi M}{\log \frac{1}{\delta}}.$$

Then from this estimate the equicontinuity of $\mathcal{F}$ follows easily.

We can now complete the solution to the Plateau problem. Let $\{b_n\}$ be a sequence from $\mathcal{F}$ such that $\lim d_{b_n} = d_\Gamma$. By the above proposition there exists a subsequence $\{b_{n_j}\}$ which converges uniformly to some $b \in \mathcal{F}$. By the lower semicontinuity of Dirichlet integral we have

$$D(\psi_b) \leq \liminf D(\psi_{b_{n_j}}) = d_\Gamma.$$

Consequently, $D(\psi_b) = d_\Gamma$ and we have obtained a *classical solution to the Plateau problem* for Γ:

Theorem (Douglas) Let Γ be a Jordan curve in R^n such that $\mathcal{A}_\Gamma < \infty$. Then there exists a continuous map $\psi : \Delta \to \mathrm{R}^n$ such that

" a)" ψ maps $\partial\Delta$ homeomorphically onto Γ,

" b)" ψ is harmonic and almost conformal on Δ^0,

" c)" $D(\psi) = d_\Gamma$ and $A(\psi) = \mathcal{A}_\Gamma$.

II. Nonparametric solution to the higher-dimensional Plateau problem

The solution surfaces given in Douglas' theorem will, in general, have self-intersections. In fact, if Γ is, say, a knot in R^3, every solution *must* have self-intersections. Nonetheless, it is reasonable to expect that if Γ is not too badly behaved, the minimizing surfaces for Γ will be embedded. Following the work of Rado we shall give a set of geometric conditions on Γ which will guarantee that Douglas' solution is not only embedded, but free of branch points.

Theorem (Rado) If the Jordan curve $\Gamma \subset \mathrm{R}^n$ admits a one-to-one orthogonal projection onto a convex curve in a plane $\mathrm{R}^2 \subset \mathrm{R}^n$, then the solution to the Plateau problem for Γ is free of branch points and can be expressed as the graph of a function $f : \mathrm{R}^2 \to \mathrm{R}^{n-2}$. When $n = 3$, the solution is unique.

Let us give a sketchy proof of this theorem. Let $\pi : \mathrm{R}^n \to \mathrm{R}^2$ be the orthogonal projection map, and suppose that $\psi : \Delta \to \mathrm{R}^n$ is a solution to the Plateau problem for Γ. Let $p \in \Delta^0$ and suppose that the rank of $d(\pi \circ \psi)$ at p is ≤ 1. Then there exists a line $\ell \subset \mathrm{R}^2$ through the point $\pi \circ \psi(p)$ such that the hyperplane $H = \pi^{-1}(\ell)$ is tangent to the surface $\psi(\Delta)$ at $\psi(p)$. However, $H \cap \Gamma$ consists of two points only, since $\pi(\Gamma)$ is convex and $\pi|\Gamma$ is one-to-one. On the other hand, in a small neighborhood U of $\psi(p)$, $\psi(\Delta) \cap U$ is divided by the one-dimensional set $H \cap \psi(\Delta)$, like a pie, into at least 4 regions. By the maximum principle for harmonic functions we see that the 4 curves of $H \cap \psi(\Delta)$ emanating from $\psi(p)$ never intersect each other. Instead, they intersect Γ, and it follows that $H \cap \Gamma$ consists of at least 4 points, which is a contradiction. Thus we conclude that $d(\pi \circ \psi)$ has rank 2 throughout Δ^0 and that $\psi(\Delta)$ is the graph of a function $f : \mathrm{R}^2 \to \mathrm{R}^{n-2}$. Furthermore, the graph of f is the set $\{(x, y, f_3(x, y), \ldots, f_n(x, y)) : (x, y) \in \pi \circ \psi(\Delta)\}$. Taking the Euler-Lagrange equations of the area integral for f, we see that f must satisfy the equation:

$$(1 + |f_y|^2) f_{xx} - 2\langle f_x, f_y \rangle f_{xy} + (1 + |f_x|^2) f_{yy} = 0.$$

When $n = 3$, the uniqueness is proved by applying the maximum principle to f.

Douglas and Rado cleverly exploited a nice property of conformal mappings on two-dimensional surfaces. Since higher-dimensional surfaces do not satisfy this property, their methods cannot be generalized to solve the *higher-dimensional Plateau problem*:

" " Given a set $\Gamma \subset \mathrm{R}^n$ which is homeomorphic to the boundary of an m-dimensional ball $B^m \subset \mathrm{R}^m$, find an m-dimensional submanifold Σ^m homeomorphic to B^m, with $\partial\Sigma = \Gamma$, such that the volume of Σ is the minimum among all m-dimensional submanifolds $U \subset \mathrm{R}^n$ with $\partial U = \Gamma$.

However, for certain Γ in R^{m+1}, one can find a solution, as a graph, to the higher-dimensional Plateau problem. This graphical solution is obtained by solving the Dirichlet problem for the classical nonparametric minimal surface equation. The minimal surface equation is a qusilinear equation which is elliptic but not uniformly elliptic. Its geometric significance makes its study tractable, and this equation has been the focus of much attention in the theory of partial differential equations.

Let us derive the minimal surface equation and discuss the solvability of the Dirichlet problem for the minimal surface equation. Let Ω be a bounded domain in R^m. Given a smooth function u defined on Ω, the volume of the graph of u is $\mathrm{Vol}(\mathrm{graph}\, u) = \int_\Omega \sqrt{1 + |Du|^2}\, dv$.

Let φ be a given Lipschitz function on $\overline{\Omega}$ and consider $\mathrm{Vol}(\mathrm{graph}\, u)$ for all functions u in the set

$$\mathcal{C} = \{u \in C^\infty(\overline{\Omega}) : u = \varphi \text{ on } \partial\Omega\}.$$

The *nonparametric higher-dimensional Plateau problem* we now consider is the following:

Find $u \in \mathcal{C}$ such that $\mathrm{Vol}(\mathrm{graph}\, u) \leq \mathrm{Vol}(\mathrm{graph}\, v)$ for all $v \in \mathcal{C}$.

Let us suppose that u is a solution of the nonparametric higher-dimensional Plateau problem and let η belong to the space

$$\mathcal{C}_0 = \{\eta \in C^\infty(\overline{\Omega}) : \eta = 0 \text{ on } \partial\Omega\};$$

then the function $v = u + t\eta$ must belong to $\mathcal{C}$ for every $t \in \mathrm{R}$. Thus $\mathrm{Vol}(\mathrm{graph}\, u) \leq \mathrm{Vol}(\mathrm{graph}\,(u + t\eta))$ for all $t \in \mathrm{R}$, and so $\mathrm{Vol}(\mathrm{graph}\,(u + t\eta))$ has a minimum at $t = 0$, hence

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\mathrm{graph}\,(u + t\eta)) = 0.$$

Now

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathrm{Vol}(\mathrm{graph}\,(u + t\eta)) &= \frac{d}{dt}\Big|_{t=0} \int_\Omega \sqrt{1 + |Du + tD\eta|^2}\, dv \\
&= \int_\Omega \frac{D\eta \cdot Du}{\sqrt{1 + |Du|^2}}\, dv \\
&= -\int_\Omega \eta \mathrm{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) dv = 0
\end{aligned}$$

for all $\eta \in \mathcal{C}_0$, that is, the function u is a solution of the minimal surface equation

$$\mathrm{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.$$

An equivalent form of the minimal surface equation is

$$(1 + |Du|^2)\Delta u - D_i u D_j u D_{ij} u = 0.$$

This equation is a special case of the quasilinear equations of the form

$$Qu = a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du) = 0,\ a^{ij} = a^{ji},$$

where $x = (x_1, \ldots, x_m)$ is contained in Ω.

The solvability of the classical Dirichlet problem for quasilinear equations is reduced to the establishment of certain a priori estimates for solutions. This reduction is achieved through the application of topological fixed point theorems in appropriate function spaces. The fixed point theorems required for the treatment presented here are obtained as infinite dimensional extensions of the Brouwer fixed point theorem, which asserts that a continuous mapping of a closed ball in R^n into itself has at least one fixed point. The Brouwer fixed point theorem can be extended to infinite dimensional spaces in various ways. We require first the following extension to Banach spaces.

Theorem Let $\mathcal{S}$ be a closed convex set in a Banach space $\mathcal{B}$ and let T be a continuous mapping of $\mathcal{S}$ into itself such that the image $T\mathcal{S}$ is precompact. Then T has a fixed point, that is, $Tx = x$ for some $x \in \mathcal{S}$.

For the proofs of this and the following theorems, refer to [GT, Chapter 11].

A continuous mapping between two Banach spaces is called *compact* (or *completely continuous*) if the images of bounded sets are precompact (that is, their closures are compact). The following consequence of the above theorem is the fixed point result most often applied in the approach to the Dirichlet problem for quasilinear equations.

Theorem Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself, and suppose there exists a constant M such that

$$\|x\|_{\mathcal{B}} < M$$

for all $x \in \mathcal{B}$ and $\sigma \in [0, 1]$ satisfying $x = \sigma T x$. Then T has a fixed point.

In order to apply this theorem to the Dirichlet problem for quasilinear equations, we fix a number $\beta \in (0, 1)$ and take the Banach space $\mathcal{B}$ to be the Hölder space $C^{1,\beta}(\overline{\Omega})$, where $C^{k,\beta}(\overline{\Omega})$ denotes the set of functions having all derivatives of order $\leq k$ continuous in $\overline{\Omega}$ with the k-th order partial derivatives being uniformly Hölder continuous with exponent α in Ω. Let Q be the operator given by

$$Qu = a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du)$$

and assume that Q is elliptic in $\overline{\Omega}$, that is, the coefficient matrix $[a^{ij}(x,z,p)]$ is positive for all $(x,z,p) \in \overline{\Omega} \times \mathrm{R} \times \mathrm{R}^m$. We also assume, for some $\alpha \in (0,1)$, that the coefficients a^{ij}, $b \in C^{\alpha}(\overline{\Omega} \times \mathrm{R} \times \mathrm{R}^m)$, that the boundary $\partial\Omega \in C^{2,\alpha}$ and that φ is a given function in $C^{2,\alpha}(\overline{\Omega})$. For all $v \in C^{1,\beta}(\overline{\Omega})$, the operator T is defined by letting $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\overline{\Omega})$ of the linear Dirichlet problem,

$$a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) = 0 \text{ in } \Omega,\ u = \varphi \text{ on } \partial\Omega.$$

The unique solvability of this problem is guaranteed by the linear existence result, [GT, Theorem 6.14]. The solvability of the Dirichlet problem, $Qu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, in the space $C^{2,\alpha}(\overline{\Omega})$ is thus equivalent to the solvability of the equation $u = Tu$ in the Banach space $\mathcal{B} = C^{1,\alpha}(\overline{\Omega})$. The equation $u = \sigma Tu$ in $\mathcal{B}$ is equivalent to the Dirichlet problem

$$Q_\sigma u = a^{ij}(x,u,Du)D_{ij}u + \sigma b(x,u,Du) = 0 \text{ in } \Omega, \qquad u = \sigma\varphi \text{ on } \partial\Omega.$$

By applying the above theorem we can then prove the following criterion for existence.

Theorem Let Ω be a bounded domain in R^m and suppose that Q is elliptic in $\overline{\Omega}$ with coefficients a^{ij}, $b \in C^{\alpha}(\overline{\Omega} \times \mathrm{R} \times \mathrm{R}^m)$, $0 < \alpha < 1$. Let $\partial\Omega \in C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Then, if for some $\beta > 0$ there exists a constant M, independent of u and σ, such that every $C^{2,\alpha}(\overline{\Omega})$ solution of the Dirichlet problems, $Q_\sigma u = 0$ in Ω, $u = \sigma\varphi$ on $\partial\Omega$, $0 \le \sigma \le 1$, satisfies

$$(*) \qquad \|u\|_{C^{1,\alpha}(\overline{\Omega})} < M,$$

it follows that the Dirichlet problem, $Qu = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, is solvable in $C^{2,\alpha}(\overline{\Omega})$.

Proof In view of the remarks preceding the statement of the theorem, it only remains to show that the operator T is continuous and compact. By virtue of the global Schauder estimate, T maps bounded sets in $C^{1,\alpha}(\overline{\Omega})$ into bounded sets in $C^{2,\alpha\beta}(\overline{\Omega})$ which (by Arzela's theorem) are precompact in $C^2(\overline{\Omega})$ and $C^{1,\beta}(\overline{\Omega})$. In order to show the continuity of T, we let v_n, $n = 1,2,\ldots$, converge to v in $C^{1,\beta}(\overline{\Omega})$. Then, since the sequence $\{Tv_n\}$ is precompact in $C^2(\overline{\Omega})$, every subsequence in turn has a convergent subsequence. Let $\{T\overline{v}_n\}$ be such a convergent subsequence with limit $u \in C^2(\overline{\Omega})$. Then since

$$\begin{aligned} &a^{ij}(x,v,Dv)D_{ij}u + b(x,v,Dv) \\ &= \lim_{n\to\infty}\{a^{ij}(x,\overline{v}_n,D\overline{v}_n)D_{ij}T\overline{v}_n + b(x,\overline{v}_n,D\overline{v}_n)\} = 0, \end{aligned}$$

we must have $u = Tv$, and hence the sequence $\{Tv_n\}$ itself converges to u.

Thus it only remains to derive the a priori estimate $(*)$. However, it is a well known fact in PDE theory that in order to obtain the estimate it is sufficient to determine an appropriate supersolution at each boundary point of Ω. In fact, constructing a suitable supersolution along the boundary which is geometrically nice, Jenkins and Serrin [JS] found a necessary and sufficient condition for the Dirichlet problem for the minimal surface equation to be well posed for C^2 boundary data. The following is an extension of their theorem.

Theorem Let Ω be a bounded C^2 domain in R^m. Then the Dirichlet problem for the minimal surface equation with boundary data φ is solvable for arbitrary $\varphi \in C^0(\partial\Omega)$ if and only if the mean curvature of the boundary $\partial\Omega$ is everywhere nonnegative (i.e., $\partial\Omega$ is mean convex).

Thus the higher-dimensional Plateau problem is solvable provided $\Gamma \subset \mathrm{R}^{m+1}$ admits a one-to-one orthogonal projection onto a mean convex hypersurface in R^m. The maximum principle says that this solution is unique. Moreover, this solution is area minimizing in the sense that it has the minimum volume among all n-dimensional surfaces with the same boundary Γ.

Therefore one can obtain the uniqueness of the solution to the Plateau problem only when $\Gamma \subset \mathrm{R}^n$ is of codimension two. However, Lawson and Osserman [LO] constructed examples of Γ of codimension higher than two for which the solutions of the Dirichlet problem for the minimal surface system are *not unique*. Moreover, they showed that such surfaces *need not even be stable* in contrast with the fact that nonparametric minimal surfaces of codimension one are area minimizing. Furthermore they showed that the Dirichlet problem is *not even solvable* in general in higher codimensions. Therefore the deep and beautiful results for nonparametric minimal surfaces in codimension one fail utterly in higher codimensions.

III. Geometric measure theory and the generalized Plateau problem

Geometric Measure Theory could be described as differential geometry, generalized through measure theory to deal with maps and surfaces that are not necessarily smooth, and applied to the calculus of variations. It dates from the 1960 foundational paper of Federer and Fleming on "Normal and integral currents" [FF]. An archetypal problem in geometric measure theory is the *generalized Plateau problem*: Given an $(m-1)$-dimensional boundary in R^n, find the m-dimensional surface of least area with that boundary. Progress on this problem depends crucially on first finding a good space of surfaces to work in. In the previous sections we considered surfaces to be defined as (the image of) mappings of a domain in R^m. Although this approach led to the solutions for the Plateau problem and the nonparametric higher-dimensional Plateau problem, almost no progress was made for the generalized Plateau problem. This is because (i) the conformal structures of a higher-dimensional ball are not as simple as that of a disk, and (ii) the energy and volume of maps from a higher-dimensional ball do not coincide as do those for conforl maps between 2-dimensional surfaces.

Along with its successes and advantages, the definition of a surface as a mapping has certain drawbacks:

" (1)" There is an inevitable *a priori* restriction on the types of singularities that can occur;

" (2)" There is an *a priori* restriction on the topological complexity;

" (3)" The natural topology lacks compactness properties.

An alternative to surfaces as mappings is provided by *rectifiable currents*, the m-dimensional, oriented surfaces of geometric measure theory. These sets have folds, corners, and more general singularities. The relevant functions $f : \mathrm{R}^m \to \mathrm{R}^n$ need not be smooth, but merely *Lipschitz*, i.e.,

$$|f(x) - f(y)| \leq C|x - y|,$$

for some "Lipschitz constant" C.

Fortunately there is a good m-dimensional measure on R^n, called *Hausdorff measure* $\mathcal{H}^m$. Hausdorff measure agrees with the classical mapping area of an embedded manifold, but it is defined for all subsets of R^n.

A Borel set of R^n is called $(\mathcal{H}^m, m)$-*rectifiable* if B is a countable union of Lipschitz images of bounded subsets of R^m, with $\mathcal{H}^m(B) < \infty$. (As usual, we will ignore sets of $\mathcal{H}^m$ measure 0.) That definition sounds rather general, and it includes just about any "m-dimensional surface" we can imagine. Nevertheless, these sets will support a kind of differential geometry; for example, it turns out that a rectifiable set B has a canonical tangent plane at almost every point.

The concept of currents was a generalization, due to de Rham [de R], of distributions. Normal and rectifiable currents were introduced in the historic paper [FF] of Federer and Fleming; their advantage is that they are at once able to be represented as "generalized surfaces" (in terms of a rectifiable set with an integer multiplicity) and at the same time have nice compactness properties. A *rectifiable current* is an oriented rectifiable set with integer multiplicities, finite area, and compact support. More generally, an m-dimensional rectifiable current is a continuous linear functional on the space $\mathcal{D}^m$ of smooth differential m-forms with compact support over an oriented rectifiable set with integer multiplicities. Hence the space $\mathcal{D}_m$ of m-currents is the dual space of $\mathcal{D}^m$.

Note that in case $m = 0$ the m-currents in R^n are just the Schwartz distributions on R^n. More importantly though, the m-currents, $m \geq 1$, can be interpreted as a generalization of the m-dimensional oriented manifolds. Indeed given an oriented, m-dimensional rectifiable set S (or manifold), let $\overrightarrow{S}(x)$ denote the unit m-vector associated with the oriented tangent plane to S at x. Then for any differential m-form φ, define

$$S(\varphi) = \int_S \langle \overrightarrow{S}(x), \varphi \rangle d\mathcal{H}^m.$$

That is, the m-current S is obtained by integration of m-forms over the set S in the usual sense of differential geometry. Furthermore, we will allow S to carry a positive integer multiplicity $\mu(x)$, with $\int_S \mu(x) d\mathcal{H}^m < \infty$, and define

$$S(\varphi) = \int_S \langle \overrightarrow{S}(x), \varphi \rangle \mu(x) d\mathcal{H}^m.$$

Finally, we will require that S have compact support. Such currents are called *rectifiable currents*, as we defined above.

This perspective yields a new topology on the space $\mathcal{D}_m$ of rectifiable currents, dual to an appropriate topology on differential forms. Under the *weak topology* on $\mathcal{D}_m$, $T_j \to T$ if and only if $T_j(\varphi) \to T(\varphi)$ for all forms $\varphi \in \mathcal{D}^m$. This topology has useful compactness properties, given by the fundamental Compactness Theorem below. Viewing rectifiable sets as currents also provides a boundary operator ∂ from m-dimensional rectifiable currents to $(m-1)$-dimensional currents, defined by

$$(\partial S)(\varphi) = S(d\varphi),$$

where $d\varphi$ is the exterior derivative of φ. By Stokes's theorem, this definition coincides with the usual notion of boundary for smooth, compact manifolds with boundary. In general, the current ∂S is not rectifiable, even if S is rectifiable.

The Compactness Theorem Let c be a positive constant. Then the set of all m-dimensional rectifiable currents T in a fixed large closed ball in R^n, such that the boundary ∂T is also rectifiable, and such that the areas of both T and ∂T are bounded by c, is compact in an appropriate weak topology.

This theorem deserves to be known as the fundamental theorem of geometric measure theory. It guarantees solutions to a wide class of variational problems in general dimensions.

Notice that rectifiable currents have none of the three drawbacks mentioned above. There is certainly no restriction on singularities or topological complexity. Moreover, the compactness theorem provides the ideal compactness properties. All of these results hold in all dimensions and codimensions.

One serious suspicion hangs over this new space of surfaces: The solutions they provide to the problem of least area, the so-called area minimizing rectifiable currents, may be generalized objects without any geometric significance. However, in 1962 Fleming proved a regularity result that at first sounds too good to be true.

Theorem (Fleming) A 2-dimensional area minimizing rectifiable current T in R^3 is a smooth embedded manifold on the interior.

More precisely, $\operatorname{spt} T - \operatorname{spt} \partial T$ is a smooth embedded manifold.

In the Classical theory, such complete regularity fails. The Douglas solution of the Plateau problem is not in general embedded.

Fleming's regularity theorem was generalized to 3-dimensional surfaces in R^4 by Almgren in 1966, and up through 6-dimensional surfaces in R^7 by Simons [S] in 1968:

Theorem For $m \leq 6$, an m-dimensional area minimizing rectifiable current in R^{m+1} is a smooth embedded manifold.

Simons's proof is based on the following.

Lemma For $2 \leq n \leq 6$, let B be an oriented compact smooth $(n-1)$-dimensional submanifold of the unit n-dimensional sphere, such that the cone from the origin over B is area minimizing. Then B is a great sphere.

In 1969, Bombieri, De Giorgi, and Giusti [BDG] gave an example of a 7-dimensional, area minimizing rectifiable current T in R^8 with an isolated singularity at the origin. This current T is the oriented truncated cone over $B = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}}) \subset S^7(1) \subset \mathrm{R}^8$; $\partial T = B$. It also provides a counterexample to the above lemma, which is precisely where the proof of regularity breaks down.

The complete interior regularity results for area minimizing hypersurfaces are given by the following theorem of Federer [Fe].

Theorem An m-dimensional, area minimizing rectifiable current T in R^{m+1} is a smooth embedded manifold on the interior except for a singular set of Hausdorff dimension at most $m-7$.

Regularity in higher codimension, for an m-dimensional area minimizing rectifiable current T in R^n, with $m < n-1$, is much harder. Until recently it was known only that the set of regular points, where $\operatorname{spt} T$ is a smooth embedded manifold, was dense in $\operatorname{spt} T - \operatorname{spt} \partial T$ [F, 5.3.16]. On the other hand, m-dimensional complex analytic varieties, which are automatically area minimizing, can have $(m-2)$-dimensional singular sets. In a major advance Almgren has proved the conclusive regularity theorem.

Theorem An m-dimensional area minimizing rectifiable current in R^n is a smooth embedded manifold on the interior except for a singular set of Hausdorff dimension at most $m-2$.

For example, a 2-dimensional area minimizing rectifiable current in R^n has at worst a 0-dimensional interior singular set. Even more recently Chang [C] has proved that these singularities must be isolated, "classical branch points".

In general dimensions and codimensions, very little is known about the structure of the set S of singularities. One might hope that S stratifies into embedded manifolds of various dimensions. However, for all we know, S could even be fractional dimensional.

A F. J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. 84 (1966), 277–292Be S. Bernstein, *Sur l'intégration des équations aux dérivées partielles du type elliptique*, Math. Ann. 95 (1926), 585–594BDGE. Bombieri, E. DeGiorgi, and E. Giusti, *Minimal cones and the Bernstein theorem*, Invent. Math. 7 (1969), 243–269BC. V. Boys, *Soap Bubbles : Their colors and the forces which mold them*, Dover, New York, 1959CS. X. Chang, *Two dimensional area minimizing integral currents are classical minimal surfaces*, J. AMS. 1 (1988), 699–778de RG. de Rham, *Variétés différentiables, formes, courants, formes harmoniques*, Act. Sci. et Ind. 1222 (1955)DJ. Douglas, *Solution of the problem of Plateau*, Trans. AMS. 33 (1931), 263–321FH. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969FeH. Federer, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. AMS. 76 (1970), 767–771FFH. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. 72 (1960), 458–520GTD. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order* (2nd Edition), Springer-Verlag, Berlin-Heidelberg-New York, 1984HTS. Hildebrandt and A. Tromba, *Mathematics and optimal form*, Scientific American Books, Inc., New York, 1985JSH. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. r. angew. Math. 229 (1968), 170–187LH. B. Lawson, *Lectures on minimal submanifolds*, Publish or Perish, Inc., Berkeley, 1980LOH. B. Lawson and R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Math. 139 (1977), 1–17McE. J. McShane, *Parametrization of saddle surfaces with applications to the problem of Plateau*, Trans. AMS. 35 (1933), 716–733MF. Morgan, *Geometric measure theory, a beginner's guide*, Academic Press, Inc., Boston, 1988MoC. B. Morrey, *Multiple integrals in the calculus of variations*, Springer-Verlag, New York, 1966NJ. C. C. Nitsche, *Lectures on minimal surfaces*, Vol.1., Cambridge University Press, Cambridge, 1989R1T. Radó, *On Plateau's problem*, Ann. of Math. (2) 31 (1930), 457-469R2, *Bemerkung zur Arbeit von Herrn Ch.H. Müntz über das Plateausche Problem*, Math. Ann. 96 (1927), 587–596ReE. R. Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. 104 (1960), 1–92SJ. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62–105