Index, Vision Number and Stability of Complete Minimal Surfaces

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Let Σ be a complete minimal surface in $\boldsymbol{R}^3$. Σ is said to be stable if Σ minimizes area up to second order on each compact set. If K is the Gauss curvature, then the condition that Σ be stable is expressed analytically by the requirement that on any relatively compact domain D of Σ, the first eigenvalue of the Jacobi operator $L = \Delta - 2K$ be positive. Here L is defined in the space $\Gamma_0(N(\Sigma))$ of all smooth compactly supported normal vector fields on Σ vanishing on ∂D. For a relatively compact domain D of Σ, the index of D is defined by the number of negative eigenvalues of the operator $\Delta - 2K$. Σ is said to have finite index if the index of every relatively compact domain has a uniform upper bound. Denote the least upper bound by Ind (Σ). If Σ is stable then Ind $(\Sigma) = 0$ and *vice versa*. Recently FISCHER-COLBRIE [F] showed that oriented Σ has finite index if and only if Σ has finite total curvature. If Σ has finite total curvature, it was shown by OSSERMAN [O] that Σ is conformally a compact Riemann surface $\bar{\Sigma}$ with finite punctures.

In this paper we compute a lower bound for Ind (Σ) in terms of a new geometric quantity, the *vision number* of Σ. The key idea is to apply the generalized Morse index theorem of S. SMALE [Sm] to a bounded Jacobi field on Σ constructed from a suitable Killing vector field in the ambient space. As a result, we show that

i) the index of JORGE & MEEKS' minimal surface with k ends [JM] is at least $2k - 3$,
ii) the index of HOFFMAN & MEEKS' minimal surface of genus g [HM] is at least $2g + 1$,
iii) the index of LAWSON's minimal surface $\xi_{m,k}$ of genus mk in S^3 [L1] is at least $\max(2m + 1, 2k + 1)$,
iv) the index of the minimal hypersurface $S^p\left(\sqrt{\frac{p}{p+q}}\right) \times S^q\left(\sqrt{\frac{q}{p+q}}\right)$ in S^{p+q+1} is at least 3,
v) the index of any complete immersed nonorientable minimal surface in $\boldsymbol{R}^3$ of finite total curvature which is conformally equivalent to a projective plane with finite punctures is at least 2,

vi) the plane, Enneper's surface, and the catenoid are the only three complete immersed orientable minimal surfaces of genus zero and index less than three in $\boldsymbol{R}^3$.

Here we should mention that TYSK [T] showed Ind (Σ) is bounded above by a constant $(= 7.68183)$ multiple of the degree of the Gauss map (see also [CT]).

By employing the vision number argument, we give a new proof of the theorem by DO CARMO & PENG [CP] and FISCHER-COLBRIE & SCHOEN [FS] that any complete stable oriented minimal surface in $\boldsymbol{R}^3$ is a plane. But we solve the same problem for nonorientable minimal surfaces in the negative: we show that any complete nonorientable minimal surface in $\boldsymbol{R}^3$ of total curvature -2π (*e.g.* Henneberg's minimal surface) is stable. We also prove that there is no complete immersed stable nonorientable minimal surface in $\boldsymbol{R}^3$ of finite total curvature which is conformally equivalent to a Klein bottle with finite punctures. Very recently LIMA & DA SILVEIRA [LS] showed that any complete immersed nonorientable minimal surface in $\boldsymbol{R}^3$ which is either conformally equivalent to a projective plane with finite punctures, or is finitely connected and of infinite total curvature, is not stable.

Finally, motivated by the above lower bounds for Ind (Σ), we conjecture that for any complete minimal surface Σ in $\boldsymbol{R}^3$, orientable or nonorientable,

$$\operatorname{Ind}(\Sigma) \leqq -1 - \frac{1}{2\pi} \int_\Sigma K.$$

More open problems are given at the end of this paper.

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1. Visible sets and stability

First let us introduce new geometric terminology.

Definition. Let N be a smooth submanifold of arbitrary codimension in a Riemannian manifold M, and ϕ a smooth vector field on M. The *horizon* of N with respect to ϕ, denoted by $H(N; \phi)$, is the set of all points of N at which ϕ is a tangent vector of N. We say that a connected subset D of N is *visible* with respect to ϕ if D is disjoint from $H(N; \phi)$. The number of components of $N \sim H(N; \phi)$ is called the *vision number* of N with respect to ϕ and is denoted by $v(N; \phi)$. For example, let S be a unit sphere in $\boldsymbol{R}^3$ and ϕ a nonvanishing vertical vector field in $\boldsymbol{R}^3$. Then the equator of S is the horizon of S with respect to ϕ ,each open hemisphere (northern and southern) is a visible set with respect to ϕ, and $v(S; \phi) = 2$.

The following lemma is the higher-dimensional analogue of the usual theorem for geodesics.

Lemma 1. *Let $\{f_t\}$ be a variation of an m-dimensional minimal submanifold N^m in M^n. Let $\eta = f_{t*}\left(\frac{\partial}{\partial t}\right)$ be the variation vector field of $\{f_t\}$ and $\eta^\perp$ the projection of*

η onto the normal bundle of N. Suppose each $f_t(N)$ is a minimal submanifold. Then $\eta^\perp$ is a Jacobi field on N, i.e., *$\eta^\perp$ satisfies*

$$\Delta\eta^\perp + \mathscr{B}(\eta^\perp) + \bar{R}(\eta^\perp) = 0,$$

where $\mathscr{B}$ and $\bar{R}$ are as defined in [**L2**, p. 48].

Proof. Let $\{e_1, \ldots, e_m\}$ be local orthonormal vector fields on $\bigcup_t f_t(N)$ such that $\{e_1, \ldots, e_m\}$ are tangent to $f_t(N)$ for each t. Then the mean curvature vector H of $f_t(N)$ is

$$H = \sum_{i=1}^m \nabla_{e_i} e_i = 0.$$

Therefore

$$\begin{aligned}
0 &= \left(\nabla_{\eta^\perp} \sum_{i=1}^m \nabla_{e_i} e_i\right)^\perp \\
&= \sum_{i=1}^m (\nabla_{e_i}\nabla_{\eta^\perp} e_i)^\perp + \sum_{i=1}^m (\bar{R}(\eta^\perp, e_i)\, e_i)^\perp \qquad (\bar{R}\text{: curvature of } M) \\
&= \sum_{i=1}^m (\nabla_{e_i}\nabla_{e_i}\eta^\perp)^\perp + \sum_{i=1}^m (\bar{R}(\eta^\perp, e_i)\, e_i)^\perp \\
&= \sum_{i=1}^m (\nabla_{e_i}(\nabla_{e_i}\eta^\perp)^\perp)^\perp + \sum_{i=1}^m (\nabla_{e_i}(\nabla_{e_i}\eta^\perp)^\top)^\perp + \sum_{i=1}^m (\bar{R}(\eta^\perp, e_i)\, e_i)^\perp \\
&= \Delta\eta^\perp + \mathscr{B}(\eta^\perp) + \bar{R}(\eta^\perp),
\end{aligned}$$

where $\perp$ and $\top$ mean the projections onto the normal and the tangent bundles of N, respectively.

Corollary 1. *Let $\varkappa$ be a Killing vector field on M and N a minimal submanifold of M. Then $\varkappa(N^\perp)$, the projection of $\varkappa$ onto the normal bundle of N, is a Jacobi field on N.*

Let ϕ_n, ϕ_l, and ϕ_p, respectively, be the variation vector fields in $\boldsymbol{R}^3$ associated with a 1-parameter family of translations τ_t^n in the direction of a unit vector n, a 1-parameter family of rotations ϱ_t^l around a straight line l, and a 1-parameter family of homothetic expansions μ_t^p with center at p.

Corollary 2. *Let Σ be a minimal surface in $\boldsymbol{R}^3$. Then $\phi_n(\Sigma^\perp)$, $\phi_l(\Sigma^\perp)$, and $\phi_p(\Sigma^\perp)$ (the projections of ϕ_n, ϕ_l, and ϕ_p onto $N(\Sigma)$, respectively) are Jacobi fields on Σ.*

Corollary 3. *i) Let $\varkappa$ be a Killing vector field on M and N a minimal hypersurface in M. Then every connected set $D \subset N$ which is visible with respect to $\varkappa$ is stable. ii) Let ϕ_p be the position vector field in $\boldsymbol{R}^n$ from a point $p \in \boldsymbol{R}^n$,* (i.e., *$\phi_p(x) = x - p$) and S a minimal hypersurface in $\boldsymbol{R}^n$. Then every connected set $D \subset S$ visible with respect to ϕ_p is stable.*

Proof. In either case one can construct near D a foliation by minimal hypersurfaces. The visibility of D implies that the foliation is not singular. Hence the classical argument [**HL**] shows that any variation of D with ∂D fixed increases the volume.

In [**C**] we showed that for any compact minimal submanifold M of $\boldsymbol{R}^n$ and any point p in $\boldsymbol{R}^n$, the volume of M is less than or equal to the volume of $p \# \partial M$, the cone from p over ∂M. In the following proposition we see when these volumes are equal.

Proposition 1. *Let D be a relatively compact domain in a k-dimensional minimal submanifold M of $\boldsymbol{R}^n$ and p a point in $\boldsymbol{R}^n$ such that*

$$\partial D \subset H(M; \phi_p).$$

Let ν be the outward unit conormal to ∂D on D and define

$$\Gamma_1 = \{q \in \partial D : \phi_p(q) \cdot \nu(q) \geqq 0\}, \quad \Gamma_2 = \{q \in \partial D : \phi_p(q) \cdot \nu(q) < 0\}.$$

Then we have

$$\text{Volume}\,(D) = \text{Volume}\,(p \# \Gamma_1) - \text{Volume}\,(p \# \Gamma_2).$$

Proof. Let $r(x) = |\phi_p(x)|$, $x \in M$. That M is minimal in $\boldsymbol{R}^n$ implies

$$\Delta r^2 = 2k.$$

Integrating this over D, we get

$$\text{Volume}\,(D) = \frac{1}{k} \int_{\partial D} r \frac{\partial r}{\partial \nu} = \frac{1}{k} \int_{\partial D} \phi_p \cdot \nu = \frac{1}{k} \int_{\Gamma_1} \phi_p \cdot \nu + \frac{1}{k} \int_{\Gamma_2} \phi_p \cdot \nu.$$

Since $\phi_p(x)$ is tangent to M at $x \in \partial D$, we can easily see that

$$\frac{1}{k} \int_{\Gamma_1} \phi_p \cdot \nu = \text{Volume}\,(p \# \Gamma_1), \quad \frac{1}{k} \int_{\Gamma_2} \phi_p \cdot \nu = -\text{Volume}\,(p \# \Gamma_2).$$

This completes the proof.

2. Morse index theorem and vision number

The original Morse index theorem is a formula which relates the index of a geodesic segment to its conjugate points relative to one end point. In 1965, SMALE [**Sm**] substantially generalized this result to a theorem which, in a similar way, evaluates the index of any strongly elliptic self-adjoint differential operator on the cross-sections of a Riemannian vector bundle. The theorem applies perfectly to the case of minimal submanifolds, and gives a natural generalization of the index theorem for geodesics, as was observed by SIMONS [**Si, L2**]. The following is an exposition of SMALE's result in the framework of minimal submanifolds.

Let N be a compact connected minimal submanifold with nonempty boundary in a Riemannian manifold M. Let c_t be a contraction of N into itself. In particular, assume c_t, $t \geqq 0$, is a smooth family of diffeomorphisms of N into N such that

(i) $c_0 =$ identity,

(ii) $c_t(N) \subset c_s(N) \quad$ for $t > s$,

(iii) $\lim_{t \to \infty}$ Volume $(c_t(N)) = 0$.

Theorem (SMALE).

$$\operatorname{Ind}(N) = \sum_{t>0} \operatorname{Nullity}(c_t(N)),$$

where Nullity $(c_t(N))$ *is the dimension of the space of Jacobi fields on* $c_t(N)$ *vanishing on the boundary of* $c_t(N)$.

Choosing a suitable contraction c_t which is determined by the visible components of a minimal surface, and compactifying the minimal surface, we obtain the following theorem.

Theorem 1. *For any unit vector n in $\mathbf{R}^3$ and any minimal surface Σ in $\mathbf{R}^3$ of finite total curvature, orientable or nonorientable, we have*

$$\operatorname{Ind}(\Sigma) \geqq v(\Sigma; \phi_n) - 1. \tag{1}$$

Proof. By hypothesis Σ is conformally equivalent to a Riemann surface $\bar{\Sigma}$ with punctures $\{p_1, \ldots, p_k\}$. Let ds^2 be the metric on Σ and $d\bar{s}^2$ a smooth metric on $\bar{\Sigma}$ such that

$$ds^2 = \mu \, d\bar{s}^2, \quad \mu > 0 \text{ on } \bar{\Sigma}.$$

Then

$$L = \Delta - 2K = \frac{1}{\mu}(\bar{\Delta} - 2\mu K) = \frac{1}{\mu}\bar{L}, \tag{2}$$

where $\bar{\Delta}$ is the Laplacian in the normal bundle of $\bar{\Sigma}$ with the metric $d\bar{s}^2$. It is shown in [F] that

$$\operatorname{Ind}(\Sigma) = \operatorname{Ind}_{\bar{L}}(\bar{\Sigma}), \tag{3}$$

where $\operatorname{Ind}_{\bar{L}}(\bar{\Sigma})$ is the index of $(\bar{\Sigma}, d\bar{s}^2)$ with respect to $\bar{L}$. Even though the orientability of $\bar{\Sigma}$ is assumed in [F], one can easily see that the arguments of Theorem 2 and Corollary 2 of [F] are also valid for nonorientable Σ. Hence (3) holds for nonorientable Σ as well. Therefore we will show

$$\operatorname{Ind}_{\bar{L}}(\bar{\Sigma}) \geqq v(\Sigma; \phi_n) - 1.$$

Note that

$$\phi_n(\Sigma^{\perp}) = (n \cdot \nu)\, \nu, \quad \nu \text{ a unit normal of } \Sigma,$$

and that ν extends smoothly across $\{p_1, \ldots, p_k\}$. Hence $\phi_n(\Sigma^\perp)$ is a smooth bounded section of the "normal bundle" of $(\bar{\Sigma}, d\bar{s}^2)$ (*i.e.*, the vector bundle of $(\bar{\Sigma}, d\bar{s}^2)$ associated with the normal bundle of Σ). From (2) we see that

$$\bar{L}\phi_n(\Sigma^\perp) = 0 \quad \text{on } \bar{\Sigma} \sim \{p_1, \ldots, p_k\}.$$

Then by HARVEY & POLKING's removable singularity theory **[HP, P]**, we have

$$\bar{L}\phi_n(\Sigma^\perp) = 0 \quad \text{on } \bar{\Sigma}.$$

Now let us apply the generalized Morse index theorem to $(\bar{\Sigma}, d\bar{s}^2)$. Set $k = v(\Sigma; \phi_n)$ and let $V_1, \ldots, V_k \subset \Sigma$ be the open visible components with respect to ϕ_n such that

$$\bigcup_{1 \leqq i \leqq k} V_i = \Sigma \sim H(\Sigma; \phi_n).$$

Let $\bar{V}_1, \ldots, \bar{V}_k \subset \bar{\Sigma}$ be the open subsets of $\bar{\Sigma}$ corresponding to $V_1, \ldots, V_k \subset \Sigma$ under the conformal equivalence between Σ and $\bar{\Sigma}$. Then we can exhaust $\bar{V}_1, \ldots, \bar{V}_k$, one by one, by a 1-parameter family of the complements of shrinking domains in $\bar{\Sigma}$. More precisely, we can find a 1-parameter family of domains D_t of $\bar{\Sigma}$ with nonempty piecewise smooth boundary $0 \leqq t \leqq k$, such that, after suitably renumbering $\bar{V}_1, \ldots, \bar{V}_k$,

i) $\bar{\Sigma} \sim D_0 \subset \bar{V}_1$ and $D_k \subset \bar{V}_k$,

ii) D_t is properly contained in D_s if $t > s$,

iii) the function $f(t) = \text{Area}\,(D_t)$ is continuous in t,

iv) for every integer $1 \leqq j \leqq k-1$, $\bigcup_{1 \leqq i \leqq j} \bar{V}_i \subset \bar{\Sigma} \sim D_j$ and $\bigcup_{j+1 \leqq i \leqq k} \bar{V}_i \subset D_j$,

v) the area of D_k is less than ε ,with ε as in **[Sm]**.

Obviously $\bar{L}$ is a self-adjoint strongly elliptic operator and has uniqueness in the Cauchy problem; that is, if u is smooth, $\bar{L}u = 0$, and $u \mid D = 0$ for some open set D, then $u = 0$ everywhere. Let $\alpha(t)$ be the dimension of the space of all sections u of the normal bundle of $\bar{\Sigma}$ restricted on D_t such that $\bar{L}u = 0$ on D_t and $u \mid \partial D_t = 0$. SMALE proved that if ∂D_t is smooth for all t then

$$\operatorname{Ind}_{\bar{L}}(D_0) \geqq \sum_{0 < t \leqq k} \alpha(t), \tag{4}$$

and the equality holds in (4) if ∂D_t depends in a smooth manner on t. However, Lemma 1 (discreteness of eigenvalues) and Lemma 2 ($\lambda_k \leqq \lambda_k'$ if $D \supset D'$) of **[Sm]** (except for the continuity of eigenvalues in t) are valid even for domains with piecewise smooth boundaries. Therefore (4) holds also in our setting. Since $\phi_n(\Sigma^\perp)$ vanishes on $H(\Sigma; \phi_n)$, and for $t = 1, \ldots, k-1$, ∂D_t corresponds to a subset of $H(\Sigma; \phi_n)$ under the conformal equivalence, we have

$$\phi_n(\Sigma^\perp) \mid \partial D_t = 0 \quad \text{for } t = 1, \ldots, k-1,$$

where $\phi_n(\Sigma^\perp)$ is viewed as a section of the normal bundle of $\bar{\Sigma}$. Thus

$$\alpha(t) \geqq 0 \quad \text{for } t = 1, \ldots, k-1.$$

This completes the proof.

3. Lower bound for index

Corollary 4. *Let Σ_k be the* JORGE-MEEKS' *immersed minimal surface which is conformally equivalent to the sphere minus k points* [JM]. *Then*

$$\operatorname{Ind}(\Sigma_k) \geqq 2k - 3.$$

Proof. Σ_k is the image of an immersion f_k from $S^2 \sim \{\pi^{-1}(\theta) : \theta \in C, \theta^k = 1\}$, where $\pi : S^2 \to \boldsymbol{R}^2$ is the stereographic projection, into $\boldsymbol{R}^3$ which is defined by the Weierstrass representation with $f(z) = (z^k - 1)^{-2}$ and $g(z) = z^{k-1}$. f_k is well-defined at $\pi^{-1}(\infty)$. We know that

$$g = \pi \circ N \circ f_k \circ \pi^{-1},$$

where $N : \Sigma_k \to S^2$ denotes the Gauss map [L2]. Choose a unit horizontal vector n in $\boldsymbol{R}^3$. The normal vectors of Σ at the two points $f_k(\pi^{-1}(0))$ and $f_k(\pi^{-1}(\infty))$ in Σ_k are vertical and thus these points are contained in $H(\Sigma_k; \phi_n)$. We claim that $f_k^{-1}(H(\Sigma_k; \phi_n))$ is the union of $k-1$ great circles in S^2 meeting at north and south poles at equal angles $\dfrac{\pi}{k-1}$. Let C be the great circle on $S^2 \subset \boldsymbol{R}^3$ passing through the two poles which is the intersection of S^2 with the plane orthogonal to n. Then

$$f_k^{-1}(H(\Sigma_k; \phi_n)) = f_k^{-1}(N^{-1}(C)) = (Nf_k)^{-1}(C) = (\pi^{-1} g \pi)^{-1}(C).$$

Hence the claim follows from the fact that $\pi^{-1} g\pi$ is a holomorphic branched covering map from S^2 onto itself of degree $k-1$ with branch points at north and south poles. Furthermore, it follows that the total curvature of Σ_k is $-4\pi(k-1)$. Since $f_k^{-1}(H(\Sigma_k; \phi_n))$ divides S^2 into $2k-2$ components and f_k is a homeomorphism, we have

$$v(\Sigma_k; \phi_n) = 2k - 2.$$

Thus Corollary 4 follows from Theorem 1.

Theorem 2. *Let Σ be a complete minimal surface in $\boldsymbol{R}^3$ of finite total curvature. If each end of Σ is embedded* (*see* [Sc] *for the definition of the end of Σ*) *and the normal vectors at the points of Σ at infinity are all parallel to a line l, then*

$$\operatorname{Ind}(\Sigma) \geqq v(\Sigma; \phi_l) - 1.$$

Proof. By Proposition 1 of [Sc], Σ is regular at infinity and hence there is a compact subset $K \subset \boldsymbol{R}^3$ such that $\Sigma \sim K$ consists of r components $\Sigma_1, \ldots, \Sigma_r$ such that each Σ_i is the graph of a function u_i with bounded slope over the exterior of a bounded region in some plane Π_i which is perpendicular to the line l. Moreover, if x_1, x_2 are coordinates in Π_i with the origin at $\Pi_i \cap l$, the u_i has the following asymptotic behavior for $|x|$ large:

$$u_i(x) = a \log |x| + b + \frac{c_1 x_1}{|x|^2} + \frac{c_2 x_2}{|x|^2} + O(|x|^{-2}), \tag{5}$$

where constants a, b, c_1, c_2 depend on i, and $O(|x|^{-2})$ indicates a term which is bounded in absolute value by a constant times $|x|^{-2}$ for $|x|$ large. By Corollary 2 $\phi_l(\Sigma^\perp)$ is a Jacobi field on Σ, and one can easily see that up to a constant multiple

$$\phi_l(\Sigma^\perp) = [\lambda \cdot (r \times \nu)]\, \nu,$$

where λ is a unit vector on l, r is the position vector from the origin, and ν is a unit normal vector of Σ. Hence

$$|\phi_l(\Sigma^\perp)\,(x, u_i(x))| \leqq |x| \cdot \frac{|\nabla u_i|}{\sqrt{1 + |\nabla u_i|^2}} \leqq |x| \cdot |\nabla u_i| \leqq a.$$

However the first two terms in the right-hand side of (5) are rotationally invariant and thus they do not contribute to $\phi_l(\Sigma^\perp)$. Hence

$$|\phi_l(\Sigma^\perp)\,(x, u_i(x))| \leqq \frac{c}{|x|}. \tag{6}$$

Now, as in Theorem 1, we have

$$\bar{L}\phi_l(\Sigma^\perp) = 0 \quad \text{on } \bar{\Sigma} \sim \{p_1, \ldots, p_r\}.$$

From (6) one can see that $\phi_l(\Sigma^\perp)$ is continuous at $p_1, \ldots, p_r$,and therefore $p_1, \ldots, p_r$ are removable singularities for $\bar{L}$ (see [P]). Thus Theorem 2 follows from the same arguments as does Theorem 1.

Corollary 5. *If Σ_g is the complete embedded minimal surface of genus g constructed by* HOFFMAN & MEEKS [HM], *then*

$$\operatorname{Ind}(\Sigma_g) \geqq 2g + 1.$$

Proof. Σ_g has total curvature $-4\pi(g + 2)$ and of course each end of Σ_g is embedded. Moreover, Σ_g is symmetric with respect to $g + 1$ planes $\Pi_1, \ldots, \Pi_{g+1}$ which meet each other at equal angles $\frac{\pi}{g+1}$ along the line $l = \bigcap_{1 \leqq i \leqq g+1} \Pi_i$. Therefore the normal vectors of Σ_g at the points at infinity are parallel to the line l. Then by the previous theorem we have

$$\operatorname{Ind}(\Sigma_g) \geqq v(\Sigma_g; \phi_l) - 1. \tag{7}$$

Since Σ_g is orthogonal to each Π_i, three vectors λ, $r(x)$, and $\nu(x)$ all lie in Π_i for every $x \in \Sigma_g \cap \Pi_i$, and thus we have

$$\phi_l(\Sigma_g^\perp) = [\lambda \cdot (r \times \nu)]\, \nu = 0 \quad \text{on } \Sigma_g \cap \Pi_i.$$

It follows that

$$H(\Sigma_g; \phi_l) = \{x \in \Sigma_g : \phi_l(\Sigma_g^\perp)(x) = 0\} \supset \Sigma_g \cap \Bigl(\bigcup_{1 \leqq i \leqq g+1} \Pi_i\Bigr).$$

Since $\bigcup_{1 \leq i \leq g+1} \Pi_i$ divides Σ_g into $2g+2$ nonempty components none of which is a subset of $H(\Sigma_g; \phi_l)$, we get

$$v(\Sigma_g; \phi_l) \geqq 2g+2. \tag{8}$$

(7) and (8) give the desired result.

The following proposition gives a relationship between the vision number and the total curvature of a complete minimal surface.

Proposition 2. *i) If Σ is a complete minimal surface in $\mathbf{R}^3$ of finite total curvature then for any unit vector n in $\mathbf{R}^3$,*

$$v(\Sigma; \phi_n) \leqq -\frac{1}{2\pi} \int_\Sigma K.$$

ii) If Σ is a complete minimal surface in $\mathbf{R}^3$ of finite total curvature, each end of Σ is embedded, and the normal vectors at the points of Σ at infinity are parallel to a line l, then

$$v(\Sigma; \phi_l) \leqq -\frac{1}{2\pi} \int_\Sigma K.$$

Proof. As we have seen in Theorems 1 and 2, $\phi_n(\Sigma^\perp)$ and $\phi_l(\Sigma^\perp)$ can extend to smooth bounded vector fields on $\overline{\Sigma}$ satisfying $\overline{L}u = 0$. Let D be a component of $\Sigma \sim H(\Sigma; \phi_n)$ or $\Sigma \sim H(\Sigma; \phi_l)$. Suppose $\int_D -K < 2\pi$. Then there exists a connected domain $D' \subset \Sigma$ such that $D \subset D'$, Area $(D' \sim D) > 0$, and $\int_{D'} -K < 2\pi$. Let $\overline{D}$ and $\overline{D}'$ be the subsets of $\overline{\Sigma}$ corresponding to D and D' under the conformal equivalence of Σ into $\overline{\Sigma}$. Then

$$\lambda_1(\overline{D}') < \lambda_1(\overline{D}) = 0,$$

where λ_1 denotes the first eigenvalue of the operator $\overline{L}$ on $(\overline{\Sigma}, d\overline{s}^2)$. However, by the theorem of Barbosa & do Carmo [BC], D' is stable and hence

$$\lambda_1(\overline{D}') \geqq 0,$$

which is a contradiction. Therefore

$$\int_D -K \geqq 2\pi$$

and this gives the desired results.

Now it is not difficult to extend the above arguments to more general case; minimal submanifolds in space forms. We state the following theorem and corollary without proof.

Theorem 3. *Let N be a complete minimal submanifold in a space form M and ϕ a Killing vector field on M. If N is compact, then*

$$\operatorname{Ind}(N) \geqq v(N; \phi) - 1,$$

and otherwise,

$$\operatorname{Ind}(N) \geqq \bar{v}(N; \phi),$$

where $\bar{v}(N; \phi)$ is the number of bounded components of $N \sim H(N; \phi)$.

Corollary 6. *Let ϕ_p be the position vector field in $\boldsymbol{R}^n$ from a point $p \in \boldsymbol{R}^n$. If S is a complete minimal submanifold in $\boldsymbol{R}^n$, then*

$$\operatorname{Ind}(S) \geqq \bar{v}(S; \phi_p).$$

Corollary 7. *Let $\xi_{m,k}$ be the embedded minimal surface of genus mk in S^3 which is constructed by* LAWSON **[L1]**. *Then*

$$\operatorname{Ind}(\xi_{m,k}) \geqq \max(2m+1, 2k+1).$$

Proof. LAWSON constructed $\xi_{m,k}$ in the following way. Let C_1 and C_2 be the great circles in $S^3 \subset \boldsymbol{R}^4 = \boldsymbol{C}^2$ defined by

$$C_1 = \{(o, \omega) \in \boldsymbol{C}^2 : |\omega| = 1\}, \quad C_2 = \{(z, o) \in \boldsymbol{C}^2 : |z| = 1\}.$$

Let $P_1, \ldots, P_{2k+2}$ and $Q_1, \ldots, Q_{2m+2}$ be equally spaced points on C_1 and C_2 respectively. Then there exists a geodesic polygon $P_1Q_1P_2Q_2$ which consists of the geodesic segments $\overline{P_1Q_1}$, $\overline{Q_1P_2}$, $\overline{P_2Q_2}$, and $\overline{Q_2P_1}$. The polygon $P_1Q_1P_2Q_2$ spans a minimal surface which is the Morrey's solution to the Plateau problem for $P_1Q_1P_2Q_2$. Extending this surface by reflection across its geodesic boundary arcs, one can get $\xi_{m,k}$.

Let us assume without loss of generality that

$$m \geqq k.$$

Let ϱ_t be the one-parameter family of rotations in S^3 around C_1 by angle t. Then the variation vector field ϕ of ϱ_t is a Killing vector field vanishing on C_1. It follows from Theorem 3 that

$$\operatorname{Ind}(\xi_{m,k}) \geqq v(\xi_{m,k}; \phi) - 1.$$

Let $S_1, \ldots, S_{m+1}$ be the unit spheres in S^3 containing C_1 and bisecting the geodesic segments $\overline{Q_1Q_2}, \overline{Q_2Q_3}, \ldots, \overline{Q_{2m+1}Q_{2m+2}}$, and $\overline{Q_{2m+2}Q_1}$. Clearly $\xi_{m,k}$ is symmetric with respect to each S_i and hence $\xi_{m,k}$ is perpendicular to each S_i. Since ϕ is orthogonal to S_i, we have

$$\phi(x) \in T_x\xi_{m,k} \quad \text{for all } x \in \xi_{m,k} \cap S_i.$$

Therefore

$$\xi_{m,k} \cap \left(\bigcup_{1 \leqq i \leqq m+1} S_i \right) \subset H(\xi_{m,k}; \phi).$$

Since $\bigcup_{1 \leq i \leq m+1} S_i$ divides $\xi_{m,k}$ into $2m + 2$ nonempty components, none of these being a subset of $H(\xi_{m,k}; \phi)$, we have

$$v(\xi_{m,k}; \phi) \geqq 2m + 2,$$

which completes the proof.

Corollary 7. *For all minimal submanifolds N in S^{n+k-1} of the form* $N = S^{p_1}\left(\sqrt{\frac{p_1}{n}}\right) \times \ldots \times S^{p_k}\left(\sqrt{\frac{p_k}{n}}\right)$, $p_1 + \ldots + p_k = n$,

$$\operatorname{Ind}(N) \geqq 3.$$

Proof. Let $(x_1, \ldots, x_{n+k})$ be the Euclidean coordinate of a point q in S^{n+k-1}. Then at $q \in N$ every normal vector ν of N in $T_q S^{n+k-1}$ is of the form

$$\nu = (t_1 x_1, \ldots, t_1 x_{p_1+1}, t_2 x_{p_1+2}, \ldots, t_2 x_{p_1+p_2+2}, t_3 x_{p_1+p_2+3}, \ldots, t_k x_{n+k}),$$

where

$$\sum_{i=1}^{k} t_i p_i = 0.$$

The vector field

$$\phi = (x_{n+k}, 0, \ldots, 0, -x_1)$$

is a Killing vector field on S^{n+k-1} associated with the rotation in $x_1 x_{n+k}$-plane. Then

$$\nu \cdot \phi = (t_1 - t_k)\, x_1 x_{n+k}.$$

Since $t_1 - t_k$ can be arbitrary, it follows that

$$H(N; \phi) = \{(x_1, \ldots, x_{n+k}) \in N;\ x_1 = 0 \text{ or } x_{n+k} = 0\}.$$

Therefore $v(N; \phi) = 4$, and the corollary follows from Theorem 3.

4. Instability and counterexample

In this section we use Theorem 1 to give a new proof of the following theorem.

Theorem 4. (DO CARMO & PENG, FISCHER-COLBRIE & SCHOEN). *The only complete stable oriented minimal surface Σ in $\boldsymbol{R}^3$ is the plane.*

Proof. Since Σ is stable, $\operatorname{Ind}(\Sigma) = 0$ and so by [F] Σ has finite total curvature. Suppose Σ is not a plane. Then the normals to Σ cover S^2 except for a finite number of points. Hence for any unit vector n in $\boldsymbol{R}^3$ we have

$$v(\Sigma; \phi_n) \geqq 2. \tag{9}$$

Then by Theorem 1 $\operatorname{Ind}(\Sigma) \geqq 1$ and thus Σ is not stable.

In showing (9), we have used the orientability of Σ. If Σ is nonorientable we get only $v(\Sigma; \phi_n) \geqq 1$ and $\operatorname{Ind}(\Sigma) \geqq 0$, and thereby gaining nothing toward the instability of Σ. On the other hand, every section E of the normal bundle of Σ gives rise to a piecewise smooth function f on Σ such that $E = f\nu$, ν a unit normal of Σ. But not every function f gives rise to a section E of the normal bundle of Σ if Σ is nonorientable (*e.g.* constant function on Σ). This is the main reason why the proofs of DO CARMO & PENG and FISCHER-COLBRIE, & SCHOEN do not work for nonorientable Σ. As a matter of fact, this very reason will lead us to a counterexample; a complete stable nonorientable minimal surface in $\boldsymbol{R}^3$. The following lemma is an extension of BARBOSA & DO CARMO's result [BC] to non-orientable minimal surfaces.

Lemma 2. *Let $D \subset N$ be a bounded nonorientable domain with piecewise smooth boundary on a nonorientable minimal surface N in $\boldsymbol{R}^3$. If the image of the oriented double cover of D under the Gauss map has area less than 4π, then D is stable.*

Proof. Let ϕ be a section of the normal bundle of D vanishing on ∂D. Then there must exist a nonempty subset C of D on which, and nowhere else, ϕ vanishes since otherwise D would be orientable with ϕ as a nonvanishing normal vector field on it. Denote $D' = D \sim C$. Then ϕ gives an orientation to D'. The image of D' under the Gauss map has area less than 2π and so, by BARBOSA & DO CARMO's result, D' is stable. Hence the second variation of the area of D with respect to ϕ is nonnegative. Since ϕ is arbitrary, D is stable.

The lemma above gives the following as an immediate consequence.

Theorem 5. *Any complete nonorientable minimal surface in $\boldsymbol{R}^3$ of total curvature -2π is stable.*

MEEKS [M] proved that the total curvature of evera complete immersed non-orientable minimal surface in $\boldsymbol{R}^3$ is at most -6π. Therefore any complete non-orientable minimal surface in $\boldsymbol{R}^3$ of total curvature -2π has at least one branch point. In fact, Henneberg's minimal surface [**Sp**, p. 405] has total curvature -2π and has two branch points. But we do not know whether or not Henneberg's surface is the only complete nonorientable minimal surface in $\boldsymbol{R}^3$ of total curvature -2π.

Theorem 6. *Let Σ be a complete immersed nonorientable minimal surface in $\boldsymbol{R}^3$ of finite total curvature. If Σ is conformally equivalent to a projective plane punctured at a finite number of points, then*

$$\operatorname{Ind}(\Sigma) \geqq 2.$$

Proof. Since Σ is nonorientable, the Gauss map $g: \Sigma \to S^2$ is not well defined. However the map $g_1 = p \circ g: \Sigma \to P^2$, for the covering map $p: S^2 \to P^2$, is a well defined branched covering map. g_1 then extends to a holomorphic map $\bar{g}$ from $\bar{\Sigma}$ onto P^2. Let $\hat{\Sigma}$ (Σ', respectively) be the orientable double cover of $\bar{\Sigma}$ (Σ, respectively) and $\hat{p}$ (p' respectively) the projection map of $\hat{\Sigma}$ (Σ' respectively) onto $\bar{\Sigma}$ (Σ, respectively) (see Figure). Here we think of Σ' as Σ with multiplicity two in $\boldsymbol{R}^3$. One can lift $\bar{g}$ (g_1, respectively) to $\hat{g}:\hat{\Sigma}\to P^2$($g':\Sigma'\to P^2$, respectively) such that $\hat{g}=\bar{g}\circ\hat{p}$ ($g'=g_1\circ p'$, respectively). By taking g to be the Gauss map on Σ' which is obviously well-defined, one can again lift g' to $g:\Sigma'\to S^2$ and so that $g'=p\circ g$. Accordingly, one can lift $\hat{g}$ to $\tilde{g}:\hat{\Sigma}\to S^2$ such that $\hat{g}=p\circ\tilde{g}$.

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, "\tilde{g}"] \arrow[d, "\hat{p}"'] \arrow[dr, "\hat{g}"] & S^2 \arrow[d, "p"] \\ \bar{\Sigma} \arrow[r, "\bar{g}'"'] & P^2 \end{tikzcd} \qquad \begin{tikzcd} \Sigma' \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, "g'"] & S^2 \arrow[d, "p"] \\ \Sigma \arrow[r, "g_1"'] & P^2 \end{tikzcd}$$

Figure

Let $\bar{p}_1,\dots,\bar{p}_k\in\bar{\Sigma}$ be the branch points of the map $\bar{g}$ and β_i the branching order of $\bar{g}$ at $\bar{p}_i$. Then β_i+1 is the local degree of $\bar{g}$ near $\bar{p}_i$. First the Riemann-Hurwitz formula [GH], applied to $\tilde{g}:\hat{\Sigma}\to S^2$, gives

$$2\sum_{i=1}^{k}\beta_i=-\chi(\hat{\Sigma})+\deg(\tilde{g})\cdot\chi(S^2),$$

where $\chi(M)$ denotes the Euler characteristic of M. From this we obtain

$$\sum_{i=1}^{k}\beta_i=-\chi(\bar{\Sigma})+\deg(\tilde{g})\cdot\chi(P^2). \tag{10}$$

Secondly, by [M], the total curvature of Σ is at most -6π. Hence

$$\deg(\tilde{g})=\deg(g)\geqq 3. \tag{11}$$

Thirdly, by hypothesis we have

$$\chi(\bar{\Sigma})=1. \tag{12}$$

From (10), (11), and (12) it follows that

$$\sum_{i=1}^{k}\beta_i\geqq 2.$$

Therefore there are two possibilities: either (i) $\beta_1=1$ and $\beta_2\geqq 1$, or (ii) $\beta_1\geqq 2$. Let p_j, $j=1,2$, be the point of Σ corresponding to $\bar{p}_j\in\bar{\Sigma}$ under the conformal equivalence of Σ into $\bar{\Sigma}$. Hence p_j can even be a point at infinity. Let ν_j be a unit normal vector of Σ at p_j, that is, $\nu_j=g(p_j)$. Then there exists a unit vector n such that

$$n\cdot\nu_j=0,\quad j=1,2\ (\text{or } j=1 \text{ only in case of (ii)}).$$

Thus $p_j\in H(\Sigma;\phi_n)$ and any sufficiently small neighborhood of p_j is divided by $H(\Sigma;\phi_n)$, like a pie, into $2\beta_j+2$ regions. Similarly a small neighborhood of $\bar{p}_j$ is divided by $\bar{H}(\Sigma;\phi_n)$, the subset of $\bar{\Sigma}$ corresponding to $H(\Sigma;\phi_n)$ under the conformal equivalence, into $2\beta_j+2$ regions.

Let $U_1, \ldots, U_J$ be the connected components of $U = \bar{\Sigma} \sim \bar{H}(\Sigma; \phi_n)$. If we denote by $\bar{K}$ the Gauss curvature of $(\bar{\Sigma}, d\bar{s}^2)$, then the Gauss-Bonnet formula for the domain U gives

$$\int_U \bar{K} + \sum_l \alpha_l = 2\pi \sum_{i=1}^{J} \chi(U_i), \tag{13}$$

where α_l's are the exterior angles ($= \pi$ minus the interior angles) of vertices of U_i's. Here we dropped the geodesic curvature term because $\bar{\Sigma}$ is closed. Obviously, we have

$$\int_U \bar{K} = \int_{\bar{\Sigma}} \bar{K} = 2\pi\chi(\bar{\Sigma}) = 2\pi. \tag{14}$$

Moreover, one can easily see that the sum of exterior angles at $\bar{p}_1$ and $\bar{p}_2$ in case of (i) or at $\bar{p}_1$ only in case of (ii) is at least 4π. Therefore

$$\sum_l \alpha_l \geqq 4\pi. \tag{15}$$

It follows from (13), (14), and (15) that

$$6\pi \leqq 2\pi \sum_{i=1}^{J} \chi(U_i).$$

Since each U_i is not closed we have

$$\chi(U_i) \leqq 1. \tag{16}$$

Therefore

$$3 \leqq \sum_{i=1}^{J} \chi(U_i) \leqq J.$$

Thus $v(\Sigma; \phi_n) \geqq 3$ and $\operatorname{Ind}(\Sigma) \geqq 2$.

Corollary 8. *If Σ is a complete immersed nonorientable minimal surface in $\boldsymbol{R}^3$ of finite total curvature which is conformally equivalent to a Klein bottle punctured at a finite number of points, then Σ is unstable.*

Proof. By hypothesis

$$\chi(\bar{\Sigma}) = \frac{1}{2\pi} \int_U \bar{K} = 0. \tag{17}$$

(10), (11), and (17) imply that

$$\sum_{i=1}^{k} \beta_i \geqq 3.$$

Even though here we have a better estimate than in Theorem 6, we cannot improve (15) because in general no more than two points of S^2 can determine a great circle on S^2. From (13), (15), (16), and (17) it follows that

$$2 \leqq \sum_{i=1}^{J} \chi(U_i) \leqq J.$$

Therefore $v(\Sigma; \phi_n) \geqq 2$ and Σ is unstable.

Very recently LOPEZ & ROS [**LR**] proved that Enneper's surface and the catenoid are the only complete immersed orientable minimal surfaces of index one in $\boldsymbol{R}^3$. Using a different argument, CHENG & TYSK [**CT**] obtained a similar result with an additional assumption that the minimal surface has embedded ends. Here we prove a stronger theorem in case the minimal surface has genus zero.

Theorem 7. *Let Σ be a complete immersed orientable nonplanar minimal surface of genus zero in $\boldsymbol{R}^3$. If Σ is neither Enneper's surface nor the catenoid, then*

$$\operatorname{Ind}(\Sigma) \geqq 3.$$

Proof. By [**F**] it suffices to prove the theorem for minimal surfaces of finite total curvature. We use the arguments of the proof of Theorem 6. Since Enneper's surface and the catenoid are the only complete immersed orientable minimal surfaces in $\boldsymbol{R}^3$ of total curvature -4π, we have for the Gauss map g of Σ

$$\deg(g) \geqq 2.$$

Hence, again by the Riemann-Hurwitz formula,

$$\sum \beta_i \geqq 2$$

and so (15) follows. By hypothesis,

$$\int_{\Sigma} \bar{K} = 4\pi. \tag{18}$$

Then from (13), (15), (16), and (18) it follows that

$$4 \leqq \sum_{i=1}^{J} \chi(U_i) \leqq J.$$

Therefore

$$\operatorname{Ind}(\Sigma) \geqq 3.$$

Open problems. (i) Show that if Σ is a complete minimal surface in $\boldsymbol{R}^3$, orientable or nonorientable, with Gauss curvature K, then

$$\operatorname{Ind}(\Sigma) \leqq -1 - \frac{1}{2\pi} \int_{\Sigma} K.$$

In view of Corollary 4, Proposition 2, Theorem 5, Theorem 6, and BARBOSA & DO CARMO's result [**BC**], this conjecture seems quite plausible. If true, this conjecture will give affirmative answers to the problems (ii) and (iv) below.

(ii) Prove that there is no complete immersed stable nonorientable minimal surface in $\mathbf{R}^3$.

(iii) Show that for any immersed nonorientable surface S in $\mathbf{R}^3$ without boundary there exists a unit vector n in $\mathbf{R}^3$ such that

$$v(S; \phi_n) \geqq 2.$$

If this is true we can obtain the nonexistence of complete immersed stable nonorientable minimal surfaces in $\mathbf{R}^3$ of finite total curvature. The following problem is in the same spirit as (i): Given a compact smooth surface $S \subset \mathbf{R}^3$ without boundary, orientable or nonorientable, prove that there exists a unit vector n in $\mathbf{R}^3$ such that

$$v(S; \phi_n) \geqq 4 - \chi(S).$$

(iv) Is it true that a complete nonorientable minimal surface in $\mathbf{R}^3$ has finite index if and only if it has finite total curvature?

(v) Prove that in $\mathbf{R}^3$ there are no complete orientable minimal surfaces of index two and no complete nonorientable minimal surfaces of index one. Prove also that no minimal surface in S^3 has index two. This may be related to the fact that it is impossible to immerse minimally the real projective plane into S^3.

(vi) Show that if Σ is a complete minimal surface in H^3 with more than two ends, then Σ is unstable. In H^3 there exist an unstable catenoid and an area minimizing catenoid, both having the same boundary at infinity.

(vii) Show that if Σ is a complete orientable minimal surface in H^3 of genus $g \geqq 1$, then Σ is unstable.

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Note added in proof. Several people have informed me of their new relevant results as follows.

(i) EJIRI & MICALLEF showed that if $\Sigma \subset \boldsymbol{R}^n$ is a complete branched minimal surface of genus g with finite total curvature, then

$$\operatorname{Ind}(\Sigma) \leqq g - 1 - \frac{1}{2\pi} \int_\Sigma K.$$

This result and Corollary 4 imply that $\operatorname{Ind}(\Sigma_k) = 2k - 3$. Moreover, combined with Theorem 7, their result shows that if $\Sigma \subset \boldsymbol{R}^3$ is a complete immersed oriented minimal surface of genus zero and total curvature -8π, then $\operatorname{Ind}(\Sigma) = 3$.

(ii) LI & TAM proved that if $\Sigma \subset \boldsymbol{R}^3$ is a complete minimal surface whose Gauss mag $g: \Sigma \to S^2$ is a d-sheeted branched cover with branch points located on a great circle of S^2, then

$$\operatorname{Ind}(\Sigma) = 2d - 1.$$

(iii) LI, STENGER, TAM & TREIBERGS showed that the index of COSTA's minimal surface, Σ_1 of Corollary 5, is either 4 or 5.
(iv) M. ROSS solved the open problem (ii) affirmatively with the additional assumption that the compactified double covering of the minimal surface is hyperelliptic.
(v) As for the open problems (vi) and (vii), G. DE OLIVEIRA constructed counterexamples in H^3: area minimizing surfaces with more than two ends, and area minimizing surfaces of genus $g \geqq 1$.