CLASSICAL MECHANICS OF MINIMAL TORI IN $\mathbb{S}^3$

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ABSTRACT. We formulate a class of minimal tori in $\mathbb{S}^3$ in terms of classical mechanics, reveal a curious property of the Clifford torus, and note that the question of periodicity can be made more explicit in a simple way.

1. INTRODUCTION

The Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is geometrically the simplest torus. It is a flat square torus. When embedded in the three-dimensional unit sphere $\mathbb{S}^3$, it becomes a minimal surface. It divides $\mathbb{S}^3$ into two congruent solid tori. Moreover, it is doubly ruled, i.e. it can be foliated by two orthogonal families of great circles. The Clifford torus is the unique algebraic minimal surface of degree 2 and is characterized even locally as the only (non-totally-geodesic) minimal surface of contant curvature in $\mathbb{S}^3$. A long-standing conjecture by Lawson [L2] that the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$ was recently solved affirmatively by S. Brendle [B1]. Also, more recently the Clifford torus has been used as a building block for constructing infinitely many compact embedded minimal surfaces in $\mathbb{S}^3$ [KY], [CS]. For a good recent overview concerning minimal surfaces in $\mathbb{S}^3$ see [B2] (and for constant mean curvature tori see [AL] and references therein).

In this note, we will report on two previously unnoticed aspects of minimal tori in $\mathbb{S}^3$: as deformations of the Clifford torus, and a relation to classical mechanics.

2. HAMILTONIAN DESCRIPTION OF SURFACES

Originally starting out to find minimal surfaces of higher genus via stationary points of the functional

$$(1) \qquad S[w] := \int \sqrt{\eta^{KL} \partial_K w \partial_L w} \sqrt{\eta} d^D u$$

i.e. solutions of (c.p. [BH], e.g.)

$$(2) \qquad (\eta^{IJ}\eta^{KL} - \eta^{IK}\eta^{JL})\partial_I w \partial_J w (\partial^2_{KL} - \Gamma^M_{KL}\partial_M w) = 0,$$

we found a class of minimal tori of the form

$$(3) \qquad \vec{x}(\varphi^1, \varphi^2) = \begin{pmatrix} \cos\theta(\varphi^1,\varphi^2)\cos(\varphi^1) \\ \cos\theta(\varphi^1,\varphi^2)\sin(\varphi^1) \\ \sin\theta(\varphi^1,\varphi^2)\cos(\varphi^2) \\ \sin\theta(\varphi^1,\varphi^2)\sin(\varphi^2) \end{pmatrix} \in \mathbb{S}^3 \subset \mathbb{R}^4,$$

where θ (when depending on φ^1 and φ^2 only via the combination $k\varphi^1 + l\varphi^2 =: t$) can be determined as the solution of a classical mechanics problem,

$$(4) \qquad \dot{\theta}^2 + \frac{c^2 s^2}{k^2 s^2 + l^2 c^2}\left(1 - \frac{c^2 s^2}{E^2}\right) = 0,$$

i.e. a zero energy solution of a point-particle (having "position" θ at "time" t) moving in a potential $V_E(\theta)$. ($c := \cos\theta$, $s := \sin\theta$)

In order to see that solutions of (4) give minimal surfaces of the form (3), let us calculate the first and second fundamental forms corresponding to (3):

$$(5) \qquad (g_{ab}) \;=\; \begin{pmatrix} c^2 + (\partial_1\theta)^2 & \partial_1\theta\partial_2\theta \\ \partial_1\theta\partial_2\theta & s^2 + (\partial_2\theta)^2 \end{pmatrix}$$

$$(6) \qquad (h_{ab}) \;=\; \frac{1}{|\vec{m}|}\begin{pmatrix} s^2 c + sc\theta_{11} + 2S^2\theta_1^2 & sc\theta_{12} + (s^2 - c^2)\theta_1\theta_2 \\ sc\theta_{12} + (s^2 - c^2)\theta_1\theta_2 & -s^2 c^2 + sc\theta_{22} - 2c^2\theta_2^2 \end{pmatrix}$$

with

$$(7) \qquad \vec{m} := sc\begin{pmatrix} -sc_1 \\ -ss_1 \\ cc_2 \\ cs_2 \end{pmatrix} - \begin{pmatrix} -ss_1\theta_1 \\ sc_1\theta_1 \\ -cs_2\theta_2 \\ cc_2\theta_2 \end{pmatrix}, \quad c_i := \cos\varphi^i, \; s_i := \sin\varphi^i$$

being orthogonal to $\partial_1\vec{x}$, $\partial_2\vec{x}$, and $\vec{x}$; hence one finds that (3) has zero mean curvature in $\mathbb{S}^3$ if and only if

$$(8) \qquad \begin{array}{c} \left(s^2 + \theta_2^2\right)\left(s^2 c^2 + sc\theta_{11} + 2s^2\theta_1^2\right) + \left(c^2 + \theta_1^2\right)\left(-s^2 c^2 + sc\theta_{22} - 2c^2\theta_2^2\right) \\ -2\theta_1\theta_2\left(sc\theta_{12} + (s^2 - c^2)\theta_1\theta_2\right) = 0, \end{array}$$

where $\theta_a := \partial_a\theta$.

Note that, equivalently, one could have obtained (8) by varying

$$(9) \qquad S[\theta] := \int \sqrt{g}\,d\varphi^1 d\varphi^2 = \int \sqrt{c^2 s^2 + s^2\theta_1^2 + c^2\theta_2^2}\,d\varphi^1 d\varphi^2.$$

For $\theta = \theta(k\varphi^1 + l\varphi^2) = \theta(t)$ one gets, from the Lagrangian (the '$-$' sign put in for later convenience)

$$(10) \qquad L = -\sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2)\dot\theta^2}$$

as well as directly from (8), the second order equation

$$(11) \qquad \begin{array}{c} sc\ddot\theta(k^2 s^2 + l^2 s^2) + \dot\theta^2\left[(l^2 - k^2)s^2 c^2 + 2s^4 k^2 - 2c^4 l^2\right] \\ + s^2 c^2(s^2 - c^2) = 0, \end{array}$$

which can also be shown to follow from (4) by simply differentiating. A possible way to systematically derive (4) from (11) is provided by a standard Legendre-transformation, obtaining from (10) a Hamiltonian

$$(12) \qquad H := \frac{\partial L}{\partial\dot\theta}\dot\theta - L = \cdots = \frac{c^2 s^2}{\sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2)\dot\theta^2}} = E.$$

Expressing H in terms of θ and the canonical momentum

$$(13) \qquad \pi := \frac{\partial L}{\partial\dot\theta} = -\frac{(k^2 s^2 + l^2 s^2)\dot\theta}{\sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2)\dot\theta^2}}$$

one obtains (replacing $|cs|$ by cs, justifiable via $H = const$)

$$(14) \qquad H = \frac{1}{2}\sin(2\theta)\sqrt{1 - \frac{\pi^2}{k^2 s^2 + l^2 c^2}} = H[\theta, \pi];$$

one can check that the first-order equations

$$(15) \qquad \dot{\theta} = \frac{\partial H}{\partial \pi} = -\frac{1}{2} \sin 2\theta \frac{\frac{\pi}{k^2 s^2 + l^2 c^2}}{\sqrt{1 - \frac{\pi^2}{k^2 s^2 + l^2 c^2}}} ; \quad \dot{\pi} = -\frac{\partial H}{\partial \theta}$$

reproduce (11).

3. $k = l$: CLIFFORD TORUS IN DISGUISE

Whereas the axially symmetric $k = 0$ (or $l = 0$) case, except for our mechanical interpretation, is well known (S.Brendle [B2] quotes R. Kusner, when discussing an elliptic integral solution of the form (3) for $l = 1, k = 0$), the above mentioned $k = l$ case can actually be integrated in terms of elementary functions, leading to the curious fact that the Clifford-Torus can be viewed as a "non-trivial" graph over itself (in infinitely many different ways). Separating variables, and letting

$$\alpha \left(\varphi := \varphi^1 + \varphi^2 \right) = 2\theta \left(k \left(\varphi^1 + \varphi^2 \right) \right),$$

$a := \frac{1}{2E} = \cosh \gamma > 1$, one deduces from (4) that

$$(16) \qquad \frac{t - t_0}{k} = \varphi - \varphi_0 = \pm \int \frac{d\alpha}{\sin \alpha \sqrt{a^2 \sin^2 \alpha - 1}}$$

$$= \mp \arctan \frac{\cos \alpha}{\sqrt{a^2 \sin^2 \alpha - 1}}$$

i.e. that $\alpha(\varphi)$ is given via

$$(17) \qquad \frac{\mp \cos \alpha}{\sqrt{a^2 \sin^2 \alpha - 1}} = \tan(\varphi - \varphi_0) =: T$$

as a solution of

$$(18) \qquad (\alpha')^2 = \sin^2 \alpha \left(a^2 \sin^2 \alpha - 1 \right).$$

Using this, respectively

$$(19) \qquad \sin \alpha = \sqrt{\frac{1 + T^2}{1 + a^2 T^2}} = \frac{1}{\sqrt{\cos^2(\varphi - \varphi_0) + \cosh^2 \gamma \sin^2(\varphi - \varphi_0)}}$$

$$\cos \alpha = \frac{-\sin(\varphi - \varphi_0) \sinh \gamma}{\sqrt{\cos^2(\varphi - \varphi_0) + \cosh^2 \gamma \sin^2(\varphi - \varphi_0)}},$$

one can check that $\alpha(\varphi)$, given as in (19), does satisfy the second order equation (cp. (11)) that is equivalent to the vanishing of the mean curvature,

$$(20) \qquad (\sin \alpha)\alpha'' - 2\alpha'^2 \cos \alpha = (\cos \alpha)(\sin \alpha)^2,$$

whereby it is useful to note that

$$\alpha' \cot \alpha = -\frac{T(a^2 - 1)}{1 + a^2 T^2}$$

$$(21) \qquad \alpha'' = \frac{-\sin(\varphi - \varphi^0) \sinh \gamma (\cosh^2 \gamma + \sinh^2 \gamma \cos^2(\varphi - \varphi^0))}{(\cos^2(\varphi - \varphi^0) + \cosh^2 \gamma \sin^2(\varphi - \varphi^0))^2}.$$

The derived solution(s) read(s)

$$(22) \qquad \vec{x}\left(\varphi^1 \varphi^2\right) = \frac{1}{2} \begin{pmatrix} \sqrt{1 - \frac{e\sin(\varphi - \varphi^0)}{\sqrt{1 + e^2 \sin^2(\varphi - \varphi^0)}}} & \cos\varphi^1 \\ & \sin\varphi^1 \\ \sqrt{1 + \frac{e\sin(\varphi - \varphi^0)}{\sqrt{1 + e^2 \sin^2(\varphi - \varphi^0)}}} & \cos\varphi^2 \\ & \sin\varphi^2 \end{pmatrix}$$

with $e := \sinh\gamma \in \mathbb{R}$ an arbitrary constant.

One easily sees that, for each value of $e \in \mathbb{R}$, the solution (22) defines an embedded minimal torus in S^3, which at first sight is rather puzzling, as every embedded minimal torus in $\mathbb{S}^3$ must be congruent to the Clifford torus [B1]. Let us give three direct proofs for the above concrete case, (22).

Firstly, for constant φ both x_3 and x_4 can be expressed as a linear combination of x_1 and x_2, defining 2 hyper-planes in $\mathbb{R}^4$, respectively. Since the intersection of $\mathbb{S}^3$ with two hyper-planes containing the origin of $\mathbb{R}^4$ gives a great circle, the minimal torus is ruled. It is known [L1] that the Clifford torus is the only embedded surface among all infinitely many ruled minimal surfaces in $\mathbb{S}^3$. Hence (22) describes a Clifford torus.

Secondly, calculating the determinants of (6) and (5) for $\theta = \theta\left(k\varphi^1 + l\varphi^2\right)$, one finds

$$(23) \qquad h = \frac{-\dot{\theta}^4 - 2s^2 c^2 \dot{\theta}^2 - s^4 c^4}{\dot{\theta}^2 + s^2 c^2} = -\left(\dot{\theta}^2 + s^2 c^2\right)$$

$$g = \dot{\theta}^2 + s^2 c^2, \qquad \frac{h}{g} = -1,$$

hence for the intrinsic Gaussian curvature

$$(24) \qquad \begin{aligned} R &= (TrW)^2 - \left(TrW^2\right) + (Tr\tilde{W})^2 - \left(Tr\tilde{W}^2\right) \\ &= 0 - 2 + 4 - 2 = 0 \end{aligned}$$

$$\left(\tilde{h}_{ab} := \vec{x} \cdot \partial_{ab}^2 \vec{x} = -\partial_a \vec{x} \cdot \partial_b \vec{x} = -g_{ab} \Rightarrow \tilde{W}_b^a := g^{ac} \tilde{h}_{cb} = -\delta_b^a\right),$$

proving that the minimal tori (22) are flat (hence the Clifford torus).

Thirdly, one can construct an explicit isometry to

$$(25) \qquad \underset{\sim}{\vec{x}}(\tilde{\varphi}^1, \tilde{\varphi}^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\tilde{\varphi}^1) \\ \sin(\tilde{\varphi}^1) \\ \cos(\tilde{\varphi}^2) \\ \sin(\tilde{\varphi}^2) \end{pmatrix}$$

by finding a re-parametrization $\varphi^1 \varphi^2 \to \tilde{\varphi}^1 \tilde{\varphi}^2$ such that $(k = l = 1, \varphi^0 = 0)$

$$(26) \qquad 2\left(g_{ab}\right) = \mathbb{1} + \begin{pmatrix} \cos\alpha & 0 \\ 0 & -\cos\alpha \end{pmatrix} + \frac{1}{2}\sin^2\alpha\left(a^2\sin^2 - 1\right)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \mathbb{1} - \frac{e\sin\varphi}{\sqrt{1 + e^2 \sin^2\varphi}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\frac{e^2\sin^2\varphi}{(1 + e^2\sin^2\varphi)^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= J^T\left(2\tilde{g}_{..}\right)J = J^T J; \qquad J_a^{\tilde{a}} = \frac{\partial\tilde{\varphi}^{\tilde{a}}}{\partial\varphi^a}.$$

The Ansatz

$$(27) \qquad \tilde{\varphi}^1 = \varphi^1 + \int^{\varphi} u, \qquad \tilde{\varphi}^2 = \varphi^2 + \int^{\varphi} v$$

gives

$$(28) \quad J = \begin{pmatrix} 1+u & u \\ v & 1+v \end{pmatrix},$$

$$J^T J = \mathbb{1} + \begin{pmatrix} 2u & u+v \\ u+v & 2v \end{pmatrix} + (u^2+v^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that (26) will be satisfied when choosing

$$(29) \quad u = \frac{1}{2}\cos\alpha(\varphi) + w(\varphi)$$

$$v = -\frac{1}{2}\cos\alpha(\varphi) + w(\varphi)$$

$$u^2 + v^2 + 2w = w^2 + 2w + \frac{1}{4}(\cos\alpha)^2 = \frac{1}{2}\frac{c^2\sin^2\varphi}{(1+c^2\sin^2\varphi)^2}$$

$$w(\varphi) = -1 + \frac{\sqrt{1 + \frac{3}{4}e^4 s^4 + \frac{9}{4}e^2 s^2}}{(1+e^2 s^2)}(s = \sin\varphi).$$

4. $k \neq l$

Analogously one could explicitly construct isothermal coordinates also for arbitrary k and l via $(s = \sin\theta, c = \cos\theta)$

$$(30) \quad \begin{aligned} (g_{ab}) &= \begin{pmatrix} \cos 2\theta & 0 \\ 0 & \sin 2\theta \end{pmatrix} + \begin{pmatrix} k^2 & kl \\ kl & l^2 \end{pmatrix} \left(\frac{c^2 s^2}{E^2} - 1\right) \left(\frac{c^2 s^2}{k^2 s^2 + l^2 c^2}\right) \\ &= \rho^2 J^T J, \qquad J = \begin{pmatrix} 1+ku & lu \\ kv & 1+lv \end{pmatrix}. \end{aligned}$$

With

$$Y(\theta) := \left(\frac{c^2 s^2}{E^2} - 1\right)\left(\frac{c^2 s^2}{k^2 s^2 + l^2 c^2}\right) = -V_E(\theta)$$

one gets

$$(31) \quad \begin{aligned} c^2 + k^2 Y &= \left(1 + k^2(u^2+v^2) + 2ku\right)\rho^2 \\ s^2 + l^2 Y &= \left(1 + l^2(u^2+v^2) + 2lv\right)\rho^2 \\ klY &= \left(lu + kv + kl(u^2+v^2)\right)\rho^2. \end{aligned}$$

Substracting $2kl(klY)$ from $l^2(c^2 + k^2 Y) + k^2(s^2 + l^2 Y)$ and dividing by $k^2 + l^2$ one obtains

$$(32) \quad \rho^2 = \frac{k^2 s^2 + l^2 c^2}{k^2 + l^2}$$

while $l^2(c^2 + k^2 Y) - k^2(s^2 + l^2 Y)$ yields an (inhomogeneous) linear! relation between u and v, which (when substituted into any of the 3 equations in (31)) gives a simple quadratic equation for u (or v).

In order to study the periodicity properties (for general k and l), it is convenient to still write the solution into the form (22), but with φ replaced by $\tilde{\varphi}$, which is defined by the absorption of the factor $k^2 s^2 + l^2 c^2$, i.e.

$$(33) \quad \frac{\partial\varphi}{\partial\tilde{\varphi}} = \sqrt{k^2\sin^2\tilde{\theta} + l^2\cos^2\tilde{\theta}}$$

$$= \sqrt{\frac{k^2 + l^2}{2}} \sqrt{1 + \frac{(k^2 - l^2)}{k^2 + l^2} \frac{e \sin \tilde{\varphi}}{\sqrt{1 + e^2 \sin^2 \tilde{\varphi}}}}.$$

In the case discussed by Brendle ($k = 0, l = 1$) one would get

$$(34) \qquad \varphi^2 = \frac{1}{\sqrt{2}} \int^{\tilde{\varphi}} \sqrt{1 - \frac{e \sin u}{\sqrt{1 + e^2 \sin^2 u}}} du,$$

which is a somewhat simpler elliptic integral than the formula for the period in [B2]. As for $e = 0$,

$$(35) \qquad \delta \varphi^2 = \frac{1}{\sqrt{2}} \int_{\tilde{\varphi}}^{\tilde{\varphi} + 2\pi} = \sqrt{2} \pi = \delta_0 \varphi^2,$$

while for $e \to +\infty$,

$$\delta \varphi^2 = \frac{1}{\sqrt{2}} \int_{\pi}^{2\pi} \sqrt{2} = \pi$$

it is, because of the continuity in e, clear that there will be infinitely many values of e for which $\delta \varphi^2$ will be a rational multiple of π, $(p/q)\pi$. Alternatively, with $\sin^2 \theta(\varphi^2) = r^2(\varphi^2) = v(\varphi^2)$ one has

$$(36) \qquad \varphi^2 - \varphi_0^2 = \frac{1}{2} \frac{1}{\sqrt{1 - v_-}} \pi \left(\arcsin \sqrt{\frac{1 - v}{1 - v_+}}, 1 - v_+, \sqrt{\frac{1 - v_+}{1 - v_-}} \right)$$

$$1 > v \geq v_+ > v_- > 0, v_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - E^2},$$

an elliptic integral of the third kind.

Acknowledgment

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