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New minimal surfaces in $\mathbb{S}^3$ desingularizing the Clifford tori

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Abstract For each integer $m \geq 2$ and $\ell \geq 1$ we construct a pair of compact embedded minimal surfaces of genus $1+4m(m-1)\ell$. These surfaces desingularize the m Clifford tori meeting each other along a great circle at the angle of π/m. They are invariant under a finite group of screw motions and have no reflection symmetry across a great sphere.

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1 Introduction

One can construct a complete minimal surface in $\mathbb{R}^3$ by suitably choosing a holomorphic 1-form $f(z)dz$ and a meromorphic function $g(z)$ for the Weierstrass representation formula. But there is no such an efficient tool in $\mathbb{S}^3$. This is why not many minimal surfaces are known to exist in $\mathbb{S}^3$. So far only three types of minimal surfaces have been constructed and their methods of construction are all different. In 1970 Lawson [5] constructed infinitely many compact minimal surfaces in $\mathbb{S}^3$; in 1988 Karcher et al. [4] found nine new compact embedded minimal surfaces; in 2010 Kapouleas and Yang [3] obtained new minimal surfaces by doubling the Clifford torus.

In memory of Professor Hyo Chul Myung, former president of KIAS.

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Lawson starts from a piecewise geodesic Jordan curve Γ, finds a minimal disk D spanning Γ, and extends D across Γ by 180°-rotations to obtain a compact immersed minimal surface. Lawson's Jordan curve Γ consists of four geodesic segments and is a subset of the 1-skeleton of a tetrahedron in $\mathbb{S}^3$. This tetrahedron is a fundamental piece of a tessellation of $\mathbb{S}^3$.

On the other hand, Karcher–Pinkall–Sterling start from a tetrahedron T which gives rise to a different type of tessellation of $\mathbb{S}^3$. Then they find a minimal disk D in T which is perpendicular to ∂T along ∂D, and extend D by the reflections across ∂T to obtain a compact embedded minimal surface.

Kapouleas–Yang's minimal surfaces resemble two parallel copies of the Clifford torus, joined by m^2 small catenoidal bridges for sufficiently large m symmetrically arranged along a square lattice of points on the torus.

In this paper we construct infinitely many compact embedded minimal surfaces by desingularizing m Clifford tori which meet each other along a great circle at the angle of π/m. Our desingularization does not employ the gluing method, the possibility of which was independently discussed by Kapouleas in [2]. Instead we use a tessellation of $\mathbb{S}^3$ by $16m^2\ell$ ($m \geq 2$, $\ell \geq 1$) pentahedra and apply Lawson's method for the Jordan curve of 6 geodesic segments which is a subset of the 1-skeleton of a pentahedron. The resulting compact embedded minimal surface has genus $1+4m(m-1)\ell$ (Theorem 1).

Given a great circle C_1 in $\mathbb{S}^3$, there is the polar great circle C_2 of C_1, that is, $\mathrm{dist}(p, q) = \pi/2$ for any $p \in C_1$ and $q \in C_2$. C_1 and C_2 are linked in $\mathbb{S}^3$. If m Clifford tori meet each other along C_1, then they intersect each other along C_2 as well. Therefore once m Clifford tori are desingularized along C_1, there are two ways of desingularizing the tori along C_2. Thus we obtain the second type(even) of minimal surfaces desingularizing m Clifford tori for each genus $1 + 4m(m - 1)\ell, \ell \geq 2$ (Theorem 2).

All the embedded minimal surfaces constructed by Lawson, Karcher–Pinkall–Sterling, Kapouleas–Yang satisfy the reflection symmetry, i.e., they are invariant under a reflection across a great sphere in $\mathbb{S}^3$. But our new minimal surfaces have no reflection symmetry.

2 Clifford torus

The Clifford torus T is the building block of our new minimal surfaces. So we start by investigating its two characteristic properties: it has the *equidistance property* and is *doubly ruled*. Define

$$T = \mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2}) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2\}.$$

Let C_{12}, C_{34} be the two linked great circles in $\mathbb{S}^3$ defined by $C_{12} = \{(x_1, x_2, 0, 0) : x_1^2 + x_2^2 = 1\}$, $C_{34} = \{(0, 0, x_3, x_4) : x_3^2 + x_4^2 = 1\}$. Throughout this paper "dist" denotes the distance in $\mathbb{S}^3$. Then

$$\mathrm{dist}(p, q) = \pi/2, \ \forall p \in C_{12}, \ \forall q \in C_{34},$$

and one gets the equidistance property:

$$\mathrm{dist}(T, C_{12}) = \mathrm{dist}(T, C_{34}) = \pi/4.$$

Also it is easy to see that

$$\overline{pq} \perp T, \ \forall p \in C_{12}, \ \forall q \in C_{34}.$$

Let $\gamma_1 = \{(x_1, x_2, 1/\sqrt{2}, 0) : x_1^2 + x_2^2 = 1/2\}$, $\gamma_2 = \{(1/\sqrt{2}, 0, x_3, x_4) : x_3^2 + x_4^2 = 1/2\}$. Cutting out γ_1 and γ_2 from T, one can obtain a flat square $Q \subset T$. Then one can consider two 1-parameter families of lines on T which are parallel to the two diagonals of the square Q. These lines of T are in fact the great circles of $\mathbb{S}^3$. For this reason T is called doubly ruled. Let's see why these lines are great circles.

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 \text{ becomes } (x_1 + x_3)(x_1 - x_3) = (x_4 + x_2)(x_4 - x_2).$$

Hence if we rotate x_1x_3-plane and x_2x_4-plane by $\pi/4$ and by $-\pi/4$, respectively, and use x_1, x_2, x_3, x_4 again for the new coordinates, then we get

$$x_1x_3 = x_2x_4.$$

Hence T can be represented by the coordinate map $\Psi : [0, 2\pi) \times [0, 2\pi) \to \mathbb{S}^3$,

$$\Psi(x, y) = (\cos x \sin y, \cos x \cos y, \sin x \cos y, \sin x \sin y).$$

Here we claim that T is ruled by the two families of great circles $\{x = \text{const}\}$ and $\{y = \text{const}\}$.

Let ρ_{ij}^t be the counterclockwise rotation of $\mathbb{S}^3$ by the angle t along the x_ix_j-plane and define

$$\Phi_{ijkl}^t = \rho_{ij}^t \circ \rho_{kl}^t,$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$ as a set. We will call Φ_{ijkl}^t a *screw motion* because it can be viewed as the composition of a rotation and a translation, ρ_{kl}^t being the translation along the great circle $x_k^2 + x_l^2 = 1$. Note that

$$\begin{aligned}\Psi(x, y) &= \cos x(\sin y, \cos y, 0, 0) + \sin x(0, 0, \cos y, \sin y)\\ &= \cos y(0, \cos x, \sin x, 0) + \sin y(\cos x, 0, 0, \sin x).\end{aligned}$$

Hence T is foliated by the great circles $\{\Phi_{1423}^t(C_{21})\}$ which are $\{x = \text{const}\}$ and by the great circles $\{\Phi_{2134}^t(C_{23})\} = \{y = \text{const}\}$. Here C_{21} is the great circle C_{12} with the opposite orientation and $C_{23} = \{(0, x_2, x_3, 0) : x_2^2 + x_3^2 = 1\}$.

These two families of great circles are orthogonal to each other. The orthogonality can be observed more easily on the fundamental piece $\hat{T}$ of the Clifford torus T as in Fig. 1. T consists of eight congruent pieces of $\hat{T}$ and $\hat{T}$ is Morrey's solution to the Plateau problem for a geodesic polygon $\Gamma = \hat{C}_{12} \cup \hat{C}_{23} \cup \hat{C}_{34} \cup \hat{C}_{41}$ as used by Lawson in [5]. $\hat{C}_{ij}$ is a subarc of length $\pi/2$ of C_{ij} such that $\hat{C}_{12}$ is from $(1, 0, 0, 0)$ to $(0, 1, 0, 0)$, $\hat{C}_{23}$ from $(0, 1, 0, 0)$ to $(0, 0, 1, 0)$, $\hat{C}_{34}$ from $(0, 0, 1, 0)$ to $(0, 0, 0, 1)$, and $\hat{C}_{41}$ from $(0, 0, 0, 1)$ to $(1, 0, 0, 0)$.

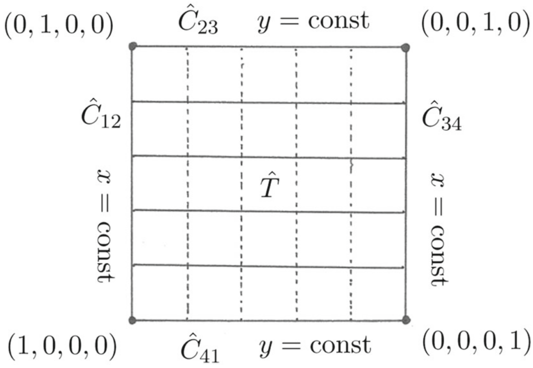

Fig. 1 Doubly ruled Clifford torus

Finally it is not difficult to see that T is the equidistance set from the two great circles $\Phi_{1342}^{\pi/4}(C_{12})$ and $\Phi_{1342}^{3\pi/4}(C_{12})$ (remember that the original x_1x_3-plane and x_2x_4-plane have been rotated by $\pi/4$ and $-\pi/4$, respectively). Also it should be mentioned that T is invariant under the screw motions Φ_{1234}^t and Φ_{1423}^t for any t. And if a great circle of $\mathbb{S}^3$ lies in a Clifford torus, so does its polar circle.

3 Odd surfaces

Given two orthogonal planes in $\mathbb{R}^3$, the minimal surface that desingularizes them is Scherk's second surface. Similarly, there exists a minimal surface that desingularizes m vertical planes making an equal angle of π/m along a vertical line. For $m+1$ great spheres making an angle of $\pi/(m+1)$ along a great circle in $\mathbb{S}^3$, the minimal surfaces that desingularize them are Lawson's minimal surfaces $\xi_{m,k}$ of genus mk. Then, given m Clifford tori in $\mathbb{S}^3$, is there a minimal surface that desingularizes them? We are motivated by this question and led to the following.

Theorem 1 *Let $T_1, \dots, T_m \subset \mathbb{S}^3$ be the Clifford tori intersecting each other along a great circle C_1 at an angle of π/m. Then there exists a compact minimal surface $T_{m,k}^o$ desingularizing $T_1 \cup \cdots \cup T_m$ for each $k = 2m\ell$ with integers $m \geq 2,\ \ell \geq 1$:*

(i) *$T_{m,k}^o$ is embedded and has genus* $1 + 2k(m-1) = 1 + 4m(m-1)\ell$*;*
(ii) *$T_{m,k}^o$ is invariant under a finite group of screw motions;*
(iii) *$T_{m,k}^o$ has no reflection symmetry across a great sphere;*
(iv) $\mathrm{Area}(T_{m,k}^o) < 2m\pi^2$.

Proof Let C_2 be the polar great circle of C_1, that is, the set of all points of distance $\pi/2$ from C_1. Then $T_1 \cap \cdots \cap T_m = C_1 \cup C_2$. We claim that $T_1, \dots, T_m$ also meet each other along C_2 at the angle of π/m. Introduce the coordinates x_1, x_2, x_3, x_4 of $\mathbb{R}^4 \supset \mathbb{S}^3$ such that $C_1 = C_{12} : x_1^2 + x_2^2 = 1,\ C_2 = C_{34} : x_3^2 + x_4^2 = 1$. Then $T_1, \dots, T_m$ are invariant under Φ_{1234}^t or Φ_{1243}^t. Suppose without loss of generality that $\Phi_{1234}^t(T_1) = T_1$. Then

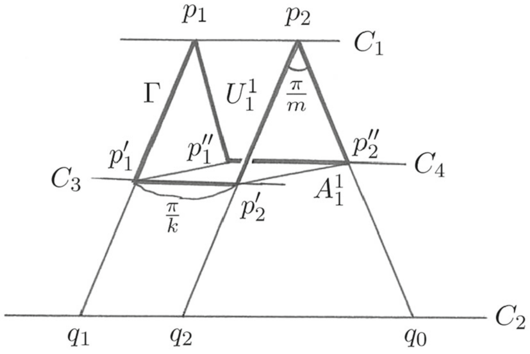

Fig. 2 Pentahedron in the tessellation of $\mathbb{S}^3$

$$\{T_1, \ldots, T_m\} = \left\{T_1, \rho_{34}^{\pi/m}(T_1), \rho_{34}^{2\pi/m}(T_1), \ldots, \rho_{34}^{(m-1)\pi/m}(T_1)\right\}.$$

Since $T_1 = \rho_{34}^{-t} \circ \rho_{12}^{-t}(T_1)$, it follows that

$$\{T_1, \ldots, T_m\} = \left\{T_1, \rho_{12}^{-\pi/m}(T_1), \ldots, \rho_{12}^{-(m-1)\pi/m}(T_1)\right\},$$

hence the claim follows.

Let T_0 be the Clifford torus which is the equidistance set from C_1 and C_2. T_0 divides $\mathbb{S}^3$ into the two domains denoted D_1 and D_2 containing C_1 and C_2, respectively. Choose equally spaced points $p_1, \ldots, p_{2k}$ on C_1 such that $\text{dist}(p_j, p_{j+1}) = \pi/k$ and let $S_1^1, \ldots, S_{2k}^1$ be the great spheres such that S_j^1 contains p_j and is perpendicular to C_1 at p_j. $T_1, \ldots, T_m$ and $S_1^1, \ldots, S_{2k}^1$ divide D_1 into congruent domains $\{U_j^i\}_{1\le i\le 2m,\, 1\le j\le 2k}$. $\{U_j^i\}$ are numbered in such a way that $\cup_{i=1}^{2m} U_j^i$ is a component of $D_1 \sim (S_j^1 \cup S_{j+1}^1)$ and $\cup_{j=1}^{2k} U_j^i$, $\cup_{j=1}^{2k} U_j^{i+m}$ are components of $D_1 \sim (T_i \cup T_{i+1})$, and U_j^i, U_j^{i+m} are symmetric about C_1, that is, $U_j^{i+m} = \rho_{C_1}(U_j^i)$, ρ_C denoting the 180°-rotation about the great circle C. These domains are in fact congruent pentahedra bounded by three Clifford tori and two great spheres as in Fig. 2. Recall that the two great spheres are perpendicular to the base Clifford torus T_0. Each $\bar{U}_j^i \cap T_0$ is a parallelogram on T_0. Hence the tessellation of D_1 by the pentahedra $\{U_j^i\}$ gives rise to a tessellation of T_0 by the parallelograms $\{A_j^i\}_{1\le i\le 2m,\, 1\le j\le 2k}$.

Note that U_1^1 is bounded by T_0, T_1, T_2 and S_1^1, S_2^1, with $p_1' \in T_1$, $p_1'' \in T_2$. Denote by p_1', p_2', p_1'', p_2'' the vertices of A_1^1 (also of U_1^1). Even though $\overline{p_1' p_1''}$ and $\overline{p_2' p_2''}$ are not great circles, they are geodesics on T_0. Let $c(s)$ be the arclength parametrization of the geodesic $\overline{p_1 p_1'}$ with $c(0) = p_1, c(\pi/4) = p_1'$. Then the angle between T_1 and S_1^1 at $c(s)$ equals $\pi/2 - s$ because the tangent plane to T_1 at $c(s)$ is rotating around $\overline{p_1 p_1'}$ under a screw motion as s increases. Hence the vertex angles of A_1^1 are $\pi/4, 3\pi/4, \pi/4, 3\pi/4$. Let C_3 (C_4, respectively) be the great circle containing $\overline{p_1' p_2'}$ ($\overline{p_1'' p_2''}$, resp.). Set $p_j' = C_3 \cap S_j^1$ ($p_j'' = C_4 \cap S_j^1$, resp.). Then $p_1', \ldots, p_{2k}'$ ($p_1'', \ldots, p_{2k}''$, resp.) are equally spaced on C_3 (C_4, resp.) and $\overline{p_j p_j'} \subset T_1 \cap S_j^1$ ($\overline{p_j p_j''} \subset T_2 \cap S_j^1$, resp.) is an edge of the pentahedron U_j^1. Note that $\overline{p'_j p''_j}$ is not a geodesic in $\mathbb{S}^3$ but a geodesic on T_0: it is part of a latitude on S_j^1 with p_j a pole and C_2 the equator.

Let $q_1, \dots, q_{2k}$ be the equally spaced points on C_2 such that $p'_j \in C_3$ is the midpoint of $\overline{p_j q_j}$, $1 \le j \le 2k$. And let $S_1^2, \dots, S_{2k}^2$ be the great spheres containing $q_1, \dots, q_{2k}$, respectively, and perpendicular to C_2. Then $T_1, \dots, T_m$ and $S_1^2, \dots, S_{2k}^2$ divide D_2 into congruent domains $\{V_j^i\}_{1\le i\le 2m,\, 1\le j\le 2k}$. $\{V_j^i\}$ are numbered in the same way as $\{U_j^i\}$ such that $\cup_{i=1}^{2m} V_j^i$ is a component of $D_2 \sim (S_j^2 \cup S_{j+1}^2)$ and $\cup_{j=1}^{2k} V_j^i$, $\cup_{j=1}^{2k} V_j^{i+m}$ are components of $D_2 \sim (T_i \cup T_{i+1})$, and V_j^i, V_j^{i+m} are symmetric about C_2, that is, $V_j^{i+m} = \rho_{C_2}(V_j^i)$. Again the pentahedra V_j^i's give a tessellation of T_0 by the parallelograms $\{B_j^i\}_{1\le i\le 2m, 1\le j\le 2k}$. A_j^i and B_j^i have the same vertex angles. However, they are not congruent in T_0, but symmetric. More precisely, let A_j^i, B_j^i be the parallelograms between C_3 and C_4 with the same base $\overline{p'_j p'_{j+1}} \subset C_3$. Let $C \subset T_0$ be the great circle passing through the midpoint of $\overline{p'_j p'_{j+1}}$ and orthogonal to C_3. Then the 180°-rotation ρ_C about C in $\mathbb{S}^3$ defines a reflection $\rho_C|_{T_0}$ across C in T_0. Since $\rho_C(A_j^i) = B_j^i$, A_j^i and B_j^i are symmetric about C.

So far we know that $\{U_j^i, V_j^i\}_{1\le i\le 2m, 1\le j\le 2k}$ forms a tessellation of $\mathbb{S}^3$. But we need more information than this between U_j^i and V_j^i. Let $C_{ij}^n = T_i \cap S_j^n$, $1 \le i \le m$, $1 \le j \le 2k$, $n = 1, 2$. C_{ij}^1 is a great circle passing through p_j and perpendicular to C_1, while $C_{ij}^2 \ni q_j$ and $C_{ij}^2 \perp C_2$. Let q_0 be the point on C_2 such that p''_2 is the midpoint of $\overline{p_2 q_0}$ as in Fig. 2. If k is divisible by m, that is, $k = m\ell$ for some integer ℓ, then $q_0 = q_{\ell+2}$ since $\operatorname{dist}(q_2, q_0) = \pi/m$. Hence one can easily see that

$$C_2 \cap \bigcup_{i,j} C_{ij}^1 = \{q_1, \dots, q_{2k}\} \quad \text{and} \quad C_1 \cap \bigcup_{i,j} C_{ij}^2 = \{p_1, \dots, p_{2k}\}.$$

It follows that

$$\bigcup_{i,j} C_{ij}^1 = \bigcup_{j=1}^{2k} \bigcup_{i=1}^{2m} \overline{p_j q_{j+(i-1)\ell}} = \bigcup_{a=1}^{2k} \bigcup_{i=1}^{2m} \overline{p_{a-(i-1)\ell} q_a} = \bigcup_{i,j} C_{ij}^2. \tag{1}$$

For $1 \le i \le 2m$, $1 \le j \le 2k$ and $b = 1, 2, 3, 4$, let U_{jb}^i be the edges of U_j^i perpendicular to T_0, and V_{jb}^i those of V_j^i perpendicular to T_0. Then one sees that

$$\left(\bigcup_{i,j,b} U_{jb}^i\right) \bigcup \left(\bigcup_{i,j,b} V_{jb}^i\right) = \bigcup_{i,j} C_{ij}^1 = \bigcup_{i,j} C_{ij}^2.$$

Hence one can conclude that $\bigcup_{i,j} C_{ij}^1$ becomes a lattice (grid) of $\mathbb{S}^3$ consisting of the edges of $\{U_j^i, V_j^i\}$. This observation is rather surprising, considering that the parallelograms A_j^i and B_j^i are not congruent in T_0.

Let $\Gamma \subset \partial U_1^1$ be the piecewise geodesic Jordan curve with six ordered vertices $p_1, p_1', p_2', p_2, p_2'', p_1'', p_1$. U_1^1 is mean convex because it is bounded by three minimal quadrilaterals and two totally geodesic triangles. Then Jost [J] shows that Γ spans an embedded minimal disk $H \subset U_1^1$. In a sense, H is a minimal graph over A_1^1, and so it is believed to be unique; for its proof one needs a generalization of Radó's theorem [6] in $\mathbb{S}^3$. Define $H' = \rho_{C_3}(H) \subset V_1^{2m} = \rho_{C_3}(U_1^1)$.

Denote by $\rho_{\overline{pq}}$ the 180°-rotation of $\mathbb{S}^3$ around the great circle $\overline{pq}$. Since H is bounded by six geodesic arcs $\overline{p_1p_1'}, \overline{p_1'p_2'}, \overline{p_2'p_2}, \overline{p_2p_2''}, \overline{p_2''p_1''}, \overline{p_1''p_1}$, H can be analytically extended across the boundary by 180°-rotations. Note that the six corresponding rotations $\rho_{\overline{p_1p_1'}}, \rho_{\overline{p_1'p_2'}}, \rho_{\overline{p_2'p_2}}, \rho_{\overline{p_2p_2''}}, \rho_{\overline{p_2''p_1''}}, \rho_{\overline{p_1''p_1}}$ generate a finite group G^o of isometries of $\mathbb{S}^3$. Hence one can perform those analytic extensions for all members of G^o to obtain a compact minimal extension $T_{m,k}^o$ of H without boundary. Obviously $T_{m,k}^o$ is invariant under G^o.

Now we claim that $T_{m,k}^o$ has no self intersection. Let $\bar{p}_1, \ldots, \bar{p}_{4mk}$ be the vertices of the parallelograms A_j^i (such as p_1', p_2', p_1'', p_2''), and $\bar{q}_1, \ldots, \bar{q}_{4mk}$ the vertices of B_j^i. Define $\rho_{\bar{p}_c}$ to be the 180°-rotation about the great circle through $\bar{p}_c$ and perpendicular to T_0, $c = 1, \ldots, 4mk$. Define $\rho_{\bar{q}_c}$ similarly. Extend H analytically by applying $\rho_{\bar{p}_1}, \ldots, \rho_{\bar{p}_{4mk}}$ to obtain $T_{m,k}^1 \subset D_1$. Also extend H' by applying $\rho_{\bar{q}_1}, \ldots, \rho_{\bar{q}_{4mk}}$ to get $T_{m,k}^2 \subset D_2$. Clearly $\partial T_{m,k}^1 \subset T_0$ and $\partial T_{m,k}^2 \subset T_0$. Since $T_{m,k}^1$ and $T_{m,k}^2$ are embedded, $T_{m,k}^o$ will be embedded if one can prove $T_{m,k}^o = T_{m,k}^1 \cup T_{m,k}^2$. Or equivalently, $T_{m,k}^o$ will be embedded if $\rho_{C_4}(T_{m,k}^1) = T_{m,k}^2$.

Since

$$T_0 \cap \bigcup_{i,j} C_{i,j}^1 = \{\bar{p}_1, \ldots, \bar{p}_{4mk}\} \quad \text{and} \quad T_0 \cap \bigcup_{i,j} C_{i,j}^2 = \{\bar{q}_1, \ldots, \bar{q}_{4mk}\},$$

it follows from (1) that

$$\{\bar{p}_1, \ldots, \bar{p}_{4mk}\} = \{\bar{q}_1, \ldots, \bar{q}_{4mk}\}.$$

Here we show that the divisibility of k by m is not sufficient for the embeddedness of $T_{m,k}^0$.

The invariance of $T_{m,k}^1$ under the rotations $\rho_{\bar{p}_c}$ implies that $T_{m,k}^1$ occupies every other pentahedron U_j^i alternatingly. Similarly $T_{m,k}^2$ does V_j^i. Hence

$$T_{m,k}^1 \subset \bigcup_{i+j=\text{even}} U_j^i \quad \text{and} \quad T_{m,k}^2 \subset \bigcup_{i+j=\text{odd}} V_j^i.$$

The length of the arc $\overline{p_2'p_2''}$ is $\frac{\pi}{\sqrt{2m}}$ in Fig. 2. Since the vertex angles of A_j^i, B_j^i are $\pi/4, 3\pi/4, \pi/4, 3\pi/4$, the length of $\overline{pq}$ is π/m in Fig. 3, $q = p_2''$. Hence if π/m is an even multiple of π/k, that is, $k = 2m\ell$ for some integer ℓ, then one sees from Fig. 3 that $\rho_{C_4}(T_{m,k}^1) = T_{m,k}^2$. One cannot draw the same conclusion in case k is an odd multiple

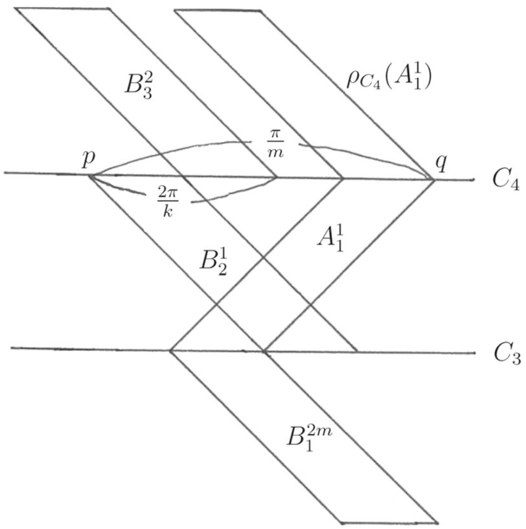

Fig. 3 Embeddedness of the odd surface

of m due to $T^1_{m,k}$'s alternating occupancy in U^i_j. Therefore $T^o_{m,k} = T^1_{m,k} \cup T^2_{m,k}$, and thus $T^o_{m,k}$ is embedded.

There are $2mk$ congruent copies of H in $T^1_{m,k}$. Similarly for $T^2_{m,k}$, hence $T^o_{m,k}$ contains a total of $4mk$ congruent copies of H when it is embedded. Now let's apply the Gauss-Bonnet theorem to H. Note that the external angles of H are $\pi/2$ at its vertices p'_1, p'_2, p''_1, p''_2 and $(m-1)\pi/m$ at p_1, p_2. Hence

$$\int_H K dA + \left(4 - \frac{2}{m}\right)\pi = 2\pi.$$

Therefore

$$2\pi\chi(T^o_{m,k}) = \int_{T^o_{m,k}} K dA = 4mk\left(-2\pi + \frac{2\pi}{m}\right),$$

and so

$$g = 1 + 2k(m-1).$$

For (ii) note that $\Phi^{\frac{2\pi}{k}}_{1234}$ maps $\overline{p_1 p'_1}$ onto $\overline{p_3 p'_3}$ and

$$\Phi^{\frac{2\pi}{k}}_{1234}(T^o_{m,k}) = T^o_{m,k}.$$

So $T^o_{m,k}$ is invariant under the finite cyclic group generated by $\Phi^{\frac{2\pi}{k}}_{1234}$.

For (iii) remember that the parallelogram $A^1_1 = \square p'_1 p'_2 p''_2 p''_1$ has vertex angles of $\pi/4, 3\pi/4, \pi/4, 3\pi/4$. Hence the fundamental piece H can have no reflection symmetry across a great sphere and neither can $T^o_{m,k}$. However, $T^o_{m,k}$ has 180°-rotation symmetries.

For (iv) note that both the minimal disk H and the union H_0 of two flat rectangles $\Box p_1 p_1' p_2' p_2$ and $\Box p_1 p_1'' p_2'' p_2$ span the same Jordan curve Γ. Hence

$$\mathrm{Area}(H) < \mathrm{Area}(H_0).$$

Since

$$\bigcup_{\rho \in G^o} \rho(H_0) = T_1 \cup \cdots \cup T_m$$

and $\mathrm{Area}(T_i) = 2\pi^2$, the conclusion follows. $\Box$

4 Even surfaces

$T^o_{m,k}$ is a desingularization of $T_1 \cup \cdots \cup T_m$ along $C_1 \cup C_2$. But there is another way of desingularizing $T_1 \cup \cdots \cup T_m$ because once $T_1 \cup \cdots \cup T_m$ is desingularized along C_1, there are two ways of desingularization along C_2. The new desingularization can be done by replacing H with K which is obtained by freeing the edges $\overline{p_1' p_2'}$, $\overline{p_1'' p_2''}$ of H into the curves on A_1^1, decreasing its area.

Theorem 2 *Let $T_1, \ldots, T_m \subset \mathbb{S}^3$ be the Clifford tori meeting each other along a great circle C_1 at an angle of π/m. Then there exists a compact minimal surface $T^e_{m,k}$ desingularizing $T_1 \cup \cdots \cup T_m$ for each $k = 2m\ell$ with integers $m \geq 2$, $\ell \geq 2$:*

(i) *$T^e_{m,k}$ is embedded and has genus $1 + 2k(m-1)$;*
(ii) *$T^e_{m,k}$ is invariant under a finite group of screw motions;*
(iii) *$T^e_{m,k}$ has no reflection symmetry across a great sphere;*
(iv) $\mathrm{Area}(T^e_{m,k}) < \mathrm{Area}(T^o_{m,k})$.

Proof First let's diversify the screw motion Φ^t_{ijkl}. Let C_a, C_b be two great circles with appropriate orientations which are polar to each other. Denote by ρ^t_C the rotation of $\mathbb{S}^3$ around the polar great circle of C by the angle t. Then $\rho^t_C|_C$ is the translation on C by distance t. Now define a screw motion $\Phi^t_{C_a}$ by

$$\Phi^t_{C_a} = \rho^t_{C_a} \circ \rho^t_{C_b}.$$

We also define a screw motion with distinct speeds $\Phi^{t,s}_{C_a}$ by

$$\Phi^{t,s}_{C_a} = \rho^t_{C_a} \circ \rho^s_{C_b}.$$

Let C_5 be the great circle on T_0 which connects the midpoint of $\overline{p_1' p_1''}$ to that of $\overline{p_2' p_2''}$ and denote by C_6 the great circle polar to C_5. Put

$$\varphi = \Phi^{-\frac{\pi}{2m}, \pi - \frac{\pi}{2m}}_{C_5} = \rho^{-\frac{\pi}{2m}}_{C_5} \circ \rho^{\pi - \frac{\pi}{2m}}_{C_6} = \Phi^{-\frac{\pi}{2m}}_{C_5} \circ \rho^{\pi}_{C_6}$$

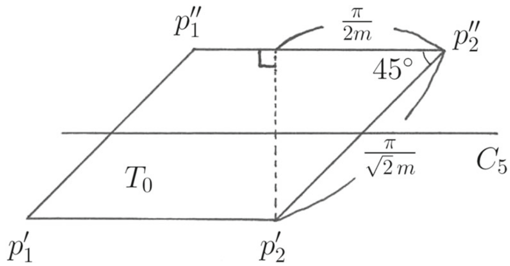

Fig. 4 Parallelogram in the tessellation of T_0

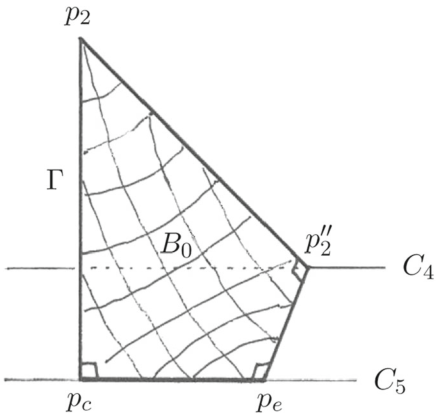

Fig. 5 Fundamental piece of a barrier

and define G^e to be the cyclic group generated by φ. It should be noted that even though $\rho_{C_5}^{-\frac{\pi}{2m}}$ is a translation on C_5 by $-\frac{\pi}{2m}$, it rotates T_0 around C_5 by the angle $\frac{\pi}{2m}$. So φ translates C_5 by $-\frac{\pi}{2m}$ and rotates T_0 around C_5 by π. Hence from Fig. 4 we see that

$$\varphi(C_5) = C_5, \quad \varphi(T_0) = T_0 \quad \text{and} \quad \varphi(p_2'') = p_2', \quad \varphi(p_1'') = p_1'.$$

Moreover, we have (see Fig. 2)

$$\varphi(\overline{p_2'' p_1''}) = \overline{p_2' p_1'}, \quad \varphi(\overline{p_2'' p_2}) = \overline{p_2' q_2}, \quad \varphi(\overline{p_1'' p_1}) = \overline{p_1' q_1} \quad \text{and} \quad |G^e| = 4m.$$

Let $p_c \in C_5$ be the midpoint of $\overline{p_2' p_2''}$ and $p_e \in C_5$ the point closest to p_2'' as in Fig. 5. Then $\overline{p_c p_e}$ is perpendicular to $\overline{p_e p_2''}$ and to $\overline{p_c p_2}$, and $\mathrm{dist}(p_c, p_e) = \pi/(4m)$. Define a Jordan curve Γ by $\Gamma = \overline{p_2 p_c} \cup \overline{p_c p_e} \cup \overline{p_e p_2''} \cup \overline{p_2'' p_2}$ as in Fig. 5. Obviously Γ bounds an area minimizing disk B_0. B_0 extends analytically across $\overline{p_2 p_c}$ to $B_0 \cup \rho_{\overline{p_2 p_c}}(B_0)$by the 180°-rotation $\rho_{\overline{p_2 p_c}}$.

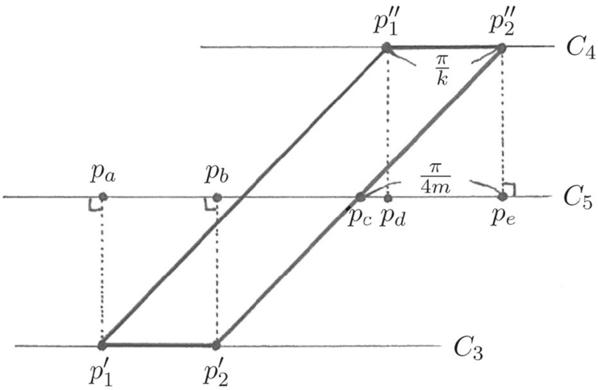

Fig. 6 Narrower parallelogram to guarantee mean convexity

Define

$$\Gamma_1 = \bigcup_{n=1}^{4m} \varphi^n(\overline{p_2'' p_2} \cup \overline{p_2 p_2'}),$$

$$B_1 = \bigcup_{n=1}^{4m} \varphi^n(B_0 \cup \rho_{\overline{p_2 p_c}}(B_0)),$$

$$F = \bigcup_{n=1}^{4m} \varphi^n(\Box p_1 p_1'' p_2'' p_2 \cup \Box p_1 p_1' p_2' p_2).$$

Since $\Phi_{C_1}^{-\pi/k}$ takes p_2, p_2', p_2'' to p_1, p_1', p_1'', respectively, let's also define

$$\Gamma_2 = \Phi_{C_1}^{-\pi/k}(\Gamma_1) \quad \text{and} \quad B_2 = \Phi_{C_1}^{-\pi/k}(B_1).$$

Then Γ_1, Γ_2 are helical Jordan curves consisting of $4m$ geodesic arcs and winding around C_5, and B_1, B_2 are embedded minimal annuli spanning $\Gamma_1 \cup C_5, \Gamma_2 \cup C_5$, respectively. Let W be the domain bounded by $F \cup B_1 \cup B_2$. Obviously $\Gamma_1, \Gamma_2, B_1, B_2, F$ and W are all invariant under the cyclic group G^e.

We now claim that W is mean convex and that there exists an embedded minimal annulus K^e in W spanning $\Gamma_1 \cup \Gamma_2$. For the mean convexity of W it suffices to show that B_1 and B_2 make an angle $\leq \pi$ along their intersection, C_5. Since B_1 and B_2 are invariant under G^e, we have only to prove this angle condition on an arc ($= \overline{p_a p_d}$) of length $\pi/(2m)$ in C_5.

Let p_a, p_b, p_d be the points of C_5 closest to p_1', p_2', p_1'', respectively (see Fig. 6). Then $\text{dist}(p_a, p_d) = \text{dist}(p_b, p_e) = \pi/(2m)$. It is here that we need the hypothesis $k \geq 4m$, i.e., $\ell \geq 2$. Then

$$\text{dist}(p_a, p_b) \leq \text{dist}(p_b, p_d). \tag{2}$$

On $\overline{p_b p_d}$ both B_1 and B_2 are on the same side of T_0 because $\Phi_{C_1}^{-\pi/k}(T_0) = T_0$. Hence they make an angle $\leq \pi$ along $\overline{p_b p_d}$. On the other hand, note that along $\overline{p_c p_e}$ B_0 makes an acute angle with the component T_0^4 of $T_0 \sim (C_4 \cup C_5)$ containing $\overline{p_e p_2''}$ (see Fig. 5). Hence along $\overline{p_a p_b}$ (2) implies that B_2 makes an acute angle with T_0^3, where T_0^3 is the component of $T_0 \sim (C_3 \cup C_5)$ containing $\overline{p_2' p_c}$. Moreover T_0^3 makes an acute angle with $\varphi(B_0) \subset B_1$ along $\overline{p_a p_b}$. Therefore B_1 and B_2 make an angle $\leq \pi$ along $\overline{p_a p_b}$. So W is locally mean convex along $\overline{p_a p_d}$, and it follows that W is mean convex. Let U be the component of $W \sim T_0$ such that $\bar{U} \supset \overline{p_1 p_2}$. Then U is also mean convex.

Denote by $\mathcal{A}$ the set of all curves $\alpha \subset \partial U \cap T_0^4$ from p_1'' to p_2'' with no self intersection. For $\alpha \in \mathcal{A}$, let Γ_α be the Jordan curve $\alpha \cup \overline{p_2'' p_2} \cup \overline{p_2 p_2'} \cup \varphi(\alpha) \cup \overline{p_1' p_1} \cup \overline{p_1 p_1''}$. The mean-convexity of U guarantees the existence of an embedded minimal disk in U spanning Γ_α. Let $\mathcal{K}$ be the family of all such embedded minimal disks in U spanning Γ_α for all $\alpha \in \mathcal{A}$.

We now show that there exists an area minimizer K in $\mathcal{K}$:

$$\mathrm{Area}(K) = \inf_{\hat{K} \in \mathcal{K}} \mathrm{Area}(\hat{K}).$$

Let $\{\hat{K}_i\}$ be a minimizing sequence in $\mathcal{K}$ with

$$\lim_{i \to \infty} \mathrm{Area}(\hat{K}_i) = \inf_{\hat{K} \in \mathcal{K}} \mathrm{Area}(\hat{K}).$$

The limit K of $\{\hat{K}_i\}$ as an area minimizing current obviously exists in U. And it is easy by [1] to see that K is a smooth embedded minimal disk. K should be unique, but its proof is harder than the uniqueness of H.

Denote again by α the part of $\partial K \cap T_0^4$ connecting p_1'' to p_2''. Then $\alpha \cup \varphi(\alpha) = \partial K \cap T_0$, and one can see that K analytically extends to $K \cup \varphi(K)$ across $\varphi(\alpha)$. This is because if K makes an angle $\neq \pi$ with $\varphi(K)$ along $\varphi(\alpha)$, then $K \cup \varphi(K)$ can be perturbed along $\varphi(\alpha)$ decreasing its area, which contradicts the assumption that K is a minimizer. Similarly K should extend analytically to $K \cup (\varphi)^{-1}(K)$ across α. Therefore one can extend K analytically to a smooth minimal annulus

$$K^e := \bigcup_{n=1}^{4m} \varphi^n(K) \subset W$$

which spans $\Gamma_1 \cup \Gamma_2$. Note that

$$\Gamma_1 \cup \Gamma_2 \subset \bigcup_{i,j} C_{ij}^1.$$

Therefore K^e can be indefinitely extended across $\Gamma_1 \cup \Gamma_2$ by 180°-rotations $\rho_{\bar{p}_1}, \ldots, \rho_{\bar{p}_{4mk}}$ to produce a complete minimal surface $T_{m,k}^e$. Since the group generated by $\rho_{\bar{p}_1}, \ldots, \rho_{\bar{p}_{4mk}}$ is finite, one can conclude that $T_{m,k}^e$ is compact. As K^e is

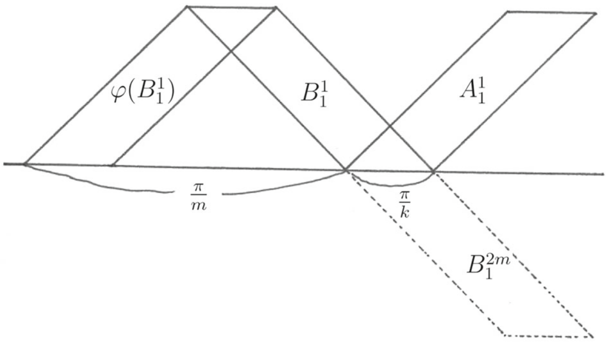

Fig. 7 Embeddedness of the even surface

invariant under the cyclic group G^e generated by the screw motion φ, so is $T^e_{m,k}$, proving (ii).

Let's show that $T^e_{m,k}$ is embedded. Clearly $\varphi(U_1^1) = V_1^1$ and $\varphi(A_1^1) = B_1^1$ as in Fig. 7. Suppose k is divisible by $2m$, i.e., $k = 2m\ell$. Then

$$\varphi^2(U_1^1) = U^1_{2k-2\ell+1} \subset \bigcup_{i+j=\text{even}} U_j^i \quad \text{and} \quad \varphi^2(V_1^1) = V^1_{2k-2\ell+1} \subset \bigcup_{i+j=\text{even}} V_j^i.$$

Since K^e is invariant under φ^n, $1 \le n \le 4m$, one sees that

$$K^e \cap D_1 \subset \bigcup_{i+j=\text{even}} U_j^i \quad \text{and} \quad K^e \cap D_2 \subset \bigcup_{i+j=\text{even}} V_j^i.$$

Hence the embeddedness of $T^e_{m,k}$ follows from the fact that for $c = 1, \ldots, 4mk$

$$\rho_{\bar{p}_c}\left(\bigcup_{i+j=\text{even}} U_j^i\right) = \bigcup_{i+j=\text{even}} U_j^i \quad \text{and} \quad \rho_{\bar{p}_c}\left(\bigcup_{i+j=\text{even}} V_j^i\right) = \bigcup_{i+j=\text{even}} V_j^i.$$

There are $2mk$ congruent copies of K in $T^e_{m,k} \cap D_1$, and there are the same number of copies of $\varphi(K)$ in $T^e_{m,k} \cap D_2$. It should be remarked that the sum of the geodesic curvatures of α and $\varphi(\alpha)$ at $p \in \alpha$ and $\varphi(p) \in \varphi(\alpha)$, respectively, vanishes. And the external angles of K at p_1', p_2', p_1'', p_2'' are $\pi/2$ and $(m-1)\pi/m$ at p_1, p_2. So by the same argument as in Theorem 1 we see that

$$g = 1 + 2k(m-1).$$

(iii) follows from the same arguments as in Theorem 1. For (iv) just note that $H \in \mathcal{K}$. $\square$

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