The metric properties of Lagrangians

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Dedicated to Professor Richard Schoen's 60th birthday

[1]

Abstract

This is a contribution to a special volume in honor of Professor R. Schoen for his sixtieth birthday. In the article, I introduce and survey some developments related to the metric properties of Lagrangian submanifolds. I would like to thank Professor Schoen for introducing me to this fascinating direction twenty years ago, and for his constant support and encouragements. The title here is almost the same as that in my thesis [40]. After so many years, the field is still full of interesting and important unanswered questions.

1 Introduction

A *symplectic manifold* M is a $2n$-dimensional manifold with a closed non-degenerated two form ω, which is called a *symplectic form*. Important symplectic manifolds include $(\mathbb{R}^{2n}, \sum_{j=1}^{n} \mathrm{d}x^j \wedge \mathrm{d}y^j)$, cotangent bundles with $\omega = \sum_{j=1}^{n} \mathrm{d}p^j \wedge \mathrm{d}q^j$, and Kähler manifolds with Kähler forms. A class of submanifolds on symplectic manifolds of special interests are *Lagrangian submanifolds* which are n-dimensional and on which the restriction of ω vanishes. More generally, we can consider immersions or other general maps from an n-dimensional manifold into M, and the Lagrangian condition will be the pull back of ω vanishing. The notion of *Lagrangian integral currents* can also be defined. We usually will not make distinctions between submanifolds and immersions.

The Kähler structure and Riemannian structure on a Kähler manifold are related by $\omega(u, v) = g(Ju, v)$, where J is its complex structure. There always exist compatible almost complex structures on a symplectic manifold and one can define compatible metrics from the above relation. We are very interested

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in understanding the interplay of symplectic structure and Riemannian structure, and that is this paper about. However, we will mainly focus on issues related to the metric properties of Lagrangians. Another important topic, Mirror Symmetry or more generally Homological Mirror Symmetry that concerns the duality of the space of complex structures on one manifold with the space of symplectic structures on another manifold, is not discussed. Even for issues concerned in this paper, the results mentioned here are still very selective and far from complete. There are many long papers in the field, and it is hard to follow sometimes. My original plan is to go through the literature, and write a comprehensive introduction and survey. But due to the limit of time and my current knowledge, the plan is hard to realize at this moment and I can only discuss things that I am more familiar with. I apologize for the incompleteness and hope that a complete survey will be available in the future. However, because many of the results appearing in this note are due to Professor Schoen, his collaborators, former students and post doctors, it can still somehow reflect Professor Schoen's great influence in this direction.

To make the article easier to read and not too lengthy, I will not list all definitions or state theorems in their complete forms. Most of the time, they are discussed in words and I may refer to material in later sections. The results are not stated in a chronical order, but in their related contents instead. This article should be considered as an informal introduction from the author's point of view, and the main purpose is to give a general picture without getting into many details. In section 2, I will do a short survey to give an overview and some rough ideas. A few definitions and properties are given in §3. Some consequences and implications of these properties are also included. Many different techniques have been applied to study the problems concerned here. I will explain two of them, which are Geometric Measure Theory in §4 and singular perturbation in §5.

I am very happy and honored to be a descendant of Schoen's academic family. He is not only our role model as a mathematician, but also as a human being. I like to express my deep gratitude to him for his constant support and encouragements, and wish him have a very Happy Birthday.

2 A short survey

There are different notions related to the area of Lagrangians. A *minimal Lagrangian submanifold* is a Lagrangian submanifold with zero mean curvature. That is, it is a critical point of the area functional with respect to all variations. (It is called Lagrangian minimal in [40, 41, 42].) On a Calabi-Yau manifold, a minimal Lagrangian submanifold is equivalent to being a *special Lagrangian*, which is defined by the notion of *calibration* [16]. One can restrict variations to lie in Lagrangian category, and critical points of the area functional among such variations are called *Lagrangian stationary.* To get a nice expression of the Euler-Lagrangian equation, we usually further restrict the variations to be *Hamiltonian* which necessarily give Lagrangian variations. A critical point of the area functional among Hamiltonian variations is called *Hamiltonian stationary* [53]. Special Lagrangians have attracted much attention in recent years due to their critical role in string theory, particularly in constructing mirror manifolds and proving mirror symmetry [65]. Since special Lagrangians are calibrated, they are area minimizing in their homology class [16]. Minimal Lagrangian submanifolds in Kähler manifolds also admit many nice properties (see §3) which make them very canonical representatives in their homology classes, and can help us to obtain a deeper understanding on minimal submanifolds in higher co-dimension. Hamiltonian stationary Lagrangians are related models for incompressible elasticity theory and occur naturally when we take the variational approach initiated by Schoen and Wolfson in [58] to find minimal Lagrangians or special Lagrangians.

The existence of these objects is a fundamental and still wildly open question. Most of the known results are in $\mathbb{C}^n$, spaces with special structure, and $n = 2$ case. In the seminal paper of Harvey and Lawson [16], they give various ways of constructing special Lagrangians in $\mathbb{C}^n$. An important generalization is made by Lawlor to prove the angle conjecture [38]. These examples are generalized by Joyce in a series of papers ([24]–[27], [33]–[34]) and others. There are also a lot of research on the construction of special Lagrangian cones in $\mathbb{C}^n$. A variant of the special Lagrangian cones will also give a family of regular special Lagrangians in $\mathbb{C}^n$. Many of these studies use symmetry and group action to simplify the problem. When $n = 3$, the intersection of the special Lagrangian cone with the unit sphere is a two-dimensional *minimal Legendrian surface* in S^5 and theories in integrable system are used heavily in constructing T^2 special Lagrangian cones. We refer to [5, 9, 11, 17, 27, 47, 64] for works related to these results. Haskins and Kapouleas apply gluing techniques to obtain higher genus special Lagrangian cones in $\mathbb{C}^3$ [18]. They recently obtain a profound generalization of this construction to higher dimension [19, 20]. Special Lagrangian cones are determined by their links in S^{2n-1}, which project to minimal Lagrangians in $\mathbb{C}P^n$. Hence the existence of minimal Lagrangian submanifolds in $\mathbb{C}P^n$ are also obtained. Gluing and singular perturbation is a very powerful method which can also lead to other existence results. In [43], the author uses it to show that we can desingularize the transversal self-intersections of a compact special Lagrangian in Calabi-Yau manifolds to obtain a family of new embedded special Lagrangians which converge to the original special Lagrangian (an angle condition at intersections is necessary for $n > 3$). The situation for two special Lagrangians intersecting transversally are studied by Butscher for $\mathbb{C}^n$ in [6] and by D. Lee for Calabi-Yau manifolds in [39]. Joyce uses the method to study the desingularization of conical singularities in details in [28, 31, 32]. We will discuss more about the method and results in §5.

A $K3$ surface, which is a 2-dimensional Calabi-Yau manifold, admits a S^2 family of complex structures. Every special Lagrangian in $K3$ surfaces is holomorphic with respect to a different complex structure [16]. Therefore, special Lagrangians in this case are the same as holomorphic curves. This also motivates us to study minimal Lagrangians/special Lagrangians with the hope that they can serve as canonical representatives for their homology classes and even reveal indications on studying Hodge conjecture which is to find algebraic cycle representatives for Hodge classes [57]. The higher dimensional generalization of $K3$ surfaces are hyperkähler manifolds which also admit a S^2 family of complex structures. These manifolds are expected to have better properties, and can serve as test cases in many occasions. For instance, we use these examples to show that additional condition is necessary for $n > 3$ in the above mentioned desingularization result [43].

When $n = 2$, there is one additional advantage that we can study the problem from a mapping point of view. By analyzing the energy density of harmonic maps, Schoen proves in [56] the existence and uniqueness of *minimal Lagrangian diffeomorphisms* between hyperbolic surfaces with the same genus. The graph of a minimal Lagrangian diffeomorphism is a minimal Lagrangian in the product space. This result is generalized by the author to show the existence and uniqueness of minimal Lagrangian surfaces in some particular classes of the product of two hyperbolic surfaces (of different topology) by same techniques [41]. These surfaces are area minimizing in their homotopy classes because we in fact show that every minimal surface in the class is Lagrangian and hence unique. Also based on two dimensional analysis, the author has the following theorem in [42]: if there exist minimal Lagrangian branch immersions for one Kähler-Einstein surface with negative curvature, then there also exist minimal Lagrangian branch immersions for the whole connected component of the Kähler-Einstein metrics. Here we fix the Kähler class and vary its complex structure. In Calabi-Yau manifolds, one can also deform a special Lagrangian immersion to obtain the existence of special Lagrangian immersions for nearby Calabi-Yau metrics (a variant of [49]). However, we need to prove the result for singular cases as well to obtain a global deformation as above.

The most general construction for the existence problem is due to Schoen and Wolfson [58]. They minimize area among Lagrangian integral currents to obtain Lagrangian minimizers. Although all special Lagrangians must be among Lagrangian minimizers, the minimizers obtained are not a-priori minimal because they are constructed in a restricted class. It is a very important issue to develop a regularity theory for these Lagrangian minimizers. When $n = 2$, by setting the problem from the mapping point of view, Schoen and Wolfson show that *branch points* and *cone singularities* are the only possible singularities in their celebrated paper [58]. Their discussions are on symplectic 4-manifolds with compatible metrics. When the ambient space is a Kähler-Einstein surface, they show that the minimizer will be minimal when it admits no cone singularities. Schoen and Wolfson's results in particular shows the existence of (singular) Hamiltonian stationary Lagrangians, and the cone singularities appear to be the obstruction to the existence of special Lagrangians or minimal Lagrangians in Kähler-Einstein surfaces. The higher dimensional analogy of Schoen and Wolfson's regularity results is still not available. Hélein and Romon use integrable system theory to find a Weierstrass-type representation for all Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$ and $\mathbb{C}P^2$ [21, 22, 23]. Joyce, Schoen, and the author obtain families of smooth embedded small Hamiltonian stationary Lagrangians in every compact symplectic manifold with given compatible metric in [36]. Local criterion for the existence and characterization of a family of Hamiltonian stationary tori in Kähler manifolds are derived by the author in [44], and Butscher and Carvino in [7].

There are some other results on the existence of minimal Lagrangians: for instance, the case of Kähler-Einstein manifolds with positive scalar curvature and a T^n-action is studied by Goldstein [15], complex hyperquadrics studied by Palmer [54], $\mathbb{C}H^n$ studied by Castro, Montealegre, and Urbano [10], and etc. The with boundary case is also investigated by several authors in different settings. An existence result is obtained by Caffarelli, Nirenberg, and Spruck for an equation related to special Lagrangians in $\mathbb{C}^n$ under Dirichlet conditions [8]. The existence and uniqueness of minimal Lagrangian diffeomorphisms between strictly convex domains in $\mathbb{H}^2$ with the same area is proved by Brendle in [2]. The existence of minimal Lagrangian diffeomorphisms between uniformly convex domains in $\mathbb{R}^n$ is shown by Brendle and Warren in [3]. Earlier studies on the problem is made by Wolfson [69]. Another potential approach to the construction of special Lagrangians/minimal Lagrangians is the mean curvature flow- as the negative gradient flow of the area functional. The Lagrangian condition will be preserved when the solution is smooth and the ambient manifold is Kähler-Einstein [60]. Lagrangian mean curvature flow has been studied by various authors and we refer to [61, 63, 66, 67] for some of the discussions. Issues and references related to singularities of Lagrangian mean curvature flow will be mentioned in §4.

Besides the problem of existence, understanding singularities is another fundamental and challenging problem. From above discussions, we can see its importance and inevitability. To compactify the moduli space of special Lagrangians, one needs to enlarge the objects from smooth to singular. Integral currents which are generalizations of manifolds are very suitable for this purpose. It is very interesting to characterize the structure of singular set of special Lagrangian currents, which is expected to be better behaved than general area minimizing currents. The deformation of these singular special Lagrangians, both when the ambient Calabi-Yau structure is fixed or varied, plays a critical role in studying the moduli spaces and in developing a global existence theory of special Lagrangians. The simplest case that only conical singularities occur is studies by Joyce in [28]–[32]. The desingularization of transversal self-intersections in [43] can also be interpreted as moving from one boundary point to the interior of moduli space. In utilizing the variational approach of Schoen and Wolfson [58], the most crucial part is to develop a regularity theory. To generalize their beautiful results from 2-dimension to higher dimension, we must first face this important and difficult issue. Many two dimensional techniques used in [58] are not eligible any more, so substantial modification on the argument and new ideas are needed in studying the higher dimensional case. As mentioned in last paragraph, cone singularities are the obstruction to the existence of minimal Lagrangian surfaces in a Kähler-Einstein surface. Many efforts have been made trying to exclude them. But unfortunately, cone singularities do occur. Wolfson find a (non-algebraic) K3 surface where every Lagrangian minimizer there admits cone singularities [70]. It is important to understand the cone singularities and find ways to deal with them. Schoen and Wolfson analyze the local models of the cone singularities and find that they are cones of the curves in (7), which will be called *Schoen-Wolfson cones.* They also prove that the cone is stable if and only if $p = q + 1$ (assuming $p > q$) and conjecture that among these cones, only $(2, 1)$ cone can occur. In [45], Wang and the author construct *eternal solutions* of *Lagrangian Brakke flow* without mass loss to resolve a (p, q) Schoen-Wolfson cone with $p > q > 1$. It in particular distinguishes $(2, 1)$ cone from all other cones, and shows that other cones are not minimizing at infinitesimal level. There also exist eternal solutions of Lagrangian Brakke flow without mass loss that resolve any (p, q) cone with another pairing Schoen-Wolfson cone [45]. The results and Schoen-Wolfson cones are generalized to higher dimension in [46]. Other higher dimensional *Hamiltonian stationary cones* are also obtained in [20].

3 Definitions and properties

As mentioned in the introduction, we have the symplectic structure and Riemannian structure related by

$$\omega(u, v) = g(Ju, v). \tag{1}$$

If M is not Kähler, here J and g will be a compatible almost complex structure and a compatible metric of ω respectively. The Lagrangian condition then can be characterized as that JTL is equal to the normal bundle $T^{\perp}L$. On a Lagrangian immersion in a Kähler manifold, (1) implies that the second fundamental form defined by $B(X, Y, Z) = \langle \nabla_X Y, JZ \rangle$ is fully symmetric for any $X, Y, Z \in T_xL$. We can also define an isomorphism from T_xM to T_x^*M by

$$\alpha_v(\cdot) = \omega(v, \cdot). \tag{2}$$

A diffeomorphism ϕ on M is called a *symplectic map* if $\phi^*(\omega) = \omega$. Clearly symplectic maps will send Lagrangians to Lagrangians. Simple computation shows that the variational vector field v of symplectic maps is characterized by $\mathrm{d}\alpha_v = 0$. Such vector fields are called *symplectic vector fields.* When $\alpha_v = \mathrm{d}f$ or equivalently $v = -J\nabla f$, v will be called a *Hamiltonian vector field* and the variation it produce is called a *Hamiltonian deformation.* On a Lagrangian immersion L in a Kähler manifold, we have

$$\mathrm{d}\alpha_H|_L = \mathrm{Ric}\,|_L, \tag{3}$$

where H is the mean curvature vector of L, and Ric is the Ricci form of M defined by $\mathrm{Ric}(u, v) = R(Ju, v)$ from the Ricci tensor R (see [4, 14]). When consider (2) along L, we will require $v \in T^{\perp}L$ and it gives an isomorphism from $T^{\perp}L$ to T^*L. A generalization of (3) to symplectic manifolds is given in [58]. From (3), it follows that the restrictions of ω and Ric on a minimal Lagrangian will both vanish. This is an overdetermined condition and we cannot expect the existence of minimal Lagrangian immersions in Kähler manifolds except some obvious examples such as the fix set of an involution anti-holomorphic isometry. When M is Kähler-Einstein, the Ric form is proportional to ω and thus (3) implies $\mathrm{d}\alpha_H = 0$ on a Lagrangian. The class $\frac{1}{\pi}[\alpha_H] \in H^1(L;\mathbb{Z})$ is equal to the *Maslov class* of L. Moreover, it follows that H is an infinitesimal symplecic vector field. We therefore expect that the Lagrangian condition is preserved along *mean curvature flow* which deforms the submanifold along the direction of its mean curvature vector. This is indeed the case as long as the submanifold is smooth [60]. When M is a Calabi-Yau manifold, we have $\alpha_H = -\mathrm{d}\theta$ or equivalently $H = J\nabla\theta$ on an oriented Lagrangian, where θ is the *Lagrangian angle* $\theta : L \to \mathbb{R}/2\pi\mathbb{Z}$ or $\mathbb{R}$ defined by $\Omega|_L \equiv e^{i\theta} vol_L$, and Ω is a Calabi-Yau $(n,0)$ form. When L is minimal, it follows that θ is a constant θ_0 and $\mathrm{Re}(e^{-i\theta_0}\Omega)|_L \equiv vol_L$. That is, L is calibrated by $\mathrm{Re}(e^{-i\theta_0}\Omega)$. Recall that a smooth p-form φ on a Riemannian manifold M is called a *calibration* if it is closed and of comass 1 [16]. We say an oriented submanifold Σ (or more generally an integral current T) is calibrated by φ if $\varphi(T_x\Sigma) = 1$ for any $x \in \Sigma$ (or a.e. for currents). A direct consequence of Stoke Theorem shows that if T is calibrated, then

$$M(T) = T(\varphi) = T'(\varphi) \leqslant M(T') \tag{4}$$

for any integral current T' in the same homology class of T. Hence T has the least area in its homology class and if equality holds in (4), T' will also be calibrated by φ. Harvey and Lawson show in [16] that $\mathrm{Re}\,\Omega$ and $\mathrm{Re}(e^{-i\theta_0}\Omega)$ are calibrations, and hence every minimal Lagrangian integral current in a Calabi-Yau manifold is area minimizing.

Now we derive the Euler-Lagrangian equation for a Hamiltonian stationary Lagrangian in a symplectic manifold with a compatible metric as in [53]. We have

$$\begin{aligned} 0 = \frac{\mathrm{d}A_t}{\mathrm{d}t}|_{t=0} &= -\int_L \langle H, v\rangle \,\mathrm{d}V = -\int_L \langle \alpha_H, \alpha_v\rangle \,\mathrm{d}V \\ &= -\int_L \langle \alpha_H, \mathrm{d}f\rangle \,\mathrm{d}V = -\int_L \langle \delta\alpha_H, f\rangle \,\mathrm{d}V \end{aligned} \tag{5}$$

for any function f, and it implies $\delta\alpha_H = 0$. Here we use the Lagrangian condition that JTL is equal to $T^{\perp}L$ in the third equality above. Combining with (3), it follows that α_H is a harmonic 1-form for a Hamiltonian stationary Lagrangian in Kähler-Einstein manifolds. The condition for L to be Hamiltonian stationary in a Calabi-Yau manifold then becomes $\Delta_L\theta = 0$.

The second variational formula of the area at a minimal Lagrangian immersion in a Kähler manifold has the following nice expression [12, 53]:

$$\frac{\mathrm{d}^2A_t}{\mathrm{d}t^2}|_{t=0} = \int_L (|\mathrm{d}\alpha_v|^2 + |\delta\alpha_v|^2 - R(v,v))\mathrm{d}V, \tag{6}$$

where v is the variational field, α_v is defined as (2), δ is the Hodge dual of d, and R is the Ricci tensor of M. It implies that L is stable if M has nonpositive Ricci curvature, and L is unstable if M has positive Ricci curvature and

$H^1(L, \mathbb{Z}) \neq 0$. When M is Ricci flat, we have that L is stable and the Jocobi fields are those vectors whose associated 1-forms α_v on L are harmonic. McLean shows that in Calabi-Yau manifolds these Jacobi fields are indeed realized by deformations of special Lagrangians, and therefore the moduli space of special Lagrangians is smooth at a special Lagrangian immersion and the dimension is $b_1 = \dim H^1(L, \mathbb{Z})$ [49].

The property that a minimal Lagrangian immersion is strictly stable on M with negative Ricci curvature is used in [43, 56] to prove the uniqueness of minimal surfaces/minimal Lagrangians in some Lagrangian homotopy classes $[\Sigma, M]$. To prove this result, we first show that every minimal surface in the depicted class is Lagrangian and is therefore strictly stable. Minimal surfaces in a homotopy class of $[\Sigma, M]$ can be characterized as critical points of an energy functional $\mathcal{E}$ on the Teichmüller space of Σ, which is a proper function on finite dimensional space. Since every critical point of $\mathcal{E}$ is strictly stable, it is therefore unique and the result follows. The existence part is obtained by a theorem of Schoen and Yau [59]. The uniqueness will imply that the minimal Lagrangians in these homotopy classes of $[\Sigma, M]$ are area minimizing. We can also use the strict stability to deform a minimal Lagrangian immersion for nearby Kähler-Einstein metrics. It first gives the deformation as a minimal immersion. The Lagrangian condition is preserved because a nearby deformation of a Lagrangian immersion is totally real and Wolfson has proven that a totally real minimal (branch) immersion in Kähler-Einstein surfaces of negative scalar curvature must be Lagrangian [68]. However, when we take a limit of these surfaces, branch points may develop. We need to show (4) for minimal Lagrangian branch immersions, their deformations for nearby metrics first as minimal branch immersions, then show them are totally real, and therefore Lagrangian. These are done in [42] to obtain the deformation of minimal Lagrangian branch immersions in the whole connected component of Kähler-Einstein metrics.

The notion of Hamiltonian stable and the corresponding second variational formulae for the area at a minimal Lagrangian, and at a Hamiltonian stationary Lagrangian are defined and derived by Oh in [52, 53]. There are a lot of studies on the Hamiltonian stabilities of different examples, and on whether the Clifford torus in $\mathbb{C}P^n$ minimizes volume in its Hamiltonian class.

4 Singularities and geometric measure theory

As discussed in Section 2, we need to study objects with singularities for questions concerning the existence or moduli space. The most natural class to work on is integral currents, which are defined and studied in Geometric Measure Theory. There are many important problems and techniques arisen from Geometric Measure Theory. We will use some examples in this section to discuss the connection and related issues.

Schoen and Wolfson in [58] minimize area among all Lagrangians. They take the 2-dimensional advantage to study the problem from the mapping point of view, and find minimizers in $W^{1,2}$ Lagrangian classes. Unlike the classical stationary case, it is very tricky to derive a related monotonicity formula which is a fundamental issue and starting point to develop a regularity theory in Geometric Measure Theory. We remark that although in most places only Hamiltonian stationary condition is needed, here they do use the Lagrangian stationary property of the minimizers in deriving the monotonicity formula. They need the Lagrangian being locally exact to lift the problem to a contact setting and include more variations which project to Lagrangian, but not Hamiltonian deformations. These variations in particular include dilations, but it still involves solving a wave equation to find suitable compact support variations. By many clever and beautiful arguments, they find the almost unique way to make things work. However, the mystery behind the proof still remains unraveling, not to mention the situation in higher dimensional case. After the monotonicity formula obtained, one needs to use the geometric measure theory techniques again to develop an analogy of Allard regularity theorem, and show that the only singularities are branch points and cone singularities. The tangent cones at cone singularities are Hamiltonian stationary Lagrangian cones determined by their links in S^3. These links, up to unitary transformation, can be characterized as

$$\gamma_{pq}(s) = (\sqrt{\frac{q}{p+q}}\, e^{ips}\,,\, i\sqrt{\frac{p}{p+q}}\, e^{-iqs}), \qquad 0 \leqslant s < 2\pi, \tag{7}$$

where p , q are two co-prime positive integers. Schoen and Wolfson further show in [58] that the Hamiltonian stationary cone $C_{p,q}$ with link $\gamma_{p,q}$ is Hamiltonian stable if and only if $|p-q| = 1$. We will assume $p > q$ for simplicity. They conjecture that among these stable cones, only $(2,1)$ cone is area minimizing. In [45], the author and Wang use *Lagrangian Brakke flow* to distinguish $(2,1)$ cone from other cones and show that the conjecture holds at infinitesimal level. Here Brakke flow is a generalization of mean curvature flow in Geometric Measure Theory setting [1]. Our construction also indicates that one may need to use Z_2 (or Z_p) integral currents instead of integral currents to construct Lagrangian Brakke solutions. Noting that Lagrangian minimizers are constructed among Lagrangian integral currents and Wolfson has found Lagrangian minimizers with cone singularities in a $K3$ surface, hence it is not possible to find a generalization of mean curvature flow for all Lagrangian integral currents that both decreases the area (unless it is already stationary in classical sense), and remains in the class of Lagrangian integral currents. When considering Z_2 Lagrangian integral currents, one should recall a result of Qiu [55] that the set of Z_2 Lagrangian integral currents (flat chains) in $\mathbb{R}^4$ is dense under the flat norm in the space of all Z_2 integral currents (flat chains). However, this phenomenon may not appear when we consider Lagrangian Brakke flow because the area is under better control. I would like to thank Schoen and White's suggestion on this possibility.

The generalization of Schoen and Wolfson's fundamental work [58] to higher dimension is a very challenging and important problem. Since the mapping approach is not available for higher dimension in general, new ideas and a lot of modifications are required. It will heavily depend on Geometric Measure Theory to carry out the program. As mentioned in §3, mean curvature flow is another potential approach to study the existence of minimal Lagrangians and the Lagrangian condition is preserved in Kähler-Einstein manifolds as long as the solution is smooth. However, singularities will occur in general. To have a deep understanding of the singularities, or to consider weak solutions such as Z_2 Lagrangian integral current Brakke solutions mentioned above, it again requires a lot of techniques from Geometric Measure Theory. For instance, Neves uses Geometric Measure Theory to show that when applying central blow up near the finite time singularities, the tangent flow will be special Lagrangian cones if the initial submanifold is an almost calibrated and rational Lagrangian [50]. We refer to [13, 37, 45, 46, 51, 67] for some other investigations related to singularities of mean curvature flow.

There are many other important questions one can ask. For instance, what is the structure of the possible singularities for special Lagrangian integral currents? With the additional Lagrangian condition, can we have better characterization of the set and obtain better properties such as uniqueness of the tangent cone? How can we deform a singular special Lagrangian and understand its local moduli space? All these questions are very much involved Geometric Measure Theory. By putting some further restrictions, Joyce studies the simplest conical singularity case in [28]–[32]. The method employed is gluing and singular perturbation which will be discussed in next section.

5 Gluing and singular perturbation

For the gluing and singular perturbation method, we first need to have some well understood good models, then use gluing to obtain approximate solutions, and the last step is to perturb the approximate solutions to exact solutions. The last step usually can be formulated as finding zeros of a map $\mathcal{F}$, and by applying Taylor expansion to $\mathcal{F}$ at 0, the problem becomes solving a nonlinear PDE. We need to study the linearlized operator $D\mathcal{F}|_0$ and derive some detailed estimates, including information of the initial approximate solutions. In some occasions, there are approximate kernel for the self-adjoint operator $D\mathcal{F}|_0$. There are different ways to overcome this difficulty. One way is that we first solve the projection problem which involves a map between Banach spaces perpendicular to the kernel. It improves our approximate solutions, and reduce the dimension of the space we work on to essentially the dimension of the approximate kernel. Then it usually requires additional conditions or special properties of the approximate solutions to perturb the improved ones to exact solutions. Another way is to add extra freedom and modify $\mathcal{F}$, such that new $D\tilde{\mathcal{F}}$ is surjective. We will discuss a few different results below to demonstrate the ideas and make a comparison.

When constructing approximate solutions for the desingularization problem in [6, 28, 39, 43], we need the local models to be nonsingular special Lagrangians that are asymptotic to the tangent cones at the singularities of original singular special Lagrangians. However, except Lawlor necks [38] which are diffeomorphic to $S^{n-1} \times \mathbb{R}$ and are local models for resolving intersecting points, and an example of Harvey and Lawson [16] as local models for resolving a T^{n-1} cone, we do not know other suitable models. We remark that the family of nonsingular special Lagrangians constructed from one special Lagrangian cone in [17] are asymptotic to two special Lagrangian cones instead one. Lawlor necks have been applied successfully to resolve intersecting points in different occasions [6, 28, 39, 43]. Joyce develops the theory of desingularizing conical singularities assuming the existence of related asymptotically conical special Lagrangians [31, 32]. In these situations, we first replace a small neighborhood of the intersecting/singular point by a suitable scaled Lawlor neck/local model. To make this work in Calabi-Yau manifolds, we have to find special *Darbox coordinates* near the point. Then connect the local model to original special Lagrangian outside a neighborhood of the point to construct Lagrangian approximate solutions. Note that an angle condition, which always holds for $n = 2$ and 3, is needed to find a Lawlor neck asymptotic to the pair of intersecting tangent planes [43]. From examples of complex special Lagrangians, one can see that it is not possible to obtain a desingularization without further restriction for $n > 3$.

For the gluing construction of special Lagrangian cones by Haskins and Kapouleaus, the process is done at the link level, namely for special Legendrians in S^{2n-1}. Also the approximate solutions are constructed from building blocks and need to be done carefully to admit required symmetry. It depends on a detailed study of the building blocks [18, 19, 20] and their approach needs more attentions on constructing approximating solutions. The perturbation method is also used in constructing Hamiltonian stationary Lagrangians by Joyce, Schoen and the author in [36], and by the author herself in [44]. A slightly different approach is given by Butscher and Corvino in [7]. For this problem, we first need a family version of good Darboux coordinates in symplectic manifolds and Kähler manifolds. Here "good" means that we have a nice control and approximation of the metric. Approximate solutions here are obtained by positing a scaled compact Hamiltonian stationary Lagrangian in Darboux coordinates.

Now back to the special Lagrangian cases. For the next step, we need to solve a nonlinear equation to perturb the approximate solutions to exact solutions. This is done by a quantitative version of inverse function theorem in [6, 39, 43]. Note that we in fact construct a family of approximate solutions parameterized by the neck size. Thus if we can obtain right orders in related estimates, we can then argue that the conditions needed in the inverse function theorem are satisfied when the neck size is small enough, and show the existence. In [43], we have a uniform positive lower bound for the first eigenvalues of the linearlized operators on approximate manifolds L_α. This is because $L \setminus \{x\}$ is connected, where x is a transversal self-intersecting point. If we consider two special Lagrangians intersecting transversally at a point as in [6] and [39], it becomes disconnected after deleting the intersecting point and there are approximate kernel for the linearlized operators. There is a good geometric reason to explain why in these situations it is not possible to solve the problem without introducing extra freedom. Recall that the dimension of the local moduli space of special Lagrangians at a special Lagrangian immersion L is equal to the first

Betti number $b_1(L)$. The first Betti number is increased by one when we attach one handle to a connect manifold. It is the case when we glue a Lawlor neck into a neighborhood of a self-intersection. Thus it is possible to obtain a family of special Lagrangians parameterized by the neck size α. However, when we add a handle to connect two special Lagrangians L_1 and L_2, the first Betti number of the connect sum is just $b_1(L_1) + b_1(L_2)$. Therefore it is impossible to obtain a new family of special Lagrangians parameterized by the neck size α since it will increase the dimension of the moduli space by 1. Hence Butscher allows the phase of the Lagrangian angle changing to get one extra freedom [6]. D. Lee allows the Calabi-Yau metric changing and thus the new special Lagranians constructed from the connect sum is w.r.t a different metric [39]. Some additional conditions are needed in his construction, but never the less, new examples of non-flat special Lagrangian submanifolds of Calabi-Yau tori are found. We remark that the case Butscher studies is special Lagrangians with boundary in $\mathbb{C}^n$ [6]. Both unobstructed and obstructed cases are studied by Joyce, and he shows that when certain balancing condition is satisfied, the obstructed case can also be solved [28]. Note that [43] is unobstructed, and [6], [39] are obstructed.

For the case of Hamiltonian stationary Lagrangian discussed in [36] and [44], there are a big set of approximate kernel. We remark that here the approximate solutions are parameterized by points in the ambient manifold, the unitary frame at that point, and the size of scaling. We first solve the projection problem, that is to find solutions mod the set of kernel. This always can be done. If the model is in addition chosen to be a rigid Hamiltonian stationary Lagrangian in $\mathbb{C}^n$. We prove that the problem of finding Hamiltonian stationary Lagrangians in M is equivalent to finding critical points of a smooth function on the unitary bundle of M. If M is a compact symplectic manifold, then the existence of critical points follows from a simple property of continuous functions on compact sets [36]. When M is Kähler and the Hamiltonian stationary Lagrangian model is T^n, we can do detailed analysis and obtain a local criterion that guarantees the existence of one smooth family of embedded Hamiltonian stationary Lagrangians near the point [44]. Furthermore, the tori in the family do not intersect each other.

References

[1] K.A. Brakke, *'The motion of a surface by its mean curvature'*, Mathematical Notes, Princeton University Press, 1978.

[2] S. Brendle, *'Minimal Lagrangian diffeomorphisms between domains in the hyperbolic plane'*, J. Diff. Geom. **80** (2008) 1–22.

[3] S. Brendle and M. Warren, *'A Boundary value Problem for minimal Lagrangian graphs'*, J. Diff. Geom. **84** (2010), 267–287.

[4] R. L. Bryant, *'Minimal Lagrangian submanifolds of Kähler-Einstein manifolds'*, Differential geometry and differential equations (Shanghai, 1985), 1–12, Lecture Notes in Math., 1255, Springer, Berlin, 1987.

[5] R. L. Bryant, *'Second order families of special Lagrangian 3-folds'*, Perspectives in Riemannian geometry, 63–98, CRM Proc. Lecture Notes **40**, Amer. Math. Soc., Providence, RI, 2006.

[6] A. Butscher, *'Regularizing a singular special Lagrangian variety'*, Comm. Anal. Geom. **12** (2004), 733–791.

[7] A. Butscher and J. Corvino, *'Hamiltonian stationary tori in Kähler manifolds'*, arXiv:0811.2829, 2008, revised 2010.

[8] L. Caffarelli, L. Nirenberg and J. Spruck, *'The Dirichlet problem for nonlinear second order elliptic equations, III: functions of the eigenvalues of the Hessian'*, Acta Math. **155** (1985), 261–301.

[9] E. Carberry and I. McIntosh, *'Minimal Lagrangian 2-tori in* $\mathbb{C}P^2$ *come in real families of every dimension'*, J. London Math. Soc. (2) **69** (2004), no. 2, 531–544.

[10] I. Castro, C.R. Montealegre and F. Urbano, *'Minimal Lagrangian submanifolds in the complex hyperbolic space'*, Illinois J. Math. **46** (2002), no. 3, 695–721.

[11] I. Castro and F. Urbano, *'New examples of minimal Lagrangian tori in the complex projective plane'*, Manuscripta Math. **85** (1994), no. 3-4, 265–281.

[12] B.-Y. Chen, *'Geometry of Submanifolds and its Applications'*, Science University of Tokyo, 1981.

[13] J. Chen and J. Li, *'Singularity of mean curvature flow of Lagrangian submanifolds'*, Invent. Math. **156** (2004), no. 1, 25–51.

[14] P. Dazord, *'Sur la géometrie des sous-fibrés et des feuilletages lagrangiens'*, Ann. Sci. Ecole Norm. Sup. (4) **14** (1981), no. 4, 465–480.

[15] E. Goldstein, *'A construction of new families of minimal Lagrangian submanifolds via torus actions'*, J. Diff. Geom. **58** (2001), no. 2, 233–261.

[16] R. Harvey and H.B. Lawson, *'Calibrated geometries'*, Acta Math. **148** (1982), 48–156.

[17] M. Haskins, *'Special Lagrangian cones'*, Amer. J. Math. **126** (2004), no. 4, 845–871.

[18] M. Haskins and N. Kapouleas, *'Special Lagrangian cones with higher genus links'*, Invent. Math. **167** (2007), 223–294.

[19] M. Haskins and N. Kapouleas, *'Gluing constructions of special Lagrangian cones'*, Handbook of geometric analysis, No. 1, 77–145, Adv. Lect. Math. (ALM),**7**, Int. Press, Somerville, MA, 2008.

[20] M. Haskins and N. Kapouleas, *'Twisted products and* $\mathrm{SO}(p) \times \mathrm{SO}(q)$*-invariant special Lagrangian cones'*, arXiv:Math.DG.1005.1419, 2010.

[21] F. Hélein and P. Romon *'Weierstrass representation of Lagrangian surfaces in four-dimensional space using spinors and quaternions'*, Comment. Math. Helv. **75** (2000), no. 4, 668–680.

[22] F. Hélein and P. Romon, *'Hamiltonian stationary surfaces in* $\mathbb{C}^2$*'*, Comm. Anal. Geom. 10 (2002), no. 1, 79–126.

[23] F. Hélein and P. Romon, *'Hamiltonian stationary tori in the complex projective plane'*, Proc. London Math. Soc. (3) 90 (2005), no. 2, 472–496.

[24] D.D. Joyce, *'Evolution equations for special Lagrangian 3-folds in* $\mathbb{C}^3$*'*, Ann. Global Anal. Geom. **20** (2001), no. 4, 345–403.

[25] D.D. Joyce, *'Constructing special Lagrangian* m*-folds in* $\mathbb{C}^m$ *by evolving quadrics'*, Math. Ann. **320** (2001), 757–797.

[26] D.D. Joyce, *'Ruled special Lagrangian 3-folds in* $\mathbb{C}^3$*'*, Proc. London Math. Soc. (3) **85** (2002), no. 1, 233–256.

[27] D.D. Joyce, *'Special Lagrangian* m*-folds in* $\mathbb{C}^m$ *with symmetries'*, Duke Math. J. **115** (2002), no. 1, 1–51.

[28] D.D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications'*, J. Diff. Geom. **63** (2003), 279–347.

[29] D.D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. I. Regularity'*, Ann. Global Anal. Geom. **25** (2004), no. 3, 201–251.

[30] D.D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces'*, Ann. Global Anal. Geom. **25** (2004), no. 4, 301–352.

[31] D.D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. III. Desingularization, the unobstructed case'*, Ann. Global Anal. Geom. **26** (2004), no. 1, 1–58.

[32] D.D. Joyce, *'Special Lagrangian submanifolds with isolated conical singularities. IV. Desingularization, obstructions and families'*, Ann. Global Anal. Geom. **26** (2004), no. 2, 117–174.

[33] D.D. Joyce, 'U(1)*-invariant special Lagrangian 3-folds. I. Nonsingular solutions'*, Adv. Math. **192** (2005), no. 1, 35–71.

[34] D.D. Joyce, 'U(1)-*invariant special Lagrangian 3-folds. II. Existence of singular solutions'*, Adv. Math. **192** (2005), no. 1, 72–134.

[35] D.D. Joyce, 'U(1)-*invariant special Lagrangian 3-folds. III. Properties of singular solutions'*, Adv. Math. **192** (2005), no. 1, 135–182.

[36] D.D. Joyce, Y.-I Lee and R. Schoen, *'On the existence of Hamiltonian stationary Lagrangian submanifolds in symplectic manifolds'*, to appear in Amer. J. Math., also available at arXiv:0902.3338, 2009.

[37] D.D. Joyce, Y.I. Lee and M.P. Tsui, *'Self-similar solutions and translating solutions for Lagrangian mean curvature flow'*, J. Diff. Geom. **84** (2010), 127–161.

[38] G. Lawlor, *'The angle criterion'*, Invent. math. **95** (1989), 437–446.

[39] D. Lee, *'Connected sums of special Lagrangian submanifolds'*, Comm. Anal. Geom. **12** (2004), 553–579.

[40] Y.-I. Lee, *'The Metric Properties of Lagrangian Submanifold'*, Stanford University Thesis, 1992.

[41] Y.-I. Lee, *'Minimal and Lagrangian Surfaces in Kähler- Einstein Surfaces of Negative Scalar Curvature'*, Comm. Anal. Geom. **2** (1994), no.4, 579–592.

[42] Y.-I. Lee, *'The Deformation of Lagrangian Minimal Surfaces in Kähler-Einstein Surfaces'*, J. Diff. Geom. **50** (1998), no. 2, 299–330.

[43] Y.-I. Lee, *'Embedded special Lagrangian submanifolds in Calabi-Yau manifolds'*, Comm. Anal. Geom. **11** (2003), 391–423.

[44] Y.-I. Lee, *'The existence of Hamiltonian stationary Lagrangian tori in Kähler manifolds of any dimension'*, arXiv:1001.3861, 2010.

[45] Y.-I. Lee and M.-T. Wang, *'Hamiltonian stationary shrinkers and expanders for Lagrangian mean curvature flows'*, J. Diff. Geom. **83** (2009), 27–42.

[46] Y.-I. Lee and M.-T. Wang, *'Hamiltonian stationary cones and self-similar solutions in higher dimension'*, Trans. Amer. Math. Soc. **362** (2010), 1491–1503.

[47] H. Ma and Y. Ma, *'Totally real minimal tori in* $\mathbb{C}P^2$*'*, Math. Z. **249** (2005), no. 2, 241–267.

[48] I. McIntosh, *'Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps'*, J. London Math. Soc. (2) **67** (2003), no. 3, 769–789.

[49] R.C. McLean, *'Deformations of special Lagrangian submanifolds'*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747.

[50] A. Neves, *'Singularities of Lagrangian mean curvature flow: zero-Maslov class case'*, Invent. math. **168** (2007), 449–484.

[51] A. Neves and G. Tian, *'Translating solutions to Lagrangian mean curvature flow'*, arXiv: 0711.4341, 2007.

[52] Y.-G. Oh, *'Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds'*, Invent. math. **101** (1990), 501–519.

[53] Y.-G. Oh, *'Volume minimization of Lagrangian submanifolds under Hamiltonian deformation'*, Math. Z. **212** (1993), 175–192.

[54] B. Palmer, *'Hamiltonian minimality and Hamiltonian stability of Gauss maps'*, Diff. Geom. Appl. **7** (1997), 51–58.

[55] W.-Y. Qiu, *'Non-orientable Lagrangian surfaces with controlled area'*, Math. Res. Lett. **8** (2001), no. 5–6, 693–701.

[56] R. Schoen, *'The role of harmonic mappings in rigidity and deformation problems'*, Complex geometry (Osaka, 1990), 179–200, Lecture Notes in Pure and Appl. Math., 143, Dekker, New York, 1993.

[57] R. Schoen, *'Minimal submanifold methods in Riemannian and Kähler geometry'*, Course given in the spring of 2010 at Stanford University.

[58] R. Schoen and J. Wolfson, *'Minimizing area among Lagrangian surfaces: the mapping problem'*, J. Diff. Geom. **58** (2001), 1–86.

[59] R. Schoen and S.-T. Yau, *'Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature'*, Ann. of Math. (2) **110** (1979), no. 1, 127–142.

[60] K. Smoczyk, *'A canonical way to deform a Lagrangian submanifold'*, dg-ga/9605005, 1996.

[61] K. Smoczyk, *'The Lagrangian mean curvature flow'*, Leipzig: Univ. Leipzig (Habil.), **102**, 2000.

[62] K. Smoczyk, *'Angle theorems for the Lagrangian mean curvature flow'*, Math. Z. **240** (2002), no. 4, 849–883.

[63] K. Smoczyk and M.-T. Wang, *'Mean curvature flows of Lagrangian submanifolds with convex potentials'*, J. Diff. Geom. **62** (2002), 243–257.

[64] R. A. Sharipov, *'Minimal tori in a five-dimensional sphere in $\mathbb{C}^3$'*, Theoret. and Math. Phys. 87 (1991), no. 1, 363–369.

[65] A. Strominger, S.-T. Yau and E. Zaslow, *'Mirror symmetry is T-duality'*, Nuclear Phys. B **479** (1996), 243–259.

[66] M.-T. Wang, *'Deforming area-preserving diffeomorphism of surfaces by mean curvature flow'*, Math. Res. Lett. **8** (2001), 651–661.

[67] M.-T. Wang, *'Mean curvature flow of surfaces in Einstein four-manifolds'*, J. Diff. Geom. **57** (2001), 301–388.

[68] J.G. Wolfson, *'Minimal surfaces in Kähler surfaces and Ricci curvature'*, J. Diff. Geom. **29** (1989), 282–294.

[69] J.G. Wolfson, *'Minimal Lagrangian diffeomorphisms and the Monge-Ampere equation'*, J. Diff. Geom. **46** (1997), no. 2, 335–373.

[70] J.G. Wolfson, *'Lagrangian homology classes without regular minimizers'*, J. Diff. Geom. **71** (2005), 307–313.