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Isoperimetric Inequalities of Minimal Submanifolds

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Abstract. In this paper we introduce various types of isoperimetric inequalities for minimal submanifolds in Euclidean space, sphere, hyperbolic space, and Riemannian manifolds. Some inequalities for compound soap films, domains, nonpositively curved surfaces and harmonic maps are also discussed.

Contents

1. Introduction

The history of the isoperimetric problem begins with its legendary origins in the Problem of Queen Dido, told by Virgil in the *Aeneid.* Dido was a Phoenician princess from the city of Tyre. She fled by ship from Tyre when King Pygmalion, her tyrannical brother, murdered her husband to usurp her possessions. When Dido arrived in Africa at the site that was to become Carthage, she sought to purchase land from the natives. They told her they would sell only as much land as she

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could surround with a bull's hide. She accepted the terms and made the most of the situation. First she had her people cut a bull's hide into thin strips and tie them together to form a single, very long, closed cord. Then, by sheer intuition, she reasoned that she could encompass the most area by shaping the cord into the circumference of a circle. In this way she acquired a larger piece of land than she had coveted.

Here let us add our own fable to history: Dido was thus able to lead a comfortable life in the big land now called Byrsa. But her peaceful life did not last long; King Pygmalion, wanting ever more power and land, invaded Carthage, and Queen Dido was forced to flee again. This time she decided to move to Wonderland, where people inhabit a big soap film (minimal surface). There she purchased land surrounded by the same cord that she had used in Carthage. And Queen Dido asked herself whether her land in Wonderland was bigger than that in Carthage...

The first proof that Dido made the optimum choice in Carthage appears in the commentary of Theon to Ptolemy's *Almagest* and in the collected works of Pappus. The author of the proof is Zenodoros. However, Zenodoros' proof still contained a gap that was not filled in until the second half of the nineteenth century. A rigorous mathematical proof was given by Weierstrass in his lectures at the University of Berlin [**Sp**]. But Dido's Wonderland question has not been completely settled yet.

This paper concerns Queen Dido's new problem. We will summarize the results so far obtained by the author and others, and will present some new results. Many kinds of minimal submanifolds will be dealt with and various types of isoperimetric inequalities for them will be introduced.

Queen Dido's characterization of the circle is most succinctly expressed in the isoperimetric inequality

$$4\pi A \le L^2,$$

where A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle. The first result in regard to Dido's new problem is due to Carleman [**Ca**] who showed in 1921 that if S is a disk type minimal surface in $\mathbb{R}^n$ with area A and perimeter L, then

$$4\pi A \le L^2 \tag{1–1}$$

with equality if and only if S is a flat disk. In other words, Queen Dido's land in Wonderland is smaller than that in Carthage. Then in 1933 Beckenbach and Radó [**BR**] generalized Carleman's method and showed that $4\pi A \le L^2$ holds also for a disk type surface of nonpositive Gaussian curvature. Note that minimal surfaces have nonpositive Gaussian curvature.

It would seem at first glance that Carleman, Beckenbach and Radó's results provide a complete solution to Queen Dido's new problem, but that is not the case for some reason. The restriction to disk type surfaces is not a natural one. For example, a simple closed curve in $\mathbb{R}^3$ may bound a minimal surface of the type of a Möbius strip, or a surface with higher genus. One would like to know if $4\pi A \le L^2$ also in those cases. A point worth noting when one drops simple connectivity is that the inequality $4\pi A \le L^2$ does not hold for general surfaces of nonpositive Gaussian curvature. For example, on a long cylinder the perimeter remains fixed while the area can be made as big as one wishes. Also one can construct an example with a single boundary curve using a flat torus: the complement of a small disk is a domain of genus one on the torus whose perimeter can be made arbitrarily small.

In view of these examples it is something of a surprise that similar examples cannot be constructed within the class of minimal surfaces. This leads us to the following.

OPEN PROBLEM 1.1. *Prove that* $4\pi A \le L^2$ *for any minimal surface in* $\mathbb{R}^n$.

The difficulty in this open problem is that the minimal surface $S \subset \mathbb{R}^n$ may have arbitrary topology, ∂S may have more than one components, and S may not be area minimizing. Here are two arguments which give some insight into why one might expect the classical isoperimetric inequality (1–1) to hold also on minimal surfaces. Consider a surface $S \subset \mathbb{R}^n$ that has least area among all surfaces with the same boundary curve ∂S. Let its area be A and its perimeter L. First, suppose that ∂S is connected. Let S' be the cone over ∂S with vertex at some point of ∂S. Then S' has the same boundary as S, and hence its area A' is not less than A. But since it is a cone, S' can be developed in $\mathbb{R}^2$ onto a domain $D \subset \mathbb{R}^2$, preserving its area and perimeter. By the classical isoperimetric inequality in the plane, $4\pi A \le 4\pi A' \le L^2$. Second, suppose ∂S is not connected, and let $C_1, \dots, C_n$ be the distinct components of ∂S, $S_1, \dots, S_n$ the Douglas–Radó solutions with $\partial S_i = C_i$, A_i the area of S_i, and L_i the length of C_i. Then $S_1 \cup \dots \cup S_n$ has the same boundary as S, and hence its area $A_1 + \dots + A_n$ is not less than A. But by the first argument for each S_i, $4\pi A_i \le L_i^2$, and so $4\pi A \le 4\pi(A_1 + \dots + A_n) \le L_1^2 + \dots + L_n^2 \le L^2$. These arguments, unfortunately, are valid only for area minimizing surfaces, while in Open Problem 1.1 S is an arbitrary minimal surface.

2. Osserman and Schiffer's Proof

First we take a look at the history of the isoperimetric inequality of a minimal surface in space. Carleman [**Ca**] used complex function theory in 1921 for the proof of (1–1). In 1959 Reid [**Re**] proved (1–1) for a minimal surface with connected boundary in $\mathbb{R}^3$. His proof is based on Wirtinger's inequality:

$$\int_0^{2\pi} \left(\frac{dy}{dt}\right)^2 dt \ge \int_0^{2\pi} y^2 dt, \tag{2–1}$$

where $y(t)$ is a smooth function with period 2π and $\int_0^{2\pi} y(t)\,dt = 0$. This proof was extended by Hsiung [**Hs**] to $\mathbb{R}^n$ in 1961. Then in 1975 a proof of (1–1) was obtained by Osserman–Schiffer [**OS**] for a doubly connected minimal surface in $\mathbb{R}^3$, and in 1977 by Feinberg [**Fe**] for a doubly connected minimal surface in $\mathbb{R}^n$. In 1983 Li–Schoen–Yau proved (1–1) for a minimal surface with two boundary components in $\mathbb{R}^3$. And in 1990 the author [**C1**] gave a proof for a minimal surface with two boundary components in $\mathbb{R}^n$.

In this section we shall give a proof for a minimal surface with connected boundary in $\mathbb{R}^n$. Also Osserman and Schiffer's result will be outlined.

Let S be an arbitrary surface in $\mathbb{R}^n$. If $\vec{H}$ is the mean curvature vector of S, and if x is the position vector, then a general formula for the area A of S is

$$2A = -\int_S \langle x-p,\, \vec{H}\rangle + \int_{\partial S} \langle x-p,\, \nu\rangle \tag{2–2}$$

where $p \in \mathbb{R}^n$ is arbitrary, and ν is the outward unit conormal to ∂S on S, i.e., the normal to ∂S which is tangent to S. This formula is a special case of the first variation formula of area, using the variation field of the 1-parameter family of homothetic expansions about p. Therefore

$$4\pi A \leq L^2 - 2\pi \int_D \langle x-p,\, \vec{H} \rangle,$$

which is the desired isoperimetric inequality for a minimal surface S, if we can show

$$2\pi \int_{\partial S} \langle x-p,\, \nu \rangle \leq L^2. \tag{2–3}$$

Osserman gave a proof of this inequality in [**O1**], p. 209. As a matter of fact, we can prove that this inequality is a consequence of an isoperimetric inequality provided p is a point of ∂S and ∂S is connected. To see why, let η be the unit normal to ∂S which makes the smallest angle with $x-p$ among the normals to ∂S. Then $\langle x-p,\, \nu \rangle \leq \langle x-p,\, \eta \rangle$ and hence

$$2\pi \int_{\partial S} \langle x-p,\, \nu \rangle \leq 2\pi \int_{\partial S} \langle x-p,\, \eta \rangle.$$

From the definition of η it is not difficult to see that

$$\tfrac{1}{2} \int_{\partial S} \langle x-p,\, \eta \rangle = \text{Area}(p ⋇ \partial S),$$

where $p ⋇ \partial S$ is the cone which is the union of the line segments from p to the points of ∂S. Therefore (2–3) follows from the isoperimetric inequality of the planar domain obtained by developing $p ⋇ \partial S$ on $\mathbb{R}^2$. Note here that in general $p ⋇ \partial S$ cannot be developed onto a planar domain if p is not from ∂S or ∂S is not connected. This kind of estimation of an integral by the area or angle of a cone will play a key role in Section 4 (see (4–1) and (4–2)).

We have thus proved the following.

THEOREM 2.1. *If S is a minimal surface in $\mathbb{R}^n$ with connected boundary, then* $4\pi A \leq L^2$.

Suppose now that S is minimal and has at least two boundary components. Let $\partial S = \bigcup C_i$, $L_i = \text{Length}(C_i)$ and choose points $p_i \in C_i$. Then by (2–3)

$$2\pi \int_{C_i} \langle x-p_i, \nu \rangle \leq L_i^2. \tag{2–4}$$

But

$$\int_{C_i} \langle x-p_1, \nu \rangle = \int_{C_i} \langle x-p_i, \nu \rangle - \int_{C_i} \langle p_1-p_i, \nu \rangle, \tag{2–5}$$

so combining (2–4) and (2–5) with (2–2) gives

$$4\pi A \leq \sum_{i=1}^{k} L_i^2 + 2\pi \sum_{i=2}^{k} \int_{C_i} \langle p_i-p_1, \nu \rangle. \tag{2–6}$$

If the integral term in (2–6) could be made to vanish, one would get

$$4\pi A \leq \sum_{i=1}^{k} L_i^2 \tag{2–7}$$

which is strictly stronger than $4\pi A \leq L^2$. In fact (2–7) does hold for area minimizing surfaces as we have seen at the end of the previous section. On the other hand, (2–7) is not true in general. Consider a catenoidal waist S bounded by parallel circles which is the *horizon* when viewed from a point p in the axis of the catenoid.

In other words, the catenoid is tangent to the position vector from p along the boundary circles of S. By [**C6**, Proposition 1], $\mathrm{Area}(S) = \mathrm{Area}(p \times\!\!\!\!\times \partial S)$ and so $\mathrm{Area}(S)$ will be greater than the sum of the areas A_i of the disks bounded by the separate circles. But for each of those disks we have $4\pi A_i = L_i^2$. Hence (2–7) fails to hold.

Returning to (2–6), let us think of geometric conditions under which the integral term of (2–6) can be made to vanish. A useful observation is the following. Applying (2–2) to a minimal surface S with $\partial S = \bigcup C_i$ yields

$$\int_{\partial S} \langle p, \nu \rangle = \int_{\partial S} \langle x, \nu \rangle - 2A,$$

for any p. Since the right hand side is independent of p, it follows that

$$\sum \int_{C_i} \nu = \int_{\partial S} \nu = 0.$$

Actually this integral is called the *flux* of S along ∂S (see [**Fa**] p. 81).

THEOREM 2.2. (a) *Let S be a minimal surface in $\mathbb{R}^n$ with $\partial S = \bigcup_{i=1}^k C_i$. If the flux $\int_{C_i} \nu$ of S along C_i vanishes for each i, then*

$$4\pi A \leq \sum_{i=1}^{k} L_i^2.$$

(b) [**O1**] *Let S be a minimal surface in $\mathbb{R}^n$ with two boundary components, i.e., $\partial S = C_1 \cup C_2$. If no hyperplane separates C_1 from C_2, then*

$$4\pi A \leq L_1^2 + L_2^2.$$

PROOF. (a) follows from (2–6) since $p_i - p_1$ is a constant vector on each C_i. The flux $\int_{C_1} \nu = -\int_{C_2} \nu$ may or may not be a zero vector. In any case, let Π be a hyperplane which is orthogonal to the flux and intersects both C_1 and C_2. Choose $p_1 \in C_1 \cap \Pi$ and $p_2 \in C_2 \cap \Pi$. Then $\left\langle p_2 - p_1, \int_{C_2} \nu \right\rangle = 0$, so (2–6) proves (b). □

Suppose that a hyperplane separates C_1 from $\bigcup_{i=2}^k C_i$. Then it is easy to see that the flux $\int_{C_1} \nu$ of S along C_1 is nonzero. This observation and Theorem 2.2 lead to the following.

OPEN PROBLEM 2.3. *If S is a minimal surface in $\mathbb{R}^n$ with $\partial S = \bigcup_{i=1}^k C_i$ such that none of $C_1, \dots, C_k$ can be separated from the others by a hyperplane, then*

$$4\pi A \leq \sum_{i=1}^{k} L_i^2.$$

The inequality of Theorem 2.2 (b) is equivalent to

$$L^2 - 4\pi A \geq 2L_1 L_2.$$

Likewise, for an arbitrary doubly connected minimal surface in $\mathbb{R}^3$, Osserman and Schiffer [**OS**] (see p. 297) proved the following.

THEOREM 2.4. [**OS**] *For any doubly connected minimal surface in $\mathbb{R}^3$,*

$$L^2 - 4\pi A \geq 2L_1 L_2 (1 - \log 2). \tag{2–8}$$

Let $S \subset \mathbb{R}^n$ be a doubly connected minimal surface. S is parametrized by a conformal harmonic map ψ of an annulus $r_1 < |z| < r_2$ into $\mathbb{R}^n$, where ψ is assumed to extend continuously to the boundary circles and to map them onto Jordan curves C_1, C_2. Let $L(r)$ be the length of the image of the circle $|z| = r$, for $r_1 < r < r_2$. The only case of interest is that where C_1 and C_2 are rectifiable, of length say, L_1 and L_2. In that case, $\lim_{r \to r_i} L(r) = L_i$ for $i = 1, 2$. The key lemma for the case $n = 3$ is this:

PROPOSITION 2.5. [**OS**] *The function $L(r)$ satisfies*

$$\frac{d^2 L}{d(\log r)^2} \geq L \tag{2–9}$$

with equality possible in only two cases: if ψ is a conformal map onto a planar annulus, or if the image of ψ is a catenoid bounded by a pair of coaxial circles in parallel planes.

Using this equality together with (2–2) and the specific expressions for $L(r)$ on a catenoid, they prove Theorem 2.4.

The proof of Proposition 2.5 depends on the Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$, and does not go through for $n > 3$. But Feinberg [**Fe**] noted that a weaker form of (2–9) will yield a weaker form of (2–8). Specifically,

$$\frac{d^2 L}{d(\log r)^2} > \frac{2}{\pi^2} L. \tag{2–10}$$

He proved this by deriving an analog of the Wirtinger inequality (2–1) but "without the squares". Namely,

$$\int_0^{2\pi} y(t)dt = 0 \;\Rightarrow\; \int_0^{2\pi} |y'(t)|\, dt > \frac{2}{\pi} \int_0^{2\pi} |y(t)|\, dt.$$

Note that unlike (2–1), the inequality here is a strict one. However, the constant $2/\pi$ is best possible. The inequality $4\pi A \leq L^2$ for all doubly connected minimal surfaces in $\mathbb{R}^n$ follows from (2–10).

A final remark for doubly connected surfaces. We have learned from Carleman, Feinberg and Beckenbach–Radó that disk type minimal surfaces, doubly connected minimal surfaces and disk type nonpositively curved surfaces all satisfy the inequality $4\pi A \leq L^2$. However, as mentioned in the Introduction, doubly connected nonpositively curved surfaces do not satisfy the inequality because there are long cylinders with fixed perimeter whose area can be arbitrarily large. But is there any way of controlling the area of a cylinder with fixed perimeter? One way would be to fix the conformal structure of the cylinder. Consider all flat doubly connected surfaces which are conformally equivalent. It is not difficult to show that among them the flat annulus A with boundary curves of equal length satisfies the best isoperimetric inequality. Motivated by this, we propose the following.

OPEN PROBLEM 2.6. *Let S be a nonpositively curved annulus which is conformally equivalent to a flat annulus A with boundary curves of equal length. Show that*

$$\frac{\text{Area}(S)}{\text{Length}(\partial S)^2} \leq \frac{\text{Area}(\mathsf{A})}{\text{Length}(\partial \mathsf{A})^2}.$$

3. Li, Schoen and Yau's proof

In this section we briefly introduce Li–Schoen–Yau's proof for a minimal surface with two boundary components [**LSY**]. Their proof works also for a minimal surface with connected boundary.

First we rederive (2–2) for minimal S. Fix a point $p \in \mathbb{R}^n$ and define $r(x) = \operatorname{dist}(p, x), x = (x_1, \dots, x_n)$. Given an m-dimensional submanifold $N \subset \mathbb{R}^n$, it is well known that

$$\triangle x = (\triangle x_1, \dots, \triangle x_n) = \vec{H},$$

where $\triangle$ is the intrinsic Laplacian on N and $\vec{H}$ is the mean curvature vector of N. Hence the rectangular coordinate functions $x_1, \dots, x_n$ are harmonic on a minimal submanifold of $\mathbb{R}^n$. If we take p as the origin, then

$$\triangle r^2 = \sum \triangle x_i^2 = 2 \sum x_i \triangle x_i + 2 \sum |\nabla x_i|^2 = 2m \text{ on } N. \tag{3–1}$$

Integrating this over $N = S$ for $m = 2$ yields

$$4 \operatorname{Area}(S) = \int_S \triangle r^2 = \int_{\partial S} 2r \frac{\partial r}{\partial \nu}, \tag{3–2}$$

where ν is the outward unit conormal to ∂S. Translating S suitably, we may assume $\int_{\partial S} x_i = 0$. Then

$$\begin{aligned} 4 \operatorname{Area}(S) = \int_{\partial S} 2r \frac{\partial r}{\partial \nu} \leq 2 \int_{\partial S} r \leq 2 \operatorname{Length}(\partial S)^{1/2} \Big(\int_{\partial S} \sum x_i^2 \Big)^{1/2} \\ \leq \frac{1}{\pi} \operatorname{Length}(\partial S)^{3/2} \Big(\int_{\partial S} \sum \Big(\frac{dx_i}{ds} \Big)^2 \Big)^{1/2} = \frac{1}{\pi} \operatorname{Length}(\partial S)^2 \end{aligned}$$

(on the last line, the inequality follows by (2–1) or the Poincaré inequality, and the equality because $\sum (dx_i/ds)^2 = 1$). This gives $4\pi A \leq L^2$ when ∂S is connected.

In case ∂S is not connected, we cannot use the Poincaré inequality as above. However, note that $\int_{\partial S} \sum (dx_i/ds)^2$ is invariant under translations in $\mathbb{R}^n$ and that $\int_{\partial S} \sum x_i^2$ may or may not be invariant depending on the choice of translations. Note also that the disconnected boundary ∂S may be made into a connected curve provided a suitable translation is applied to each component of ∂S. With these observations Li, Schoen and Yau introduced the following.

DEFINITION 3.1. *A set* $\Gamma \subset \mathbb{R}^n$ *is* weakly connected *if there exists a rectangular coordinate system* $\{x_i\}_{i=1}^n$ *of* $\mathbb{R}^n$ *such that for every affine hypersurface* $H^{n-1} = \{x_i = constant\}$ *in* $\mathbb{R}^n$, *H does not separate* Γ.

Therefore a connected set is obviously weakly connected. The key idea of [**LSY**] is that if ∂S is weakly connected, then one can find a family of suitable translations which leave $\int_{\partial S} \sum x_i^2$ unchanged and make ∂S into a connected curve C. Consequently, the Poincaré inequality can be applied to C, and thereby yielding $4\pi A \leq L^2$ for S as above. In this way they proved

THEOREM 3.2. [**LSY**] *If* S *is a minimal surface in* $\mathbb{R}^n$ *with weakly connected boundary* ∂S, *then* S *satisfies* $4\pi A \leq L^2$.

And then applying the maximum principle to the family of homothetic contractions of a catenoid, they showed that if the boundary of a minimal surface S in $\mathbb{R}^3$ has two components and is not weakly connected, then S cannot be connected.

In this way they proved that a minimal surface in $\mathbb{R}^3$ with one or two boundary components satisfies $4\pi A \le L^2$.

4. The Cone Method

The author extended Li–Schoen–Yau's theorem by taking a more geometric point of view [**C1**]. In their proof they used the inequality $\frac{\partial r}{\partial \nu} \le 1$. But if η is the unit normal to ∂S which makes the smallest angle with ∇r, then

$$\frac{\partial r}{\partial \nu} \le \frac{\partial r}{\partial \eta} \le 1.$$

In fact η is the outward unit conormal to ∂S on the cone $p \mathbin{⨳} \partial S$, the union of the line segments from p to the points of ∂S. Although $p \mathbin{⨳} \partial S$ is not minimal, the identity $\triangle r^2 = 4$ holds there too because

$$\triangle r^2 = 4 + 2\langle x, \triangle x\rangle = 4 + 2\langle x, \vec{H}\rangle = 4,$$

where $\vec{H}$ is the mean curvature vector of $p \mathbin{⨳} \partial S$, which is perpendicular to x. Therefore

(4–1)
$$4\,\mathrm{Area}(S) = \int_S \triangle r^2 = \int_{\partial S} 2r\frac{\partial r}{\partial \nu} \le \int_{\partial S} 2r\frac{\partial r}{\partial \eta} = \int_{p \mathbin{⨳} \partial S} \triangle r^2 = 4\,\mathrm{Area}(p \mathbin{⨳} \partial S),$$

which gives an area comparison between S and $p \mathbin{⨳} \partial S$.

A nice thing about the cone $p \mathbin{⨳} \partial S$ is that $p \mathbin{⨳} \partial S$ is flat and hence is locally developable. If ∂S is connected, cut along a line segment l from p to a point of ∂S and then one can develop $p \mathbin{⨳} \partial S$ into a cone $O \mathbin{⨳} C$ on $\mathbb{R}^2$. $O \mathbin{⨳} C$ has the same area as $p \mathbin{⨳} \partial S$ and C has the same length as ∂S. C may not be a closed curve, but we can show that it is a curve with self-intersection if p is an interior point of S as follows. Choose a point $p \in S$ and show that (Lemma 4.5 below)

$$\triangle \log r \ge 2\pi\delta_p.$$

Then

$$2\pi \le \int_S \triangle \log r = \int_{\partial S} \frac{1}{r}\frac{\partial r}{\partial \nu} \le \int_{\partial S} \frac{1}{r}\frac{\partial r}{\partial \eta} = \mathrm{Angle}(\partial S, p), \qquad (4\text{–}2)$$

where $\mathrm{Angle}(\partial S, p)$ is the *angle* of ∂S viewed from p. In other words, $\mathrm{Angle}(\partial S, p)$ equals 2π times the density of $p \mathbin{⨳} \partial S$ at p. This angle estimate implies that ∂S rotates around p by at least $360°$ and consequently C should intersect itself. Then cutting $O \mathbin{⨳} C$ into two pieces and pasting them appropriately give rise to a domain $D \subset \mathbb{R}^2$ with

$$\mathrm{Area}(D) \ge \mathrm{Area}(O \mathbin{⨳} C) \ge \mathrm{Area}(S), \quad \mathrm{Length}(\partial D) = \mathrm{Length}(C) = \mathrm{Length}(\partial S)$$

(see [**C1**, Lemma 1] for the construction of D). Therefore the classical isoperimetric inequality for D gives rise to (1–1) for S.

So far ∂S has been assumed to be connected. However, even if ∂S is not connected, C as defined above may behave like a connected curve. This motivates the following.

Definition 4.1. *A set* $\Gamma \subset \mathbb{R}^n$ *is said to be* radially connected *from* $p \in \mathbb{R}^n$ *if* $\{r : r = \mathrm{dist}(p, q),\ q \in \Gamma\}$ *is a connected interval.*

If ∂S is radially connected from p, then we can apply to $p \divideontimes \partial S$ the argument of "cutting and inserting and pasting" to obtain a cone $O \divideontimes C \subset \mathbb{R}^2$ with C connected. Moreover, if p is in S, then C has a self-intersection and so we can obtain the domain $D \subset \mathbb{R}^2$ as above and hence the desired isoperimetric inequality for S. See [**C1**, Theorem 1] for more details. Thus we have outlined the proof of the following.

THEOREM 4.2. [**C1**] *If S is a minimal surface whose boundary is radially connected from a point of the surface, then it satisfies $4\pi A \leq L^2$.*

Although there is no relationship between radial connectivity and weak connectivity, we have a stronger corollary than Li–Schoen–Yau's: If S is in $\mathbb{R}^n$ such that ∂S has two components then ∂S is radially connected from a point in S which is a midpoint between the two boundary components, and hence (1–1) holds for such S.

The volume comparison as in (4–1) holds for higher dimensional minimal submanifolds as well with the same proof:

PROPOSITION 4.3. [**C1**]. *If $N \subset \mathbb{R}^n$ is an m-dimensional minimal submanifold and p is a point in $\mathbb{R}^n$, then*

$$\mathrm{Vol}(N) \leq \mathrm{Vol}(p \divideontimes \partial N).$$

Similarly, the angle estimate (4–2) holds in higher dimension also. To show this we need to define the geometric quantity *angle*:

DEFINITION 4.4. [**C1**]. *Let $M \subset \mathbb{R}^n$ be a k-dimensional rectifiable set and p a point in $\mathbb{R}^n$. We define the* k-dimensional angle *of M from p,* Angle(M, p) *to be the k-dimensional mass of $(p \divideontimes M)^\infty \cap \partial B_p^n(1)$ counting multiplicity, where $(p \divideontimes M)^\infty$ is the infinite cone obtained by indefinitely extending $p \divideontimes M$ across M and $\partial B_p^n(1)$ is the unit sphere with center at p in $\mathbb{R}^n$.*

Note that

$$\mathrm{Angle}(M, p) = (k+1)\omega_{k+1}\Theta^{k+1}(p \divideontimes M, p),$$

where ω_{k+1} is the volume of a unit ball in $\mathbb{R}^{k+1}$ and $\Theta^{k+1}(p \divideontimes M, p)$ denotes the $(k+1)$-dimensional density of $p \divideontimes M$ at p. Using this, we can also define the angle of a set in a Riemannian manifold.

LEMMA 4.5. [**C1**]. (a) *$\log r$ is subharmonic on a minimal surface $N^2 \subset \mathbb{R}^n$. More precisely, $\frac{1}{2\pi}\triangle \log r \geq \delta_p$, the Dirac delta function with singularity at p.*
(b) *On an m-dimensional minimal submanifold $N \subset \mathbb{R}^n$, $m \geq 3$, $\triangle r^{2-m} \leq m\,\omega_m \delta_p$.*

PROOF. Use $\triangle r^2 = 2n$, $|\nabla r| \leq 1$ on N. And recall that in $\mathbb{R}^m$ $\triangle r^{2-m} = m\,\omega_m \delta_O$ and that near p N approximates to $\mathbb{R}^m$. □

Integrating these Laplacians and using the facts $\frac{\partial r}{\partial \nu} \leq \frac{\partial r}{\partial \eta}$ and $\int_{\partial N} r^{1-m}\frac{\partial r}{\partial \eta} =$ Angle$(\partial N, p)$, we get the following angle estimate.

PROPOSITION 4.6. [**C1**]. *If p is an interior point of an m-dimensional minimal submanifold $N \subset \mathbb{R}^n$, then*

$$\mathrm{Angle}(\partial N, p) \geq m\,\omega_m$$

and equality holds if and only if N is totally geodesic and star-shaped with respect to p.

5. Hyperbolic Case

The author and Gulliver [**CG1**], [**CG2**] investigated the possibility of extending the cone method to a minimal surface S in S^n and H^n. We showed that the area comparison $\text{Area}(S) \leq \text{Area}(p \divideontimes \partial S)$ does hold for $S \subset H^n$ but not for $S \subset S^n$ (see [**CG1**, Proposition 2, Remark 1]), whereas the angle estimate $\text{Angle}(\partial S, p) \geq 2\pi$ holds for $S \subset S^n$ as well as for $S \subset H^n$ ([**CG2**, Proposition 2]). As a result we proved the isoperimetric inequality

$$4\pi A \leq L^2 - A^2$$

for a minimal surface $S \subset H^n$ whose boundary is radially connected from a point of S.

Just as $p \divideontimes \partial S \subset \mathbb{R}^n$ is flat away from p, $p \divideontimes \partial S \subset H^n$ has constant Gaussian curvature -1 away from p. Therefore we can also develop $p \divideontimes \partial S \subset H^n$ on H^2 as we did $p \divideontimes \partial S \subset \mathbb{R}^n$ on $\mathbb{R}^2$. In H^n, however, we shall take a different approach: apply Bol's isoperimetric inequality. Bol showed that a simply connected domain D on a surface satisfies the inequality

$$4\pi A \leq L^2 + (\sup_D K) A^2,$$

where K is the Gaussian curvature of the surface [**Bo**]. In this section Bol's isoperimetric inequality will be used on a smooth surface which is very close to $p \divideontimes \partial S$.

Given a k-dimensional submanifold N in a Riemannian manifold M, one can define the *extrinsic Laplacian* $\overline{\triangle}$ and *intrinsic Laplacian* $\triangle$ on N as follows. Let $e_1, \ldots, e_k$ be orthonormal vector fields on a domain of N. Define

$$\overline{\triangle} f = \sum_i \overline{\nabla}^2 f(e_i, e_i) \qquad \text{and} \qquad \triangle f = \sum_i \nabla^2 f(e_i, e_i),$$

where $\overline{\nabla}$ and ∇ are the Riemannian connections of M and N, respectively. Then one can easily prove the following.

LEMMA 5.1. *$\overline{\triangle} f = \triangle f - \vec{H} f$, where $\vec{H}$ is the mean curvature vector of N.*

By this lemma the intrinsic Laplacian on a minimal submanifold or a cone can be replaced with the extrinsic Laplacian which is easier to compute.

Let $G(r)$ be a Green's function of S^k (or H^k), whose gradient is $\sin^{1-k} r \nabla r$ (or $\sinh r^{1-k} r \nabla r$, respectively). For instance,

$$G(r) = \log \frac{\sin r}{1 + \cos r} \;\text{ on } S^2 \quad \text{and} \quad G(r) = \log \frac{\sinh r}{1 + \cosh r} \;\text{ on } H^2.$$

Note that $\triangle G(r) = k\omega_k \delta_p$. Assuming $S^k \subset S^n$ and $H^k \subset H^n$, one can extend G to a rotationally symmetric function $G(r)$ on S^n and H^n. Then by Lemma 5.1 we have

LEMMA 5.2. [**CG2**] *$\triangle G(r) \geq k\omega_k \delta_p$ on minimal $N \subset S^n$ or H^n, and $\triangle G = k\omega_k \Theta \delta_p$ on $p \divideontimes \partial N$, where Θ is the density of $p \divideontimes \partial N$ at p.*

DEFINITION 5.3. *Let $C \subset S^n$ be a k-dimensional rectifiable set and p a point in S^n. The k-dimensional* angle $\text{Angle}(C, p)$ *of C viewed from p is defined by*

$$\text{Angle}(C, p) = \frac{\text{Vol}[(p \divideontimes C) \cap \partial B_p^n(t)]}{\sin^k t},$$

where $\partial B_p^n(t)$ is the geodesic sphere of radius $t < \operatorname{dist}(p, C)$ centered at p, and the volume is measured counting multiplicity. Clearly the angle does not depend on t. There is obviously an analogous definition for the angle of $C \subset H^n$ viewed from $p \in H^n$.

Note that

$$\operatorname{Angle}(C, p) = (k+1)\omega_{k+1}\Theta^{k+1}(p⋇C, p),$$

where $\Theta^{k+1}(p⋇C, p)$ is the $(k+1)$-dimensional density of $p⋇C$ at p. Now we integrate the Laplacians of Lemma 5.2 on N and on $p⋇\partial N$. Using $\partial r/\partial \nu \leq \partial r/\partial \eta$ as in the previous section, we can prove the following.

Proposition 5.4. [**CG2**]. *Let N be a k-dimensional minimal submanifold in S^n or H^n and p an interior point of N. In case of $N \subset S^n$, we assume that $\operatorname{dist}(p, q) \leq \pi/2$ for all $q \in N$. Then*

$$\operatorname{Angle}(\partial N, p) \geq k\omega_k.$$

Equality holds if and only if N is totally geodesic and star-shaped with respect to p.

Now let $\alpha(r)$ be the volume of the geodesic ball of radius r in H^k and let $f(r)$ be a function on H^k such that $\nabla f = \frac{\alpha}{\alpha'}\nabla r$. Then $f(r) = \log(1 + \cosh r)$ for $k = 2$. If we extend f to a rotationally symmetric function $f(r)$ on $H^n \supset H^k$, then by Lemma 5.1 we have

Lemma 5.5. [**CG1**] *$\triangle f \geq 1$ on minimal $N^k \subset H^n$ and $\triangle f = 1$ on H^k and a k-dimensional cone in H^n.*

Here the vertex of the cone should be the point p from which the distance r is measured. Therefore integrating these Laplacians, we get the following.

Proposition 5.6. [**CG1**] *Let N be a minimal submanifold of H^n and p any point in H^n. Then*

$$\operatorname{Vol}(N) \leq \operatorname{Vol}(p⋇\partial N);$$

equality holds if and only if $p \in N$, and N must be totally geodesic and star-shaped with respect to p.

If ∂S is radially connected from p, then after applying the cutting and pasting arguments one can think of $p⋇\partial S$ as a cone with connected boundary. Now Proposition 5.4 implies that for $k = 2$ and $N = S$, the Gaussian curvature of $p⋇\partial S$ at p is either $-\infty$ if $\operatorname{Angle}(\partial S, p) > 2\pi$, or -1 if $\operatorname{Angle}(\partial S, p) = 2\pi$. In the case of $-\infty$ one can slightly perturb $p⋇\partial S$ near p to a smooth surface Σ of Gaussian curvature ≤ -1; see [**CG1**, Lemma 3]. Then Bol's isoperimetric inequality for Σ implies

$$4\pi \operatorname{Area}(p⋇\partial S) \leq \operatorname{Length}(\partial S)^2 - \operatorname{Area}(p⋇\partial S)^2$$

(see [**CG1**, Lemma 4] for details). Hence using Proposition 5.6 and the monotonicity of the quadratic function $4\pi A + A^2$ for positive area A, we can prove the following.

Theorem 5.7. [**CG1**] *If S is a minimal surface in hyperbolic space H^n whose boundary is radially connected from a point of the surface, then S satisfies $4\pi A \leq L^2 - A^2$, with equality if and only if S is a geodesic disk in a totally geodesic $H^2 \subset H^n$.*

6. Variable Curvature Case

So far we have considered minimal surfaces in a Riemannian manifold $\overline{M}$ of constant sectional curvature K. In this section we extend the results of the preceding sections to minimal surfaces in a manifold M of variable curvature $\leq K$. The main obstacle to this extension is that one cannot prove the area comparison

$$\text{Area}(S) \leq \text{Area}(p \divideontimes \partial S)$$

for a minimal surface $S \subset M$. But we will get around this difficulty by comparing the area of S with that of a cone in $\overline{M}$.

When we study a Riemannian manifold of variable curvature the comparison theorems are very useful tools. Among them, the one that we need most for our purposes is the Hessian comparison theorem for the distances in M and in $\overline{M}$. Let $r(x)$ be the distance from a fixed point p to x in M and denote the Hessian of r by $\nabla^2 r$. Assume that γ is a geodesic from p to q and v is a vector at q perpendicular to γ. Then $\nabla^2 r(v,v)$ is the second variation of the length of γ associated with the Jacobi field X along γ satisfying $X(p) = 0$ and $X(q) = v$. The Jacobi field minimizes the second variation among all vector fields along γ with the same boundary conditions. Therefore if the sectional curvature of M^n is bounded from above by that of a space form $\overline{M}^n$ which has a distance function $\bar{r}$ with $\bar{r}(\cdot) = \text{dist}(\bar{p}, \cdot)$ and the Riemannian connection denoted by $\overline{\nabla}$, then one gets the Hessian comparison

$$\nabla^2 r(v,v) \geq \overline{\nabla}^2 \bar{r}(u,u), \tag{6–1}$$

where u is a vector at $\bar{q} \in \overline{M}$ with $|u| = |v|$ and $\bar{r}(\bar{q}) = r(q)$, which is perpendicular to the geodesic $\bar{\gamma} \subset \overline{M}$ from $\bar{p}$ to $\bar{q}$ (see **[SY1]** p. 4).

From this comparison one can obtain the following lemmas on the Laplacian of some functions of distance.

LEMMA 6.1. **[C4]** *Let S be a minimal surface in a simply connected Riemannian manifold M of sectional curvature bounded above by a constant K. Define $r(x) =$ $\text{dist}(p,x)$ for fixed $p \in M$.*

(I) *If $K = 0$, we have on S:*
  (a) $\triangle r^2 \geq 4$;
  (b) $\triangle r \geq \dfrac{1}{r}(2 - |\nabla r|^2)$;
  (c) $\triangle \log r \geq 2\pi\delta_p$ *if* $p \in S$.

(II) *If $K = -k^2 < 0$, then*
  (d) $\triangle r \geq k(2 - |\nabla r|^2)\coth kr$;
  (e) $\triangle \log(1 + \cosh kr) \geq -K$;
  (f) $\triangle \log \dfrac{\sinh kr}{1 + \cosh kr} \geq 2\pi\delta_p$ *if* $p \in S$;
  (g) $\triangle \log \sinh kr \geq 2\pi\delta_p - K$ *if* $p \in S$.

(III) *If $K = k^2 > 0$, then*
  (h) $\triangle r \geq k(2 - |\nabla r|^2)\cot kr$;

(i) $\triangle \log \sin kr \geq 2\pi\delta_p - K$ *if* $p \in S$ *and* $r \leq \dfrac{\pi}{2k}$;

(j) $\triangle \log \dfrac{\sin kr}{1+\cos kr} \geq 2\pi\delta_p$ *if* $p \in S$ *and* $r \leq \dfrac{\pi}{2k}$.

PROOF. Denote the metrics of $\overline{M}$ and M by $\bar{g}$ and g, respectively. Assume first that $K = 0$ and $\overline{M} = \mathbb{R}^n$. A straightforward computation in orthonormal coordinates of $\mathbb{R}^n$ gives

$$\overline{\nabla}^2 \bar{r}^2 = 2\bar{g}. \tag{6–2}$$

Since

$$\nabla^2 r^2 = 2r\nabla^2 r + 2\nabla r \otimes \nabla r \quad \text{and} \quad \overline{\nabla}^2 \bar{r}^2 = 2\bar{r}\overline{\nabla}^2 \bar{r} + 2\overline{\nabla}\bar{r} \otimes \overline{\nabla}\bar{r},$$

(6–1) and (6–2) imply

$$\nabla^2 r^2 \geq 2g.$$

This and Lemma 5.1 imply (a). For (b) we compute

$$\triangle r = \operatorname{div}\nabla(r^2)^{1/2} = \operatorname{div}\frac{1}{2r}\nabla r^2 = \frac{1}{2r}\triangle r^2 - \frac{1}{2r^2}\langle \nabla r,\ 2r\nabla r\rangle \geq \frac{1}{r}(2 - |\nabla r|^2).$$

Similarly for (c)

$$\triangle \log r = \operatorname{div}\frac{1}{r}\nabla r = \frac{1}{r}\triangle r - \frac{1}{r^2}|\nabla r|^2 \geq \frac{2}{r^2}(1 - |\nabla r|^2) \geq 0.$$

Near p, however, S can be identified with T_pS, the tangent plane of S at p, on which $\triangle \log r = 2\pi\delta_p$ with respect to the Euclidean metric. Therefore on S, $\triangle \log r \geq 2\pi\delta_p$.

Assume now that $K = -k^2 < 0$. For any circle $C \subset \overline{M}$ of radius a with center at $\bar{p}$, the length of C equals $l(a) = (2\pi/k)\sinh ka$. So the geodesic curvature of C is $l'(a)/l(a) = k \coth ka$. Hence the principal curvature of the geodesic sphere $\Sigma \subset \overline{M}$ of radius a with center at $\bar{p}$ is $k\coth ka$ everywhere on Σ in any direction. Note here that both the tangent space and the normal line to Σ are the eigenspaces of the Hessian of $\bar{r}$, $\overline{\nabla}^2\bar{r}$. Therefore one can easily see that

$$\overline{\nabla}^2 \cosh k\bar{r} = (k^2 \cosh k\bar{r})\bar{g}, \tag{6–3}$$

and hence

$$\overline{\nabla}^2 \bar{r} = k \coth k\bar{r}(\bar{g} - \overline{\nabla}\bar{r} \otimes \overline{\nabla}\bar{r}). \tag{6–4}$$

Thus Lemma 5.1, (6–1), (6–4) and the fact that $\nabla r (= \operatorname{grad} r)$ is an eigenvector of $\nabla^2 r$ with eigenvalue zero prove (d). Then

$$\begin{aligned}\triangle \log(1+\cosh kr) &= \operatorname{div}\frac{k\sinh kr}{1+\cosh kr}\nabla r = \frac{k^2}{1+\cosh kr}|\nabla r|^2 + \frac{k\sinh kr}{1+\cosh kr}\triangle r\\ &\geq \frac{2k^2\cosh kr + k^2|\nabla r|^2(1-\cosh kr)}{1+\cosh kr} \geq k^2,\end{aligned}$$

which gives (e). Now we have

$$\begin{aligned}\triangle \log \frac{\sinh kr}{1+\cosh kr} &= \operatorname{div}\frac{k}{\sinh kr}\nabla r = -\frac{k^2\cosh kr}{\sinh^2 kr}|\nabla r|^2 + \frac{k}{\sinh kr}\triangle r\\ &\geq \frac{2k^2\cosh kr(1-|\nabla r|^2)}{\sinh^2 kr} \geq 0.\end{aligned}$$

However,

$$f(r) = \frac{1}{2\pi} \log \frac{\sinh kr}{1 + \cosh kr}$$

is a fundamental solution of $\triangle$ on $H^2(K)$ since

$$\frac{1}{2\pi f'(r)} = \frac{1}{k} \sinh kr$$

is the length of a Jacobi field. So (f) follows. Adding (e) to (f) gives (g).

Assume finally that $K = k^2 > 0$. As in the case of $K < 0$ above, one can show that in $\overline{M}^n$ of constant sectional curvature K

$$\overline{\nabla}^2 \cos k\bar{r} = -(k^2 \cos k\bar{r})\bar{g}$$

and

$$\overline{\nabla}^2 \bar{r} = k \cot k\bar{r}(\bar{g} - \overline{\nabla}\bar{r} \otimes \overline{\nabla}\bar{r}),$$

from which (h) follows. And then

$$\begin{aligned} \triangle \log \frac{\sin kr}{1 + \cos kr} &= \operatorname{div} \frac{k}{\sin kr} \nabla r = -\frac{k^2 \cos kr}{\sin^2 kr} |\nabla r|^2 + \frac{k}{\sin kr} \triangle r \\ &\geq \frac{2k^2 \cos kr}{\sin^2 kr} (1 - |\nabla r|^2) \geq 0. \end{aligned}$$

As in (f),

$$f(r) = \frac{1}{2\pi} \log \frac{\sin kr}{1 + \cos kr}$$

is a fundamental solution of $\triangle$ on $S^2(K)$ since

$$\frac{1}{2\pi f'(r)} = \frac{1}{k} \sin kr$$

is the length of a Jacobi field. Thus (j) follows. For (i) we compute

$$\begin{aligned} \triangle \log \sin kr = \operatorname{div} \left(\frac{k \cos kr}{\sin kr} \nabla r \right) &= -k^2 \csc^2 kr |\nabla r|^2 + k \cot kr \triangle r \\ &\geq k^2 \csc^2 kr [2 \cos^2 kr - (1 + \cos^2 kr)|\nabla r|^2] \geq -k^2. \end{aligned}$$

Note that

$$\lim_{r \to 0} \frac{\dfrac{d}{dr} \log \sin kr}{\dfrac{d}{dr} \log \dfrac{\sin kr}{1 + \cos kr}} = 1,$$

which proves (i). $\square$

LEMMA 6.2. **[C4]** *Let* $\Gamma = \bar{p} \divideontimes \bar{C}$ *be the cone from* $\bar{p}$ *over a curve* $\bar{C}$ *in a Riemannian manifold* $\overline{M}$ *of nonpositive constant sectional curvature* $K = -k^2$ *and define* $\bar{r}(x) = \operatorname{dist}(\bar{p}, x), \bar{p} \in \overline{M}$. *Then, on* Γ,

(a) $\triangle \bar{r}^2 = 4$ *if* $K = 0$, *while* $\triangle \log(1 + \cosh k\bar{r}) = -K$ *if* $K < 0$.

(b) $\triangle \log \bar{r} = \alpha \delta_{\bar{p}}$ *if* $K = 0$, *while* $\triangle \log \dfrac{\sinh k\bar{r}}{1 + \cosh k\bar{r}} = \alpha \delta_{\bar{p}}$ *if* $K < 0$, *where* $\alpha = \operatorname{Angle}(\bar{C}, \bar{p})$.

Proof. On Γ $\nabla\bar{r}$ is perpendicular to $\vec{H}$, the mean curvature vector of Γ; hence Lemma 5.1 implies that for any function f of distance $\bar{r}$, $\overline{\triangle} f = \triangle f$. Moreover $|\nabla\bar{r}| \equiv 1$ on Γ. It follows from (6–2) and (6–3) that all the inequalities in Lemma 6.1(a), (c), (e) and (f) become equalities. This proves the lemma except for the constant α. The constant 2π that appears in the Laplacian of the fundamental solution on $\mathbb{R}^2$ and H^2 comes from the limit as $a \to 0$ of the circumference of the circle of radius a with center at $\bar{p}$ divided by a. Similarly, if $\Sigma_{\bar{p}}(a)$ denotes the geodesic sphere of radius a with center at $\bar{p}$, α equals

$$\lim_{a\to 0} \frac{1}{a} \operatorname{Length}(\Gamma \cap \Sigma_{\bar{p}}(a)),$$

which is called the *angle* of $\bar{C}$ viewed from $\bar{p}$ and denoted $\operatorname{Angle}(\bar{C}, \bar{p})$. □

Now we have the main theorem as follows.

Theorem 6.3. **[C4]** *Let S be a minimal surface in a complete simply connected Riemannian manifold M with sectional curvature bounded above by a nonpositive constant K. If ∂S is radially connected from a point of S, then S satisfies the isoperimetric inequality*

$$4\pi A \le L^2 + KA^2, \tag{6–5}$$

where equality holds if and only if S is a geodesic disk in a surface of constant Gaussian curvature K.

Proof. First suppose $K < 0$. Integrate Lemma 6.1(e) to get

$$\begin{aligned} -K\operatorname{Area}(S) &\le \int_S \triangle \log(1+\cosh kr) = \int_{\partial S} \frac{k\sinh kr}{1+\cosh kr}\frac{\partial r}{\partial \nu} \\ &\le \int_{\partial S} \frac{k\sinh kr}{1+\cosh kr}\frac{\partial r}{\partial \eta} = \int_{\partial S} \frac{k\sinh kr}{1+\cosh kr}\sqrt{1-\langle \nabla r, \tau\rangle^2}, \end{aligned}$$

where ν, η are as in the preceding sections and τ is a unit tangent to ∂S.

Now the key step in the extension to the variable curvature case is to carry the last integral above over to the simply connected space form $\overline{M}$ of sectional curvature K. Let $C_1, \dots, C_l$ be the components of ∂S. Choose $q_i \in C_i$ for each $i = 1, \dots, l$, and take $\bar{q}_1, \dots, \bar{q}_l \in \overline{M}$ in such a way that $r(q_i) = \bar{r}(\bar{q}_i)$. Suppose that each curve C_i is parametrized by $c_i(s)$ with arclength parameter s such that $q_i = c_i(0) = c_i(\lambda_i)$, $\lambda_i = \operatorname{Length}(C_i)$. Then we construct a curve $\bar{C}_i$ in $\overline{M}$ starting from $\bar{q}_i$ and parametrized by $\bar{c}_i(s)$ with arclength parameter $s \in [0, \lambda_i]$ and $\bar{c}_i(0) = \bar{q}_i$ such that the unit tangent vector $\bar{c}_i'(s)$ makes an angle of $\cos^{-1}\langle \nabla r, c_i'(s)\rangle$ with $\nabla\bar{r}$. Of course the curve $\bar{C}_i$ is not unique; but given a two-dimensional infinite cone $\bar{p} \divideontimes \bar{C}$ containing $\bar{q}_i$, one can uniquely determine a curve $\bar{C}_i$ on $\bar{p} \divideontimes \bar{C}$ with the prescribed properties. Since $\bar{p} \divideontimes \bar{C}$ is developable, one can also assume without loss of generality that $\bar{c}_i(0) = \bar{c}_i(\lambda_i)$, or equivalently, that $\bar{C}_i$ is closed. Anyhow, r on C_i coincides with $\bar{r}$ on $\bar{C}_i$ in the sense that

$$r(c_i(s)) = \bar{r}(\bar{c}_i(s)) \quad \text{and} \quad \langle \nabla r, c_i'(s)\rangle = \langle \nabla \bar{r}, \bar{c}_i'(s)\rangle.$$

Hence

$$-K\,\mathrm{Area}(S) \le \sum_{i=1}^{l} \int_{C_i} \frac{k\sinh kr}{1+\cosh kr}\sqrt{1-\langle \nabla r, c_i'(s)\rangle^2}$$
$$= \sum_{i=1}^{l} \int_{\bar{C}_i} \frac{k\sinh k\bar{r}}{1+\cosh k\bar{r}}\sqrt{1-\langle \nabla \bar{r}, \bar{c}_i'(s)\rangle^2}.$$

If $\bar{\eta}$ is the outward unit conormal to $\bar{C}_i$ on $\bar{p}⋇\bar{C}_i$, then

$$\mathrm{Area}(S) \le -\frac{1}{K}\sum_{i=1}^{l}\int_{\bar{C}_i} \frac{k\sinh k\bar{r}}{1+\cosh k\bar{r}}\frac{\partial \bar{r}}{\partial \bar{\eta}} = \sum_{i=1}^{l}\int_{\bar{p}⋇\bar{C}_i} \frac{1}{-K}\triangle \log(1+\cosh k\bar{r})$$
$$= \sum_{i=1}^{l} \mathrm{Area}(\bar{p}⋇\bar{C}_i) \quad (\text{by Lemma 6.2(a)}) = \mathrm{Area}(\bar{p}⋇\bar{C}),$$

with $\bar{C} = \bigcup_{i=1}^{l} \bar{C}_i$. Also it follows from the definition of $\bar{C}_i$ that

$$\mathrm{Length}(\partial S) = \mathrm{Length}(\bar{C}).$$

On the other hand, integrating Lemma 6.1(f) over S and Lemma 6.2(b) over $\bar{p}⋇\bar{C}$ as above, we get

$$2\pi \le \int_S \triangle \log\frac{\sinh kr}{1+\cosh kr} = \int_{\partial S}\frac{k}{\sinh kr}\frac{\partial r}{\partial \nu} \le \int_{\partial S}\frac{k}{\sinh kr}\frac{\partial r}{\partial \eta}$$
$$= \int_{\bar{C}}\frac{k}{\sinh k\bar{r}}\frac{\partial \bar{r}}{\partial \bar{\eta}} = \int_{\bar{p}⋇\bar{C}} \triangle \log\frac{\sinh k\bar{r}}{1+\cosh k\bar{r}} = \mathrm{Angle}(\bar{C}, \bar{p}).$$

Moreover, since $r|_{\partial S}$ coincides with $\bar{r}|_{\bar{C}}$, $\bar{C}$ is also radially connected from $\bar{p}$. Hence from the cutting and pasting arguments and the approximation argument as in [**CG1**, Lemma 4] it follows that

$$4\pi\mathrm{Area}(\bar{p}⋇\bar{C}) \le \mathrm{Length}(\bar{C})^2 + K\mathrm{Area}(\bar{p}⋇\bar{C})^2.$$

Therefore using the area comparison obtained above and the monotonicity of the quadratic function $4\pi A - KA^2$ of $A > 0$, we obtain the desired isoperimetric inequality for S in case $K < 0$.

If equality holds in the isoperimetric inequality, then

$$\mathrm{Area}(S) = \mathrm{Area}(\bar{p}⋇\bar{C})$$

and therefore equality should hold in Lemma 6.1(e). Consequently equality holds in (6–1) and $|\nabla r| \equiv 1$ on S as we easily see in the proof of Lemma 6.1(e). It follows that $S = p⋇\partial S$ and, by Index Lemma, S is constantly curved and hence totally geodesic. Thus Schmidt's theorem [**Sm**] completes the proof in case $K < 0$.

Second, suppose $K = 0$. Lemma 6.1(a) and Lemma 6.2(a) imply

$$\mathrm{Area}(S) \le \mathrm{Area}(\bar{p}⋇\bar{C}),$$

and Lemma 6.1(c) and Lemma 6.2(b) imply

$$2\pi \le \mathrm{Angle}(\bar{C}, \bar{p}).$$

Thus the theorem follows from the arguments of Section 4. □

In showing $\text{Area}(S) \leq \text{Area}(\bar{p} \divideontimes \bar{C})$ in the proof of Theorem 6.3, the main idea is to construct the cone $\bar{p} \divideontimes \bar{C}$ in $\overline{M}$ such that

$$\int_{\partial S} \frac{k \sinh kr}{1+\cosh kr} \frac{\partial r}{\partial \eta} = \int_{\bar{C}} \frac{k \sinh k\bar{r}}{1+\cosh k\bar{r}} \frac{\partial \bar{r}}{\partial \bar{\eta}},$$

in other words,

$$r|_{\partial S} = \bar{r}|_{\bar{C}} \text{ and } (\partial r/\partial \eta)|_{\partial S} = (\partial \bar{r}/\partial \bar{\eta})|_{\bar{C}}.$$

It is interesting to remark that this idea can be interpreted as *giving a constant curvature metric* $\hat{g}$ to $p \divideontimes \partial S$ that preserves r, the length of ∂S and the angle between ∇r and ∂S. This new metric $\hat{g}$ plays a key role in the proof of the embeddedness of a minimal surface in a Riemannian manifold in [**CG3**].

7. Varifolds and Flat Chains

Some minimal surfaces in $\mathbb{R}^3$, like compound soap films, contain singular curves. They are not smooth but smooth almost everywhere. In some literature they are called stationary varifolds or area minimizing currents. Here it is interesting to ask whether the isoperimetric inequality $4\pi A \leq L^2$ holds also for these surfaces with singularities. In [**C2**] the author gave an affirmative answer; moreover he derived a new type of optimal isoperimetric inequality for certain types of soap films with singularities.

First we state sharp isoperimetric inequalities for domains in the plane where only a specific part of the boundary counts toward the length.

LEMMA 7.1. [**C2**] *Let l_1 and l_2 be the rays emanating from a point O with an angle of $\theta \leq \pi$. Let C be a curve from a point of l_1 to a point of l_2.*

(a) *Suppose that C lies in the smaller sector of the two formed by the rays (C may lie in either sector if $\theta = \pi$). Define D as the domain bounded by l_1, l_2, and C. Then*

$$2\theta \,\text{Area}(D) \leq \text{Length}(C)^2,$$

and equality holds if and only if C is a circular arc perpendicular to the rays.

(b) *If C lies in the larger sector, then*

$$2\pi \,\text{Area}(D) \leq \text{Length}(C)^2,$$

where equality holds if and only if C is a semicircle perpendicular to one of the two rays.

DEFINITION 7.2. *A* compound Jordan curve *is a one-dimensional rectifiable connected set in $\mathbb{R}^n$ which is the union of finitely many Jordan curves (= homeomorphic images of a circle).*

LEMMA 7.3. [**C2**] *If C is a compound Jordan curve in $\mathbb{R}^n$ and p a point of C, then*

$$4\pi \,\text{Area}(p \divideontimes C) \leq \text{Length}(C)^2.$$

Equality holds if and only if $p \divideontimes C$ can be developed, by cutting and inserting, one-to-one onto a disk.

DEFINITION 7.4. (a) *Suppose V is an m-dimensional varifold of locally bounded first variation in $\mathbb{R}^n$, and that Z is the generalized boundary of V with the generalized boundary measure σ. Assume Z is $(m-1)$-rectifiable. Let*

$$\psi(x) = \lim_{\rho\to 0} \frac{\sigma(B_x(\rho))}{\mathsf{H}^{m-1}(Z \cap B_x(\rho))}, \quad x \in Z.$$

Then define the varifold boundary ∂V *of V to be the varifold $\underline{\underline{v}}(Z,\psi)$. In other words, ∂V is the $(m-1)$-dimensional rectifiable varifold with support Z and multiplicity ψ. Clearly $\mu_{\partial V} = \sigma$.*

(b) *For an m-varifold $V = \underline{\underline{v}}(M,\theta)$, the* varifold cone $p⨳V$ *from p over V is the $(m+1)$-varifold $\underline{\underline{v}}(p⨳M,\bar{\theta})$, where $\bar{\theta}(y) = \theta(x)$ whenever y lies on the line segment from p to $x \in M$.*

PROPOSITION 7.5. **[C2]** *Let V be an m-varifold of locally bounded first variation in $\mathbb{R}^n$. If the generalized boundary Z of V is rectifiable and V is stationary in $\mathbb{R}^n \sim Z$, then for any $p \in \mathbb{R}^n$*

$$\mathbf{M}(V) \le \mathbf{M}(p⨳\partial V).$$

PROOF. By the first variation formula for the mass of V with the variation field Y,

$$\delta V(Y) = \int_Z \langle \nu, Y\rangle \, d\sigma.$$

Take Y to be the radial vector field defined by $Y(x) = x-p$. Then Y is the initial velocity vector field of the 1-parameter family of homothetic expansions $\{\phi_t\}$ given by $\phi_t(x) = (1+t)(x-p)+p$. Hence

$$\mathbf{M}(\phi_{t\sharp}V) = (1+t)^m \mathbf{M}(V),$$

and so

$$\delta V(Y) = \frac{d}{dt}(1+t)^m \mathbf{M}(V)|_{t=0} = m\mathbf{M}(V).$$

On the other hand, since Z is rectifiable, Z has tangent spaces almost everywhere and ν is normal to Z. Let $\eta(x)$ be a unit vector which is perpendicular to Z at $x \in Z$ and lies in the subspace of $\mathbb{R}^n$ spanned by $Y(x) = x-p$ and the tangent space to Z at x. Taking the negative of η if necessary, one may assume $\langle \eta, Y\rangle \ge 0$. It is not difficult to see that

$$\langle \nu, Y\rangle \le \langle \eta, Y\rangle.$$

Let $r(x) = |Y(x)|$. Then dr is the 1-form dual to the unit radial vector field $Y/|Y|$. Hence

$$m\,\mathbf{M}(V) = \int_Z \langle \nu, Y\rangle \, d\sigma \le \int_Z \langle \eta, Y\rangle \, d\sigma = \int_Z Y \lrcorner \, (dr \wedge d\sigma) = m\,\mathbf{M}(p⨳\partial V),$$

where $\lrcorner$ denotes the interior multiplication. □

LEMMA 7.6. **[C2]** *Let $W = \underline{\underline{v}}(Z,\psi)$ be a rectifiable 1-varifold in $\mathbb{R}^n$ with $\psi \ge 1$ and let p be a point in Z. If Z is a compound Jordan curve, then*

$$4\pi\mathbf{M}(p⨳W) \le \mathbf{M}(W)^2.$$

THEOREM 7.7. **[C2]** *Suppose that V is a 2-varifold of locally bounded first variation in $\mathbb{R}^n$, the generalized boundary Z of V is rectifiable, and V is stationary in $\mathbb{R}^n \sim Z$. If the multiplicity of ∂V is ≥ 1 and Z is a compound Jordan curve, then*

$$4\pi\,\mathbf{M}(V) \leq \mathbf{M}(\partial V)^2.$$

PROOF. Use Proposition 7.5 and Lemma 7.6. □

The inequality of Theorem 7.7 resembles the classical isoperimetric inequality. But here we shall see a new optimal isoperimetric inequality for some specific soap films as follows.

DEFINITION 7.8. *Let $Y^k \subset \mathbb{R}^3$ be the union of k great semicircles on a sphere meeting at the north and south poles at equal angles of $2\pi/k$. Define $\mathsf{Y}_2^k \subset \mathsf{I}_2^k$ to be the set of 2-dimensional flat chains T mod k in $\mathbb{R}^n$ with multiplicity 1 almost everywhere such that* $\operatorname{spt}\partial T$ *is homeomorphic to Y^k and the associated varifold $V = \underline{\underline{v}}(\operatorname{spt}T, \theta)$ is locally of bounded first variation in $\mathbb{R}^n$.*

THEOREM 7.9. **[C2]** *Suppose that T is a 2-dimensional area minimizing flat chain mod k in Y_2^k. If $C_1, C_2, \dots, C_k$ are the curves that constitute* $\operatorname{spt}\partial T$ *and have common end points, then*

$$2\pi\,\mathbf{M}(T) \leq \sum_{i=1}^{k} \operatorname{Length}(C_i)^2.$$

And equality holds if and only if $\operatorname{spt} T$ *is the union of k flat half-disks meeting each other along the common diameter.*

Let Y be the union of three half-disks meeting each other along their common diameter at equal angles of 120 degrees. Let T be the intersection with the unit ball $B_p(1)$ of an infinite cone from p through the 1-skeleton of a regular tetrahedron whose center of mass is p. In **[Ta]** J. Taylor proved that the disk, Y and T are the only three cones that are area minimizing under Lipschitz maps leaving the boundary fixed. In view of this fact and Theorem 7.9, we would like to propose the following problem.

OPEN PROBLEM 7.10. *Suppose that V is a 2-varifold with multiplicity 1 almost everywhere and is locally of bounded first variation in $\mathbb{R}^n$ such that V is stationary outside the rectifiable boundary* $\operatorname{spt}\partial V$. *Suppose also that* $\operatorname{spt}V$ *is homeomorphic to T. Let $C_1, C_2, \dots, C_6 \subset \operatorname{spt}\partial V$ be the curves that constitute* $\operatorname{spt}\partial V$ *and lie between 4 junctions of* $\operatorname{spt}\partial V$. *Show that*

$$2\cos^{-1}(-\tfrac{1}{3})\,\mathbf{M}(V) \leq \sum_{i=1}^{6} \operatorname{Length}(C_i)^2,$$

where equality holds if and only if $\operatorname{spt}V$ *is a homothetic expansion (or contraction) of T.*

8. Higher-Dimensional Minimal Submanifolds

So far we have considered two-dimensional minimal surfaces only. In this section we study the isoperimetric inequality of higher-dimensional minimal submanifolds.

Given a domain D in $\mathbb{R}^m$ it is well known that if ω_m is the volume of a unit ball in $\mathbb{R}^m$, then

$$m^m \omega_m \mathrm{Vol}(D)^{m-1} \leq \mathrm{Vol}(\partial D)^m \tag{8–1}$$

and equality holds if and only if D is a ball. In view of Open Problem 1.1 it is tempting to conjecture

OPEN PROBLEM 8.1. *Any m-dimensional minimal submanifold N of $\mathbb{R}^n$ satisfies the classical isoperimetric inequality*

$$m^m \omega_m \mathrm{Vol}(N)^{m-1} \leq \mathrm{Vol}(\partial N)^m,$$

where equality holds if and only if N is a ball in an m-plane of $\mathbb{R}^n$.

This problem is far less settled than Open Problem 1.1. The only two cases that are known to hold are i) when ∂N lies on the $(n-1)$-dimensional sphere centered at a point of N (by monotonicity) and (ii) when N is area minimizing (by Almgren [**Al**]).

8.1. Monotonicity. Closely related to the isoperimetric inequality of a minimal submanifol N is the monotonicity property: the volume of $N \cap B_p(r)$ divided by the volume of the geodesic ball of radius r is a nondecreasing function of r. This property has been proved for $\mathbb{R}^n$ and $\mathbf{H}^n$ [**An**], but not for $\mathbf{S}^n_+$ (but see [**GS**], p. 353).

LEMMA 8.2. (Monotonicity) *Let N be an m-dimensional minimal submanifold in $\mathbb{R}^n$ and r the distance in $\mathbb{R}^n$ from $p \in \mathbb{R}^n$. Then $\mathrm{Vol}(N \cap B_p(r))/r^m$ is a monotonically nondecreasing function of r for $0 < r < \mathrm{dist}(p, \partial N)$.*

PROOF. Write $N_r = N \cap B_p(r)$. Integrate (3–1) ($\triangle r^2 = 2m$) to obtain

$$m\mathrm{Vol}(N_r) = \frac{1}{2}\int_{N_r} \triangle r^2 = \int_{\partial N_r} r|\nabla r|. \tag{8–2}$$

Denote the volume forms on N and ∂N by dv and dS_r, respectively. Then we have

$$dv = \frac{1}{|\nabla r|} dS_r dr.$$

Then

$$\frac{d}{dr}\int_{N_r} |\nabla r|^2 dv = \int_{\partial N_r} |\nabla r| dS_r.$$

Hence

$$m\mathrm{Vol}(N_r) = r\frac{d}{dr}\int_{N_r} |\nabla r|^2\, dv = r\frac{d}{dr}\mathrm{Vol}(N_r) - r\frac{d}{dr}\int_{N_r}\left(1 - |\nabla r|^2\right) dv \leq r\frac{d}{dr}\mathrm{Vol}(N_r).$$

In the inequality above we used the fact that $|\nabla r| \leq 1$ on N. Hence

$$\frac{d}{dr}\frac{\mathrm{Vol}(N_r)}{r^m} \geq 0. \qquad \square$$

THEOREM 8.3. *Let N be an m-dimensional minimal submanifold of $\mathbb{R}^n$. If ∂N lies in a sphere centered at a point p of N then*

$$m^m \omega_m \mathrm{Vol}(N)^{m-1} \leq \mathrm{Vol}(\partial N)^m.$$

Equality holds if and only if N is a ball.

PROOF. Let R be the radius of the sphere. Applying $|\nabla r| \leq 1$ to (8–2) gives

$$m\mathrm{Vol}(N) = \int_{\partial N} r|\nabla r| \leq R \cdot \mathrm{Vol}(\partial N).$$

Since $\lim_{r\to 0} \mathrm{Vol}(N_r)/r^m = \omega_m$, from Lemma 8.2 we get

$$\omega_m \leq \mathrm{Vol}(N)/R^m.$$

Hence

$$m\mathrm{Vol}(N) \leq \omega_m^{-1/m}\mathrm{Vol}(N)^{1/m}\mathrm{Vol}(\partial N),$$

which gives the desired inequality. Equality holds if and only if N is a cone with density at the center equal to 1 if and only if N is a ball. □

8.2. Almgren's Proof. We have seen at the end of Section 1 that Open Problem 1.1 is true for area minimizing surface. Likewise it is shown by Almgren [**Al**] that Open Problem 8.1 is true for area minimizing submanifolds in $\mathbb{R}^n$. A submanifold is said to be *area minimizing* if its volume is less than or equal to the volume of every other submanifold having the same boundary.

THEOREM 8.4. [**Al**] *If N is an m-dimensional area minimizing submanifold of $\mathbb{R}^n$, then*

$$m^m\omega_m\mathrm{Vol}(N)^{m-1} \leq \mathrm{Vol}(\partial N)^m$$

with equality if and only if N is a ball.

Although the proof involves some complicated technicalities, the basic idea is very elegant and not hard to understand. So we shall introduce the argument sketched by B. White in [**Wh**] for the special case of 2-dimensional surfaces in $\mathbb{R}^3$ which generalizes to higher dimension.

PROPOSITION 8.5. [**Al**], [**Wh**] *If S is an area minimizing surface in $\mathbb{R}^3$ with area π, then its perimeter is greater than or equal to 2π, with equality if and only if S is a disk.*

PROOF. Among all area minimizing surfaces with area π, let S be the one whose perimeter is as short as possible. It follows that S minimizes the ratio

$$\frac{\mathrm{Length}(\partial S)^2}{\mathrm{Area}(S)}$$

among all area minimizing surfaces (because the ratio is invariant with respect to dilations). Of course since a disk of area π and circumference 2π is area minimizing, ∂S must have length $\leq 2\pi$; our goal is to show that it has length exactly 2π.

Let C_t be a 1-parameter family of curves with $C_0 = \partial S$, and for each C_t let S_t be a surface of least area with boundary $\partial S_t = C_t$. Because $\mathrm{Length}(C_t)^2/\mathrm{Area}(S_t)$ attains its minimum at $t = 0$ we have

$$0 = \left(\frac{d}{dt}\right)_{t=0} \frac{\mathrm{Length}(C_t)^2}{\mathrm{Area}(S_t)}$$

and therefore

$$0 = \frac{2}{\pi}\,\mathrm{Length}(C_0)\left(\frac{d}{dt}\right)_{t=0}\mathrm{Length}(C_t) - \frac{1}{\pi^2}\,\mathrm{Length}(C_0)^2\left(\frac{d}{dt}\right)_{t=0}\mathrm{Area}(S_t). \tag{8–3}$$

Now if in the 1-parameter family of curves C_t, the initial velocity of each point $x \in C_0$ is $v(x)$, then

$$\left(\frac{d}{dt}\right)_{t=0} \text{Length}(C_t) = -\int_{C_0} \langle v, \kappa \rangle \quad \text{and} \quad \left(\frac{d}{dt}\right)_{t=0} \text{Area}(S_t) = \int_{C_0} \langle v, \nu \rangle,$$

where $\kappa(x)$ is the curvature vector of C_0 at x and ν is the outward unit conormal to ∂S. The first of these formulas is the first variation formula for the length of the curves C_t and the second is the first variation formula for the area of the minimal surfaces S_t. Combining these formulas with (8–3) gives

$$0 = \int_{C_0} \bigl\langle v,\, 2\pi\kappa + \text{Length}(C_0)\nu \bigr\rangle.$$

Because this holds for every vector field v, it follows that $2\pi\kappa + \text{Length}(C_0)\nu$ must be identically 0. That is, the curve C_0 has constant curvature $\frac{1}{2\pi}\text{Length}(C_0)$. Recall that $\text{Length}(C_0) \le 2\pi$ (we are trying to prove equality), so C_0 has curvature everywhere less than or equal to 1. The proposition then follows from the following result.

PROPOSITION 8.6. [**Al**], [**Wh**] *If C is a closed k-dimensional submanifold in $\mathbb{R}^m$ with mean curvature everywhere less than or equal to k, then the volume of C is greater than or equal to the volume of the unit k-sphere, with equality if and only if C is congruent to the unit k-sphere.*

PROOF FOR $k = 1$, $m = 3$. Let K be the convex hull of C. Let $n : \partial K \to S^2$ be the Gauss map, which assigns to each $x \in \partial K$ the outward unit normal $n(x)$ to K at x. Note that at a corner of K there are many outward unit normals so the map n is multivalued.

The first observation is that n maps $\partial K \sim C$ to a set of zero area in S^2. To see this, consider for example the case where C consists of two congruent circles with one above the other so that K is a cylinder. Then n maps the sides of the cylinder to a great circle and the top and bottom to a pair of points.

On the other hand, the image of ∂K under n is all of S^2, so the image of $C \cap \partial K$ under n must have area 4π. It is not too hard to see that an infinitesimal piece of C at x of length ds is mapped to a set of area at most

$$2\,|\kappa(x)|\,ds,$$

which is less than or equal to $2\,ds$ because $|\kappa(x)| \le 1$. Thus

$$\frac{\text{Area}(n(C))}{\text{Length}(C)} \le 2.$$

But the area of $n(C)$ is 4π, therefore the length of C is at least 2π. □

The key ingredients in the proof of Proposition 8.5 are the existence of the area minimizing surface S_t and the continuity of $\text{Area}(S_t)$ to $\text{Area}(S)$ as $t \to 0$, both of which would not be valid if S were a general (not necessarily area minimizing) minimal surface.

9. The Calibration Method

There are numerous proofs for the original problem of Dido; among these we will introduce the most recent one given by Hélein [**He**] in 1994. His proof, in fact, holds even for curves on a sphere and on a hyperbolic plane as follows.

THEOREM 9.1. *Let S be a surface of Gaussian curvature K ($=1$ or -1). If a closed curve C on S has length L and encloses a domain of area A, then*

$$4\pi A \le L^2 + KA^2,$$

where equality holds if and only if C is a geodesic circle.

PROOF [**He**]. Let D be the domain enclosed by the smooth curve C on S. (x, y) denotes a point in $D \times \partial D$. For fixed $y \in \partial D$, cover D with the set of all circular arcs emanating from y and perpendicular to ∂D at y. Let $V(x, y)$ be the unit tangent vector to the arc pointing away from y. Then $V(x, y)$ is a unit vector field on $D \times \partial D$. One can easily compute

$$\mathrm{div} V = \frac{1 + f'(r)}{f(r)} \langle \nu, \nabla r \rangle$$

where ν is the unit inward normal to ∂D at y, $r = \mathrm{dist}(x, y)$ and $f(r) = r$, $\sin r$ or $\sinh r$ depending on whether $K = 0$, $K = +1$ or $K = -1$. Moreover one can show that if $r = \mathrm{dist}(x, z), x, z \in D$, then

$$\mathrm{div} \frac{1 + f'(r)}{f(r)} \nabla r = 4\pi\delta_x - K.$$

Let ω, dl be the volume forms of D, ∂D, respectively. Then we have a two-form $\alpha = V \lrcorner\, \omega \wedge dl$ where $\lrcorner$ denotes the interior multiplication such that

$$d\alpha = \mathrm{div} V \omega \wedge dl = \frac{1 + f'(r)}{f(r)} \langle \nu, \nabla r \rangle \, \omega \wedge dl.$$

Therefore

$$\begin{aligned}
\int_{D\times\partial D} d\alpha &= \int_D \Big(\int_{\partial D} \frac{1 + f'(r)}{f(r)} \langle \nu, \nabla r \rangle \, dl \Big) \omega = \int_D \left(\int_D d\Big(\frac{1 + f'(r)}{f(r)} \, \nabla r \lrcorner\, \omega \Big) \right) \omega \\
&= \int_D \left(\int_D \mathrm{div} \Big(\frac{1 + f'(r)}{f(r)} \, \nabla r \Big) \omega \right) \omega = \int_D \Big(\int_D (4\pi\delta_x - K) \, \omega \Big) \omega \\
&= \int_D (4\pi - KA) \, \omega = 4\pi A - KA^2.
\end{aligned}$$

On the other hand

$$\int_{D\times\partial D} d\alpha = \int_{\partial D\times\partial D} \alpha \ \le \ \int_{\partial D\times\partial D} dl \wedge dl = L^2. \tag{9–1}$$

Thus we get $4\pi A \le L^2 + KA^2$. In (9–1) inequality becomes equality if and only if V is perpendicular to ∂D at the end $\ne y$ of the circular arc, which occurs if and only if ∂D is a circle. □

10. Weak Isoperimetric Inequalities

It would be beautiful if Hélein's argument generalized to minimal surfaces as well. But various attempts made by the author ended up with no results. In this section, however, we will exploit Simon's argument which resembles Hélein's (see [**CG2**], p. 181) and obtain an isoperimetric inequality which is not sharp but which holds for *all* minimal surfaces. We will also present Ros's argument which improves Simon's inequality, and derive Sobolev-type inequalities related with the weak isoperimetric inequalities.

10.1. Weak Inequalities.

THEOREM 10.1. **[C4]**, **[CG2]** *Let S be a minimal surface in a complete simply connected Riemannian manifold M with sectional curvature bounded above by a constant K. If $K \leq 0$, then*

$$2\pi A \leq L^2 + KA^2. \tag{10–1}$$

In case $K > 0$, (10–1) *holds under the additional assumption* $\operatorname{diam}(S) \leq \frac{\pi}{2\sqrt{K}}$.

PROOF. (i) $K = -k^2 < 0$. Integrating Lemma 6.1(g) for fixed $p \in S$, we get

$$2\pi - KA \leq \int_S \triangle \log \sinh kr \leq \int_{\partial S} k \coth kr. \tag{10–2}$$

Recall that $r(x) = \operatorname{dist}(p, x)$ for fixed $p \in M$. Since (10–2) holds for any $p \in S$ we can let p vary on S and integrate (10–2) over S and apply Fubini's theorem to obtain

$$\begin{aligned} 2\pi A - KA^2 &\leq \int_S \int_{\partial S} k \coth kr = \int_{\partial S} \int_S k \coth kr \\ &\leq \int_{\partial S} \int_S \triangle r \text{ (by Lemma 6.1(d))} \\ &= \int_{\partial S} \int_{\partial S} \frac{\partial r}{\partial \nu} \leq L^2. \end{aligned}$$

(ii) $K = 0$. Integrate Lemma 6.1(c) twice and apply Lemma 6.1(b) as above.
(iii) $K > 0$. Integrate Lemma 6.1(i) twice and apply Lemma 6.1(h). □

THEOREM 10.2. **[CG2]** *Let N be an m-dimensional minimal submanifold of a complete simply connected Riemannian manifold M^n with sectional curvature bounded above by a negative constant $-k^2$. Then*

$$k(m-1)\operatorname{Vol}(N) \leq \operatorname{Vol}(\partial N).$$

PROOF. On a space form $\overline{M}^n$ of sectional curvature $-k^2$, $\overline{\nabla}^2 r = k \coth kr(g - dr \otimes dr)$, where g is the metric tensor of $\overline{M}$. Hence the Hessian comparison (6–1) and Lemma 5.1 imply that on N

$$\triangle r \geq k(m - |\nabla r|^2) \coth kr. \tag{10–3}$$

Therefore

$$\begin{aligned} \triangle \log \cosh kr = \operatorname{div}\left(k \frac{\sinh kr}{\cosh kr} \nabla r\right) &= \frac{k^2}{\cosh^2 kr} |\nabla r|^2 + k \frac{\sinh kr}{\cosh kr} \triangle r \\ &\geq k^2(m - |\nabla r|^2) \geq k^2(m-1). \end{aligned}$$

Hence

$$k(m-1)\operatorname{Vol}(N) \leq \frac{1}{k} \int_N \triangle \log \cosh kr \leq \int_{\partial N} \frac{\sinh kr}{\cosh kr} \leq \operatorname{Vol}(\partial N). \qquad \square$$

10.2. Simon's and Ros's Methods. The weak isoperimetric inequality

$$2\pi A \le L^2$$

for all minimal surfaces in $\mathbb{R}^n$ was originally proved by L. Simon (see [**Bm**, p. 318] or [**O2**, p. 1210]). We review Simon's argument briefly. As in (3–1), we have $\triangle r^2 = 4$ on a minimal surface $S \subset \mathbb{R}^n$. Hence

$$\triangle \log r = \frac{2}{r^2}(1 - |\nabla r|^2) \ge 2\pi\delta_p \tag{10–4}$$

and

$$\triangle r = \frac{1}{r}(2 - |\nabla r|^2) \ge \frac{1}{r}. \tag{10–5}$$

Integrating (10–4) over S yields

$$2\pi \le \int_{y\in S} \triangle \log r_x(y) \le \int_{y\in\partial S} \frac{1}{r_x(y)}\frac{\partial r_x(y)}{\partial \nu} \le \int_{y\in\partial S} \frac{1}{r_x(y)}\,, \tag{10–6}$$

where $r_x(y) = \operatorname{dist}(x, y)$ for some fixed $x \in \mathbb{R}^n$ and ν is the outward unit conormal to ∂S. Integrating (10–5) over S gives

$$\int_{x\in S} \frac{1}{r_y(x)} \le \int_{x\in S} \triangle r_y(x) = \int_{x\in\partial S} \frac{\partial r_y(x)}{\partial \nu}. \tag{10–7}$$

Let x vary over S, integrate (10–6) over S and use Fubini's theorem and (10–7) to get

$$\begin{aligned} 2\pi A &\le \int_{x\in S}\int_{y\in\partial S} \frac{1}{r_x(y)} = \int_{y\in\partial S}\int_{x\in S} \frac{1}{r_x(y)} = \int_{y\in\partial S}\int_{x\in S} \frac{1}{r_y(x)} \\ &\le \int_{y\in\partial S}\int_{x\in\partial S} \frac{\partial r_y(x)}{\partial \nu} \le \int_{y\in\partial S}\int_{x\in\partial S} 1 = L^2. \end{aligned} \tag{10–8}$$

Recently A. Ros improved this inequality by the factor of $\sqrt{2}$. His idea goes as follows. First, note that one can write

$$\frac{\partial r_y(x)}{\partial \nu} = \frac{\langle x-y, \nu(x)\rangle}{|x-y|}.$$

Note also that the roles of x and y can be interchanged in (10–8). Hence by adding up each expression (10–8) for x and y, we get

$$4\pi A \le \int_{y\in\partial S}\int_{x\in\partial S} \frac{\langle x-y, \nu(x)-\nu(y)\rangle}{|x-y|}.$$

Therefore

$$\begin{aligned} 4\pi A &\le \int_{y\in\partial S}\int_{x\in\partial S} |\nu(x) - \nu(y)| \\ &\le L\left(\int_{y\in\partial S}\int_{x\in\partial S} |\nu(x) - \nu(y)|^2\right)^{1/2} \qquad \text{(by the Hölder inequality)} \\ &= L\left(\int_{y\in\partial S}\int_{x\in\partial S} (2 - 2\langle \nu(x), \nu(y)\rangle)\right)^{1/2} \\ &= L\left(\int_{y\in\partial S}\int_{x\in\partial S} 2\right)^{1/2} \qquad \left(\text{since } \textstyle\int_{x\in\partial S} \nu(x) = 0 \text{ on minimal } S\right) \\ &= \sqrt{2}L^2. \end{aligned}$$

Thus:

THEOREM 10.3. (Ros) *For any minimal surface in* $\mathbb{R}^n$,

$$2\sqrt{2}\,\pi A \leq L^2.$$

The author has recently heard that A. Stone also obtained this result [**St**].

10.3. Sobolev-type Inequalities. Due to the analytic nature of the proofs of Theorem 10.1 and 10.2 we can derive, applying the same argument, the Sobolev-type inequalities corresponding to the above isoperimetric inequality. As is well known, one can recover the isoperimetric inequality from the Sobolev-type inequality using characteristic functions as test functions. For more Sobolev-type inequalities, see [**CG2**].

PROPOSITION 10.4. [**CG2**] *Let* f *be a compactly supported smooth nonnegative function on a minimal surface* S *in a complete simply connected Riemannian manifold* M^n *with sectional curvature bounded above by a constant* $K = \pm k^2$. *If* $K = k^2$, *assume also that* $\operatorname{diam}(S) \leq \pi/(2k)$. *Then*

$$2\pi \int_S f^2 \leq \left(\int_S |\nabla f|\right)^2 + K\left(\int_S f\right)^2.$$

PROOF. For $K = k^2$ we have from Lemma 6.1(i)

$$\operatorname{div}(f\nabla \log \sin kr) \geq \langle \nabla f\,, k\cot kr\nabla r\rangle - Kf + 2\pi f\,\delta_p.$$

Integrating both sides, we see that for a fixed $p = y \in S$

$$2\pi f(y) \leq k\int_{x\in S} |\nabla f(x)|\cot kr_y(x) + K\int_{x\in S} f(x),$$

where $r_y(x) = \operatorname{dist}(y,x)$. Moreover by Lemma 6.1(h)

$$\operatorname{div}(f\nabla r) \geq \langle \nabla f, \nabla r\rangle + kf\cot kr.$$

So

$$k\int_{y\in S} f(y)\cot kr_x(y) \leq \int_{y\in S} |\nabla f(y)|.$$

Therefore

$$\begin{aligned} 2\pi \int_S f^2 &\leq \int_{y\in S} f(y)\left(k\int_{x\in S}|\nabla f(x)|\cot kr_y(x) + K\int_{x\in S} f(x)\right) \\ &= \int_{x\in S} |\nabla f(x)|\left(k\int_{y\in S} f(y)\cot kr_x(y)\right) + K\left(\int_S f\right)^2 \\ &\leq \left(\int_S |\nabla f|\right)^2 + K\left(\int_S f\right)^2. \end{aligned}$$

A similar proof is valid for $K = -k^2$. □

PROPOSITION 10.5. [**CG2**] *Let* f *be a nonnegative smooth function with compact support on an* m*-dimensional minimal submanifold* N *in a complete simply connected Riemannian manifold of sectional curvature bounded above by a negative constant* $-k^2$. *Then*

$$k(m-1)\int_N f \leq \int_N |\nabla f|.$$

PROOF. From (10–3) we have

$$\operatorname{div}(f\nabla r) \geq \langle \nabla f, \nabla r\rangle + k(m-1)f.$$

Integrate both sides over N. □

11. Modified Volume

Unlike the isoperimetric inequalities $4\pi A \leq L^2$ and $k^k\omega_k \mathrm{Vol}(D)^{k-1} \leq \mathrm{Vol}(\partial D)^k$ in space (see (1–1) and (8–1)), the inequality (6–5) $4\pi A \leq L^2 + KA^2$ for minimal surfaces in space forms of curvature K has a correction term. In this section, however, we introduce a *modified volume* $M_p(N)$ of a k-dimensional minimal submanifold N of a space form and obtain an isoperimetric inequality like (8–1) with no correction term:

$$k^k\omega_k M_p(N)^{k-1} \leq \mathrm{Vol}(\partial N)^k.$$

DEFINITION 11.1. *Let p be a point in S^n and let $r(x)$ be the distance from p to x in S^n. Given a k-dimensional submanifold N of S^n, the* modified volume $M_p(N)$ of N with center at p *is defined to be*

$$M_p(N) = \int_N \cos r.$$

Similarly for $N \subset H^n$, define

$$M_p(N) = \int_N \cosh r.$$

Obviously $M_p(N) \leq \mathrm{Vol}(N)$ for $N \subset S^n$, and $M_p(N) \geq \mathrm{Vol}(N)$ for $N \subset H^n$. Suppose that S^n is embedded in $\mathbb{R}^{n+1}$ with p the north pole $(0,\ldots,0,1)$ and that H^n is embedded as the hypersurface $\Sigma, x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0$, of $\mathbb{R}^{n+1}$ with the Minkowski metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$ such that p becomes the point $(0,\ldots,0,1) \in \Sigma$. Note that $\cos r$ is the Jacobian of the projection of S^n into $x_{n+1} = 0$ and that in the Minkowski space

$$dx_i = \cosh r\, dr, \;\; i = 1,\ldots,n.$$

Therefore we have the following.

LEMMA 11.2. [**CG2**] *$M_p(U), U \subset S^n$ or H^n, is the Euclidean volume of the orthogonal projection of U into the horizontal hyperplane $x_{n+1} = 0$, counting orientation.*

It is well known that $\sin r$ and $\sinh r$ are the lengths of Jacobi fields in S^2 and H^2, respectively. Hence it is easy to show that

$$\overline{\nabla}^2 \cos r = -(\cos r)g \text{ on } S^n, \;\; \overline{\nabla}^2 \cosh r = (\cosh r)g \text{ on } H^n$$

(see [**CG2**, Lemma 3]). Hence by Lemma 5.1, if $N \subset S^n$ or H^n is a k-dimensional minimal submanifold or a cone, then

(11–1) $$\triangle \cos r = -k\cos r, \;\; \triangle \cosh r = k\cosh r,$$

where the vertex of the cone should be the point p from which the distance function r is measured. Integrating (11–1) and following the proof of Proposition 4.3 or Proposition 5.6, we get the following.

PROPOSITION 11.3. [**CG2**] *For any minimal submanifold N in S^n or H^n and any point p in S^n or H^n,*

$$M_p(N) \le M_p(p \divideontimes \partial N). \tag{11–2}$$

We now have the comparison formulas that we need: Proposition 5.4 and Proposition 11.3. With these in our hands we can use the arguments of developing and cutting and pasting as in Section 4 and prove

THEOREM 11.4. [**CG2**] *Let S be a minimal surface in S^n and p a point of S. Assume that $r \le \pi/2$ on S. If ∂S is radially connected from p, that is, $\{s : s = \operatorname{dist}(p,q), q \in \partial S\}$ is a connected interval, then $4\pi M_p(S) \le \operatorname{Length}(\partial S)^2$. Equality holds if and only if S is a totally geodesic disk with center at p.*

The same inequality is false for minimal surfaces in H^n. In fact, among domains in H^2 with prescribed boundary length, the modified area has no upper bound. Moreover, our proof fails in H^n because the projection in Minkowski space $\mathbb{R}^{n+1}$ from H^n onto the hyperplane $x_{n+1} = 0$ is a length-increasing map. In Theorem 11.6 below, however, it will be shown that the same inequality holds, even in higher dimension, in case ∂S lies in a sphere of H^n centered at p.

As we have seen in Section 8, the monotonicity of the volume of a minimal submanifold is closely related to the isoperimetric inequalities. The monotonicity has been proved for $\mathbb{R}^n$ and for H^n [**An**], but not for S^n (see however [**GS**], p. 353). The next proposition shows that modified volume enjoys the monotonicity in all three cases.

PROPOSITION 11.5. [**CG2**] *Let N be a k-dimensional minimal submanifold in S^n (H^n, respectively) and r the distance in S^n (H^n, respectively) from p. Then $M_p(N \cap B_p(r))/\sin^k r$ $(M_p(N \cap B_p(r))/\sinh^k r$, respectively) is a monotonically nondecreasing function of r for $0 < r < \min(\pi/2, \operatorname{dist}(p, \partial N))$ $(0 < r < \operatorname{dist}(p, \partial N)$, respectively).*

PROOF. Define $N_r = N \cap B_p(r) \subset S^n$. Then

$$M_p(N_r) = -\frac{1}{k}\int_{N_r} \triangle \cos r = \frac{1}{k}\int_{\partial N_r} \sin r \frac{\partial r}{\partial \nu} = \frac{1}{k}\sin r \int_{\partial N_r} |\nabla r|.$$

Denote the volume forms of N and ∂N_r by dv and dS_r, respectively. Then $dv = \frac{1}{|\nabla r|} dS_r dr$. Hence

$$\frac{d}{dr}\int_{N_r} \cos r |\nabla r|^2 dv = \frac{d}{dr}\int_0^r \int_{\partial N_r} \cos r |\nabla r| dS_r dr = \cos r \int_{\partial N_r} |\nabla r|.$$

Therefore

$$\begin{aligned} M_p(N_r) &= \frac{\sin r}{k \cos r}\cos r \int_{\partial N_r} |\nabla r| = \frac{\sin r}{k\cos r}\frac{d}{dr}\int_{N_r} \cos r |\nabla r|^2 \\ &\le \frac{\sin r}{k\cos r}\frac{d}{dr}\int_{N_r} \cos r = \frac{\sin r}{k \cos r}\frac{d}{dr} M_p(N_r). \end{aligned}$$

Hence

$$\frac{d}{dr}\log[M_p(N_r)/\sin^k r] \ge 0.$$

Thus $M_p(N_r)/\sin^k r$ is nondecreasing; similarly for $N \subset H^n$. □

A very special case of the radially connected boundary occurs when ∂N lies in a geodesic sphere. In this case the conclusion of Theorem 11.4 may be extended to hyperbolic space, and the minimal submanifold may have any dimension.

THEOREM 11.6. [**CG2**] *Let N be a k-dimensional minimal submanifold in S^n or H^n. Assume that ∂N lies in a geodesic sphere centered at a point p of N and that r is the distance in S^n or H^n from p. Furthermore, in case of $N \subset S^n$, assume $r \le \pi/2$ on N. Then*

$$k^k \omega_k M_p(N)^{k-1} \le \mathrm{Vol}(\partial N)^k.$$

Equality holds if and only if N is a totally geodesic ball centered at p.

PROOF. Assume $N \subset H^n$ and let R be the radius of the geodesic sphere in which ∂N lies. Then

$$\begin{aligned} M_p(N) &= \frac{1}{k}\int_N \triangle \cosh r = \frac{1}{k}\int_{\partial N} \sinh r \frac{\partial r}{\partial \nu} \\ &= \frac{1}{k}\sinh R \int_{\partial N} \frac{\partial r}{\partial \nu} \le \frac{1}{k}\sinh R \cdot \mathrm{Vol}(\partial N). \end{aligned}$$

Since $\lim_{r\to 0} M_p(N \cap B_p(r))/\sinh^k r = \omega_k$, we obtain from Proposition 11.5

$$M_p(N)/\sinh^k R \ge \omega_k.$$

Hence

$$M_p(N) \le \frac{1}{k}\omega_k^{-1/k} M_p(N)^{1/k}\mathrm{Vol}(\partial N)$$

and so the desired inequality follows. Obviously equality holds if and only if N is a cone with density at the center equal to 1, or equivalently, N is a totally geodesic ball. A similar proof holds for $N \subset S^n$. □

12. Relative Isoperimetric Inequality

By the classical isoperimetric inequality (8–1) for $D \subset \mathbb{R}^n$ we have

(12–1) $$n^n \omega_n \mathrm{Vol}(D)^{n-1} \le \mathrm{Vol}(\partial D)^n.$$

An immediate consequence of this inequality is that if H is a closed half space of $\mathbb{R}^n$ and D is a subset of H then

$$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \le \mathrm{Vol}(\partial D \sim \partial H)^n$$

and equality holds if and only if D is a half ball with the flat part of its boundary contained in ∂H. This follows if one applies (12–1) to the union of D and its mirror image across ∂H. Then a natural question to ask is the following.

OPEN PROBLEM 12.1. *If $C \subset \mathbb{R}^n$ is a convex domain and D is a subset of $\mathbb{R}^n \sim C$, does D satisfy the isoperimetric inequality*

(12–2) $$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \le \mathrm{Vol}(\partial D \sim \partial C)^n?$$

Does equality hold if and only if $C = H$ and D is a half ball with the flat part of its boundary lying in ∂H?

Inequality (12–2) is called the *relative* isoperimetric inequality, C is called the supporting set of D, and $\mathrm{Vol}(\partial D \sim \partial C)$ is called the relative volume of ∂D. For $n = 2$ it is easy to prove (12–2): just reflect the convex hull of D about its linear boundary.

A partial answer for $n \geq 3$ was obtained by I. Kim [**K2**]; he showed that if $U = \{(x,y) \in \mathbb{R}^2 : y \geq f(x), f'' \geq 0\}$, then (12–2) holds for $C = U \times \mathbb{R}^{n-2}$. In this section we shall first see that the relative isoperimetric inequality holds if C is a graph which is symmetric about $n-1$ hyperplanes of $\mathbb{R}^n$ [**C5**]. In particular, (12–2) holds when C is a ball. The tools of [**C5**] are Gromov's method of using the divergence theorem and Steiner's method of symmetrization. Then we shall give an outline of the proof of the relative isoperimetric inequality which has been obtained recently by the author and Ritoré [**CR**].

12.1. Gromov's Method. In [**Gr**] Gromov gave a new proof of the classical isoperimetric inequality. As F. Morgan pointed out to us, Knothe [**Kn**] and Berger [**Bg**] also used the same method as Gromov. His proof is based on a volume-preserving map whose divergence is bigger than or equal to the dimension of space. Here we shall see how Gromov's method can be adapted for our purpose and why the convexity of the supporting set is necessary.

THEOREM 12.2. [**C5**] *Let C be a convex domain in $\mathbb{R}^n$ and D a subset of $\mathbb{R}^n \sim C$ with piecewise C^1 boundary. Suppose that every normal vector η to $\partial D \cap \partial C$ toward the exterior of D does not point upward, that is, $\langle \eta, \partial/\partial x^n \rangle \leq 0$ for the unit vertical vector $\partial/\partial x^n$. Suppose also that there exist vertical hyperplanes $\Pi_1, \ldots, \Pi_{n-1}$ which are mutually perpendicular such that C and D are symmetric about each of them. Then*

$$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \leq \mathrm{Vol}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

PARTIAL PROOF. First define a C^1 map $\phi_D : D \to [0,1]^n$ by

$$\phi_D(x_1, \ldots, x_n) = (\phi_1, \ldots, \phi_n),\ \ \phi_i = \frac{\overline{v}_i}{v_i},$$

$$v_i = L^{n-i+1}\{(a_1, \ldots, a_n) \in D : a_j = x_j,\ 1 \leq j \leq i-1,\ -\infty \leq a_k \leq \infty,\ i \leq k \leq n\},$$
$$\overline{v}_i = L^{n-i+1}\{(a_1, \ldots, a_n) \in D : a_j = x_j,\ 1 \leq j \leq i-1,\ -\infty \leq a_i \leq x_i,\ -\infty \leq a_k \leq \infty,\ i+1 \leq k\},$$

where L^k is k-dimensional Lebesgue measure. Then $\phi_i = \phi_i(x_1, \ldots, x_i)$ and the Jacobian matrix of ϕ_D, $(\partial \phi_i/\partial x_j)$, is lower triangular with diagonal entries $\partial\phi_i/\partial x_i = v_{i+1}/v_i$ and $\partial \phi_n/\partial x_n = 1/v_n$. Therefore

$$\det\left(\frac{\partial \phi_i}{\partial x_j}\right) = \frac{1}{v_1}.$$

Similarly, define $\phi_B : B \to [0,1]^n$ where B is the half ball

$$\Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0,\ \sum x_i^2 \leq (2\omega_n^{-1}\mathrm{Vol}(D))^{2/n}\Big\}. \tag{12–3}$$

Note that $\mathrm{Vol}(B) = \mathrm{Vol}(D) = v_1$. Like ϕ_D the Jacobian determinant of ϕ_B equals $1/v_1$. Let $\psi : D \to B$ be defined by $\psi = \phi_B^{-1} \circ \phi_D$. Then the Jacobian determinant of ψ equals 1. In other words, ψ is a volume-preserving map.

Now consider a vector field V on D defined by $V(x) =$ *the position vector of* $\psi(x), x \in D$. Since the Jacobian matrix of ψ is also lower triangular, it follows from the arithmetic-geometric mean inequality that

$$n = n(\det D\psi)^{1/n} \leq \mathrm{div} V. \tag{12–4}$$

Let Π_n be the horizontal hyperplane $\{x_n = 0\}$ and let $U_1, \dots, U_{2^{n-1}}$ be the congruent subsets of Π_n separated by the vertical hyperplanes $\Pi_1, \dots, \Pi_{n-1}$. Translating C and D in a suitable way we may assume that each Π_i contains $(0, \dots, 0)$. Define the projection $p : \mathbb{R}^n \to \Pi_n$ by $p(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, 0)$. By the divergence theorem applied to (12–4)

$$n\mathrm{Vol}(D) \le \int_{\partial D \sim \partial C} \langle V, \eta \rangle + \int_{\partial D \cap \partial C} \langle V, \eta \rangle, \tag{12–5}$$

where η is the outward unit normal to ∂D. By (12–3) we have

$$|V| \le (2\,\omega_n^{-1}\mathrm{Vol}(D))^{1/n} \text{ on } \partial D \sim \partial C. \tag{12–6}$$

By the symmetry of C and D about $\Pi_1, \dots, \Pi_{n-1}$ and by the convexity of C, we get

$$\langle V, \eta \rangle \le 0 \text{ on } \partial D \cap \partial C. \tag{12–7}$$

This is because if $x \in \partial D \cap \partial C$ and $p(x) \in U_k, 1 \le k \le 2^{n-1}$, then both $\psi(x)$ and $-p(q_\eta)$ lie in U_k, where $q_\eta \in \mathbb{R}^n$ is the point whose position vector is η. Therefore it follows from (12–5), (12–6), and (12–7) that

$$n\mathrm{Vol}(D) \le (2\,\omega_n^{-1}\mathrm{Vol}(D))^{1/n}\mathrm{Vol}(\partial D \sim \partial C),$$

which implies (12–2).

See **[C5]** for the case of equality. □

12.2. Steiner's Symmetrization. One of the oldest and most powerful methods in isoperimetric inequalities is Steiner's symmetrization **[S2]**. The key idea of this method is that given k functions

$$x_n = f_1(x_1, \dots, x_{n-1}), \quad \dots, \quad x_n = f_k(x_1, \dots, x_{n-1}),$$

the volume of the graph of the average function of $f_1, \dots, f_k$ is not bigger than the average of the volumes of the graphs of $f_1, \dots, f_k$. This volume estimate is based on the simple inequality for k vectors in $\mathbb{R}^n$: $|v_1 + \cdots + v_k| \le |v_1| + \cdots + |v_k|$. Here, using the symmetrization method, we shall improve Theorem 12.2.

THEOREM 12.3. **[C5]** *Let C be a convex domain in $\mathbb{R}^n$, D a subset of $\mathbb{R}^n \sim C$ with piecewise C^1 boundary, and Π_n a horizontal hyperplane $\{x_n = 0\}$. Suppose that both $\partial D \sim \partial C$ and $\partial D \cap \partial C$ are graphs over a closed set $A \subset \Pi_n$. If A is symmetric about $n-1$ vertical hyperplanes $\Pi_1, \dots, \Pi_{n-1}$ which are mutually perpendicular, then*

$$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \le \mathrm{Vol}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a half ball.

PARTIAL PROOF. Let $f_0, g_0 : A \to \mathbb{R}$ be the functions defined by

$$x_n = f_0(x_1, \dots, x_{n-1}), x_n = g_0(x_1, \dots, x_{n-1})$$

such that $\partial D \sim \partial C, \partial D \cap \partial C$ are the graphs of f_0, g_0, respectively. Let G be the group of isometries of $\mathbb{R}^n$ generated by $n-1$ horizontal reflections which leave $\Pi_1, \dots, \Pi_{n-1}$ fixed, respectively. G consists of 2^{n-1} elements, say, $r_1, \dots, r_{2^{n-1}}$. Define $f_i = f_0 \circ r_i$ and $g_i = g_0 \circ r_i, i = 1, \dots, 2^{n-1}$. Also define $f = 2^{1-n} \sum_{i=1}^{2^{n-1}} f_i$, $g = 2^{1-n} \sum_{i=1}^{2^{n-1}} g_i$. Since $f_0 \ge g_0$ on A and $f_0 = g_0$ on ∂A, we have $f \ge g$ on A

and $f = g$ on ∂A. Hence graph(f) and graph(g) enclose a domain $\hat{D}$, and it is easy to see that

$$\mathrm{Vol}(D) = \mathrm{Vol}(\hat{D}). \tag{12–8}$$

Note also that $\hat{D}$ is symmetric about $\Pi_1, \ldots, \Pi_{n-1}$ and graph$(g) \subset \partial\hat{D}$ is a subset of $\partial\hat{C}$ for some convex doamain $\hat{C}$. Moreover

$$\begin{aligned}\mathrm{Vol}(\partial\hat{D} \sim \partial\hat{C}) = \mathrm{Vol}(\mathrm{graph}(f)) &= \int_A \left|\Big(2^{1-n}\sum_i \frac{\partial f_i}{\partial x_1}, \ldots, 2^{1-n}\sum_i \frac{\partial f_i}{\partial x_{n-1}}, 1\Big)\right| \\ &= 2^{1-n}\int_A \left|\Big(\sum_i \frac{\partial f_i}{\partial x_1}, \ldots, \sum_i \frac{\partial f_i}{\partial x_{n-1}}, 2^{n-1}\Big)\right| \\ &\le 2^{1-n}\int_A \sum_{i=1}^{2^{n-1}} \left|\Big(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_{n-1}}, 1\Big)\right| \\ &= 2^{1-n}\sum_{i=1}^{2^{n-1}} \mathrm{Vol}(\mathrm{graph}(f_i)) = \mathrm{Vol}(\mathrm{graph}(f_0)) = \mathrm{Vol}(\partial D \sim \partial C).\end{aligned}$$

This plus (12–8), together with Theorem 12.2 applied to $\hat{C}, \hat{D}$, gives the desired inequality.

See [**C5**] for the proof of the equality case. □

Although the symmetry assumption is required in Theorems 12.2 and 12.3, it is not necessary in case the convex set C is a ball:

THEOREM 12.4. [**C5**] *If C is a ball in $\mathbb{R}^n$ and D is a subset of $\mathbb{R}^n \sim C$ with rectifiable boundary, then*

$$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \le \mathrm{Vol}(\partial D \sim \partial C)^n$$

with equality if and only if D is a half ball.

It is easy to prove this theorem once we know that the isoperimetric region of the complement of a ball is rotationally symmetric about a line through the center of the ball.

LEMMA 12.5. [**C5**] *Outside a ball $C \subset \mathbb{R}^n$ there exists a set $\tilde{D}$ whose boundary has the least relative volume $\mathrm{Vol}(\partial\tilde{D} \sim \partial C)$ among all sets outside C with the same volume as $\tilde{D}$. In fact, $\partial\tilde{D} \sim \partial C$ is a spherical cap perpendicular to ∂C and $\partial\tilde{D} \cap \partial C$ lies in an open hemisphere of ∂C.*

PROOF. The existence of $\tilde{D}$ can be obtained by following the compactness argument in [**Sp**], pp. 441-444. Obviously $\partial\tilde{D} \sim \partial C$ has constant mean curvature and makes 90° with ∂C. We claim that $\tilde{D}$ is rotationally symmetric about a line. Suppose not. Then there exists an $(n-3)$-dimensional great sphere S in ∂C such that $\tilde{D}$ is not symmetric about any hyperplane containing S. Choose a hyperplane Π containing S that devides $\tilde{D}$ into $\tilde{D}_1$ and $\tilde{D}_2$ of equal volume. Suppose without loss of generality that $\mathrm{Vol}(\partial\tilde{D}_1 \sim (\partial C \cup \Pi)) \le \mathrm{Vol}(\partial\tilde{D}_2 \sim (\partial C \cup \Pi))$. Let $\tilde{D}_3$ be the mirror image of $\tilde{D}_1$ across Π and define $\tilde{D}_{13}$ to be the union of the closures of $\tilde{D}_1$ and $\tilde{D}_3$. If $\partial\tilde{D} \sim \partial C$ intersects Π at 90°, then the unique continuation property of the constant mean curvature hypersurfaces implies that $\tilde{D}$ is symmetric about Π, contradicting our hypothesis. Therefore some part of $\partial\tilde{D}_{13} \sim \partial C$ should be not

C^1 along II. Then we can slightly perturb $\tilde{D}_{13}$ along this singular part to get a set $D' \subset \mathbb{R}^n \sim C$ such that

$$\mathrm{Vol}(D') = \mathrm{Vol}(\tilde{D}_{13}) = \mathrm{Vol}(\tilde{D}),$$

and

$$\mathrm{Vol}(\partial D' \sim \partial C) < \mathrm{Vol}(\partial \tilde{D}_{13} \sim \partial C) \leq \mathrm{Vol}(\partial \tilde{D} \sim \partial C).$$

But this contradicts the least relative volume property of $\partial\tilde{D}$. Hence $\tilde{D}$ must be rotationally symmetric about a line l. Now let $\{q\} = (\partial\tilde{D} \sim \partial C) \cap l$ and take a spherical cap A through q which is rotationally symmetric about l and has the same mean curvature as $\partial\tilde{D} \sim \partial C$. Since $\partial\tilde{D} \sim \partial C$ is tangent to A at q, we can apply the maximum principle and conclude that $\partial\tilde{D} \sim \partial C$ itself is a spherical cap. Then $\partial\tilde{D} \cap \partial C$ is a subset of an open hemisphere of ∂C. □

12.3. Dimensions Three and Four. The author and M. Ritoré have recently proved (12–2) in case $n = 3$ [**CR**]. Here we give an idea of the proof. Given a convex set $C \subset \mathbb{R}^n$, define the *relative isoperimetric profile* of $\mathbb{R}^n \sim \overline{C}$, $I_C : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$I_C(v) = \inf_D \{\mathrm{Area}(\partial D \sim \partial C) : D \subset \mathbb{R}^n \sim \overline{C},\ \mathrm{Vol}(D) = V\}.$$

Let $\mathbb{H}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ be the upper half space of $\mathbb{R}^n$. Then the relative profile of $\mathbb{R}^n \sim \mathbb{H}^+$ is given by

$$I_{\mathbb{H}^+}(V) = n\left(\frac{\omega_n}{2}\right)^{1/n} V^{(n-1)/n},$$

and the relative isoperimetric inequality (12–2) is equivalent to

$$I_C(\mathrm{Vol}(D)) \geq I_{\mathbb{H}^+}(\mathrm{Vol}(D)), \tag{12–9}$$

with quality if and only if D is a half ball. How can one prove (12–9)? The idea is the following. First we shall take the first variation of (12–9) to get

(12–10)
$$\sup_{p \in \partial D \sim \partial C} \{H(p) : H(p) \text{ is the mean curvature of } \partial D \text{ at } p\} \geq H_0(\mathrm{Area}(\partial D \sim \partial C))$$

where $H_0(\mathrm{Area}(\partial D \sim \partial C))$ is the mean curvature of the hemisphere of area $\mathrm{Area}(\partial D \sim \partial C)$ in $\mathbb{R}^n$.

Then (12–10) follows from

$$\int_{\partial D \sim \partial C} H^2 \geq 2\pi. \tag{12–11}$$

We prove this inequality by using a method of conformal geometry: If k_1 and k_2 are the principal curvatures of $\partial D \sim \partial C$, then $\int_{\partial D \sim \partial C} (k_1 - k_2)^2$ is invariant under the conformal change of metric in $\mathbb{R}^3$. Note that

$$\frac{1}{4}\int_{\partial D \sim \partial C} (k_1 - k_2)^2 = \int_{\partial D \sim \partial C} (H^2 - K),$$

where K is the Gaussian curvature. Then (12–11) can be obtained by conformally blowing up $\partial D \sim \partial C$ around its boundary point and by using the convexity of ∂C.

In [**CR**] a new type of relative isoperimetric inequality is also proved:

Let C_1, C_2 be convex domains in $\mathbb{R}^3$ and D a subset of $\mathbb{R}^3 \sim (C_1 \cup C_2)$. If ∂C_1 and ∂C_2 make an angle of at least θ, then

$$18\theta \mathrm{Vol}(D)^2 \leq \mathrm{Area}(\partial D \sim \partial(C_1 \cup C_2))^3$$

with equality if and only if C_1 and C_2 are half spaces with an angle of θ and $\partial D \sim \partial(C_1 \cup C_2)$ is part of a sphere perpendicular to the planes ∂C_1 and ∂C_2.

The author also proved the relative isoperimetric inequality (12–2) in $\mathbb{R}^4$ [**C7**]. Croke's arguments in [**Cr**] work nicely in the relative setting as well due to the following observation: The distance between two rays in $\mathbb{R}^n$ emanating from a point grows linearly, while the distance between two rays in D grows faster after than before the rays hit and bounce off ∂C.

To be precise, the relative isoperimetric inequalities of [**CR**] and [**C7**] hold in a Riemannian manifold M^n $(n = 3, 4)$ of nonpositive curvature. These results partially answer the Open Problem 12.6 below.

It would be interesting if one could derive a version of the relative isoperimetric inequality for minimal surfaces. Therefore, combining Open Problems 8.1 and 12.1, one can propose the following.

OPEN PROBLEM 12.6. *Given a convex domain C in $\mathbb{R}^n$ and an m-dimensional minimal submanifold N outside C such that N is orthogonal to ∂C along $\partial C \cap \partial N$, prove that*

$$\tfrac{1}{2}\, m^m \omega_m \mathrm{Vol}(N)^{m-1} \leq \mathrm{Vol}(\partial N \sim \partial C)^m,$$

where equality holds if and only if N is a half ball.

I. Kim [**K1**] obtained a partial result for this open problem when N is two-dimensional. He proved that if S is a minimal surface in a Riemannian manifold M of constant sectional curvature $K \leq 0$, S lies outside a convex set C in M and is orthogonal to ∂C, and $\partial S \sim \partial C$ is connected or radially connected from a point of $\partial S \cap \partial C$, then

$$2\pi\, \mathrm{Area}(S) \leq \mathrm{Length}(\partial S \sim \partial C)^2 + K\, \mathrm{Area}(S)^2$$

and equality holds if and only if S is a totally geodesic half disk. For the proof of this, he first showed that

$$\mathrm{Area}(S) \leq \mathrm{Area}(p \,⋇\, (\partial S \sim \partial C)) \;\text{ for any } p \in C$$

and

$$\mathrm{Angle}(\partial S \sim \partial C, p) \geq \pi \;\text{ for any } p \in \partial S \cap \partial C.$$

Then he used the method of developing and cutting and pasting as in Section 4. Kim also extended Küster's linear isoperimetric inequality; Küster [**Ku**] proved

$$\mathrm{Vol}(N) \leq \frac{R}{m}\, \mathrm{Vol}(\partial N)$$

for an m-dimensional minimal submanifold N of $\mathbb{R}^n$ contained in a closed ball of radius R. Kim obtained a linear isoperimetric inequality for a minimal submanifold N^m in a complete simply connected Riemannian manifold M^n with sectional curvature bounded above by a nonpositive constant K:

$$\mathrm{Vol}(N) \leq \frac{\alpha_{m,K}(R)}{\alpha'_{m,K}(R)}\, \mathrm{Vol}(\partial N),$$

where N is contained in a geodesic ball of radius R in M and $\alpha_{m,K}(R)$ denotes the volume of the geodesic ball of radius R in the m-dimensional space form of sectional curvature K.

13. Negatively Curved Surfaces

As we have seen in Section 1, it was Carleman [**Ca**] who first showed that the classical isoperimetric inequality $4\pi A \leq L^2$ still holds for some curved surfaces: disk type minimal surfaces in space. Then in 1926 Weil [**We**] obtained the same result for disk type surfaces of negative Gaussian curvature. Thereafter a variety of different methods were employed by a dozen mathematicians to prove the same or more general inequalities; Bol [**Bo**] used parallel curves and Alexandrov [**Ax**] used the method of polyhedral approximation. Huber [**Hu**] improved the inequality of Carleman and its generalization to subharmonic functions by Beckenbach and Radó [**BR**]. In this section we give a new simple proof of $4\pi A \leq L^2$ for nonpositively curved surfaces using the maximum principle: Given a disk type nonpositively curved surface S, we construct a domain S in $\mathbb{R}^2$ with area larger than that of S and perimeter equal to that of S. Then the inequality for S follows immediately from the classical isoperimetric inequality for S.

THEOREM 13.1. *If S is a simply connected nonpositively curved surface, then* $4\pi A \leq L^2$.

Here we will give two proofs of this theorem. One is a geometric proof given in [BZ], and the other is more analytic in nature.

GEOMETRIC PROOF (OUTLINE). Consider the following special case. Suppose that S has connected smooth boundary all of whose parallel curves $l_t = \{x \in S : \operatorname{dist}(x, \partial S) = t\}$, except the furthest one l_r, are smooth simple closed curves. Denote by $A(t)$ the area of the set $S_t = \{x \in S : \operatorname{dist}(x, \partial S) < t\}$. Under these assumptions $A'(t) = l(t)$, where $l(t)$ is the length of l_t. But the first variation formula for $l(t)$ says

$$\frac{dl}{dt} = -\int_{l_t} k\,ds$$

where k is the geodesic curvature of l_t with respect to the inward normal to l_t. Assume that $A''(t) = l'(t)$ also exists and is continuous in $[0,r)$. By the Gauss–Bonnet formula for S_t, we have

$$A''(t) = -\int_{l_t} k\,ds = -\int_{S_t} K\,dA \;-\; \int_{\partial S} k\,ds.$$

Since $K \leq 0$, the Gauss–Bonnet formula for S implies

$$A''(t) \leq -\int_{S} K\,dA \;-\; \int_{\partial S} k\,ds = -2\pi. \tag{13–1}$$

Assume here that $r = \sup\{t : l_t \neq \phi\}$. Multiplying (13–1) by $2A'(t) \geq 0$ and integrating from 0 to r, we get

$$A'(r)^2 - A'(0)^2 \leq -4\pi[A(r) - A(0)].$$

Since $A'(0) = L,\ A(0) = 0,\ A(r) = A$, this yields

$$L^2 - 4\pi A \geq A'(r)^2 \geq 0,$$

which is the desired inequality.

In general, the assumptions on the structure of the parallel curves l_t and the differentiability of $A(t)$ do not hold. Nevertheless it is possible to obtain a rigorous proof along these lines. Such a proof is presented in [**BZ**], pp. 20-27. In order to overcome the technical difficulties, the argument is carried out for polyhedra. The general case follows by passing to the limit. □

Analytic Proof. Let x and y be isothermal coordinates on S. Then the metric and the Gaussian curvature K of S can be written as

$$ds^2 = e^{2\lambda}(dx^2 + dy^2), \qquad K = -e^{-2\lambda} \triangle \lambda. \tag{13–2}$$

By the curvature assumption we have

$$\triangle \lambda \geq 0 \text{ on } S. \tag{13–3}$$

Let h be the solution to the Dirichlet problem

$$\triangle h = 0 \text{ on } S, \qquad h = \lambda \text{ on } \partial S, \tag{13–4}$$

and introduce a surface $\tilde{S}$ which is S equipped with the flat metric $\tilde{g} = e^{2h}(dx^2 + dy^2)$. Actually $\tilde{S}$ is the image of S in the complex plane under the holomorhic map $\phi(z)$ such that $\log|\phi'(z)| = h(x, y), z = x + iy$. Define $\tilde{A} = \text{Area}(\tilde{S})$, $\tilde{L} = \text{Length}(\partial\tilde{S})$. Note that the boundary condition (13–4) implies $\tilde{L} = L$. From (13–3), (13–4) and the maximum principle we get $\tilde{A} \geq A$. But we have $4\pi\tilde{A} \leq \tilde{L}^2$ for $\tilde{S} \subset \mathbb{R}^2$. Hence

$$4\pi A \leq 4\pi\tilde{A} \leq \tilde{L}^2 = L^2.$$ □

In the analytic proof we used the Dirichlet boundary value problem. By solving the mixed boundary value problem, instead, we can get a relative isoperimetric inequality:

Theorem 13.2. **[C5]** *Let S be a disk type surface of nonpositive Gaussian curvature. Suppose that ∂S is the disjoint union of Γ_1 and Γ_2 such that Γ_1 is connected and concave, i.e., if $c(s)$ is an arclength parametrization of Γ_1, then $c''(s)$ vanishes or points outward from S. Then*

$$2\pi \text{Area}(S) \leq \text{Length}(\Gamma_2)^2 \tag{13–5}$$

and equality holds if and only if S is a flat half-disk.

Proof. The proof is similar to the analytic proof of the preceeding theorem. The difference is that here we solve the mixed boundary value problem

$$\triangle h = 0 \text{ on } S, \qquad h = \lambda \text{ on } \Gamma_2, \qquad \frac{\partial h}{\partial \nu} = 0 \text{ on } \Gamma_1$$

and that the concavity of the free part Γ_1 implies

$$\frac{\partial \lambda}{\partial \nu} \leq 0,$$

where ν is the outward unit normal to Γ_1. This is because

$$\begin{aligned} 0 &\geq \Big\langle \nabla e^{-\lambda}\partial/\partial y e^{-\lambda}\frac{\partial}{\partial y}, e^{-\lambda}\frac{\partial}{\partial x}\Big\rangle = -\Big\langle e^{-\lambda}\frac{\partial}{\partial y}, \nabla_{e^{-\lambda}\frac{\partial}{\partial y}} e^{-\lambda}\frac{\partial}{\partial x}\Big\rangle \\ &= -e^{-3\lambda}\Big\langle \frac{\partial}{\partial y}, \nabla_{\partial/\partial y}\frac{\partial}{\partial x}\Big\rangle = -e^{-3\lambda}\Big\langle \frac{\partial}{\partial y}, \nabla_{\partial/\partial x}\frac{\partial}{\partial y}\Big\rangle \\ &= -\tfrac{1}{2}e^{-3\lambda}\frac{\partial}{\partial x}\left|\frac{\partial}{\partial y}\right|^2 = -e^{-\lambda}\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial \nu}. \end{aligned}$$

Then, by the maximum principle, $h \geq \lambda$ on S and hence $\tilde{A} \geq A$. Clearly $\text{Length}(\Gamma_2)$ remains the same under the new metric $\tilde{g}$ and Γ_1 is a line segment in $\tilde{S}$. Therefore

(13–5) follows from the relative isoperimetric inequality for $\tilde{S} \subset \mathbb{R}^2$. See **[C5]** for more details. □

In this section we have seen that nonpositively curved two-dimensional surfaces satisfy the same isoperimetric inequality as $\mathbb{R}^2$. In regard to higher dimensional nonpositively curved Riemannian manifolds, Aubin conjectured that in the sense of the isoperimetric inequality, $\mathbb{R}^n$ is more efficient than any complete simply connected Riemannian manifold M^n of nonpositive sectional curvature. More precisely, he conjectured that for any domain D in M^n

$$n^n \omega_n \mathrm{Vol}(D)^{n-1} \leq \mathrm{Vol}(\partial D)^n$$

and equality holds if and only if D is a Euclidean ball. Recently Kleiner **[Kl]** and Croke **[Cr]** proved this inequality in M^3 and M^4, respectively; but this conjecture is still open for $n \geq 5$. Extending Aubin's conjecture to the case of relative isoperimetric inequality, we would like to propose the following:

OPEN PROBLEM 13.3. *Let C be a convex domain in a complete simply connected Riemannian manifold M^n of nonpositive sectional curvature and D a subset of $M \sim C$. Prove that*

$$\tfrac{1}{2}\, n^n \omega_n \mathrm{Vol}(D)^{n-1} \leq \mathrm{Vol}(\partial D \sim \partial C)^n,$$

where equality holds if and only if D is a Euclidean half ball.

As mentioned in the preceding section, this open problem has been solved for $n = 3$ in **[CR]** and for $n = 4$ in **[C7]**.

14. Isoenergy Inequalities

So far we have studied many isoperimetric inequalities that relate the volume of a domain with that of its boundary. In this last section, however, we shall consider a *map* from a domain into a manifold and derive an inequality that relates the interior energy of the map with its boundary energy.

Consider a C^2 harmonic map u from a closed unit ball $\bar{B} \subset \mathbb{R}^n$ to $\mathbb{R}^k, n \geq 2$. Define $E(u)$ and $E(u|_{\partial B})$ to be the energy of the map u and of the restriction of u to ∂B, respectively. Then is there any relationship between $E(u)$ and $E(u|_{\partial B})$ that resembles the isoperimetric inequality? Here we answer this question affirmatively; we obtain a relationship in a sharp form, called the *isoenergy* inequality, for a general target manifold N^k as well as $\mathbb{R}^k$. First, if N is a nonpositively curved k-dimensional Riemannian manifold, then we show

$$(n-1)E(u) \leq E(u|_{\partial B}),$$

where equality holds when $N = \mathbb{R}^k$ and u is a homothety or an orthogonal projection composed with a homothety. Second, when N^k is any Riemannian manifold, we prove

$$(n-2)E(u) \leq E(u|_{\partial B}),$$

where equality holds if $N = S^{n-1} \subset \mathbb{R}^n$ and $u(x) = x/|x|$.

In Subsection 14.2, the method of the isoenergy inequality will enable us to estimate an upper bound for the first eigenvalue of the Laplacian on a minimal submanifold in the sphere.

14.1. Isoenergy Inequality via Monotonicity. Assume that M^n, N^k are Riemannian manifolds with N^k isometrically embedded in $\mathbb{R}^m$. We look at a bounded map $u : M \to N$ whose first derivatives are in L^2; such a map is thought of as a map $u = (u_1, \ldots, u_m) : M \to \mathbb{R}^m$ having image almost everywhere in N. Then the *energy* $E(u)$ of u is defined by

$$E(u) = \int_M |\nabla u|^2,$$

where $|\nabla u|^2 = \sum_{i=1}^m |\nabla u_i|^2$, ∇u_i being the gradient of u_i on M. $|\nabla u|^2$ is called the *energy density* of u. The critical points of $E(u)$ on the space of maps are referred to as *harmonic maps.* Thus $u \in C^2$ is harmonic if and only if

$$\Delta_M \ u \ \perp \ T_u N. \tag{14–1}$$

A harmonic map u is *stationary* if its energy is critical with respect to variations of the type $u \circ F_t$, where $F_t : M \to M$ is a smooth path of diffeomorphisms of M fixing the boundary. It can be shown that stationary harmonic maps satisfy the monotonicity property for the scale invariant energy in balls. We state an equivalent form of the monotonicity in the following lemma.

LEMMA 14.1. *Let* $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ *and* $B = B_1$. *If* $u : B_{1+\epsilon} \to N^k$, $\epsilon > 0$, *is a stationary harmonic map, we have*

$$(n-2)\int_B |\nabla u|^2 = \int_{\partial B}\left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r}\right|^2\right), \qquad r = |x|. \tag{14–2}$$

PROOF. The monotonicity formula **[Pr]**, **[Sc]** says

$$\rho^{2-n}\int_{B_\rho}|\nabla u|^2 - \sigma^{2-n}\int_{B_\sigma}|\nabla u|^2 \ = \ 2\int_{B_\rho - B_\sigma} r^{2-n}\left|\frac{\partial u}{\partial r}\right|^2,$$

for $0 < \sigma < \rho < 1 + \epsilon$. Noting that $\int_{\partial B_\rho} f = \frac{d}{d\rho}\int_{B_\rho} f$ for almost all ρ, differentiate the formula with respect to ρ and set $\rho = 1$. □

The first isoenergy inequality of this section holds for a stationary harmonic map of $B_{1+\epsilon}$ into an arbitrary target manifold.

THEOREM 14.2. **[C3]** *Let* $n \geq 3$ *and suppose that* $u : B_{1+\epsilon} \to N^k$, $\epsilon > 0$, *is a stationary harmonic map into a Riemannian manifold* N. *Then*

$$(n-2)E(u|_B) \leq E(u|_{\partial B}),$$

where equality can be attained if $N = S^{n-1} \subset \mathbb{R}^n$ *and* $u(x) = x/|x|$.

PROOF. Let $\overline{\nabla} u_i$ denote the gradient of u_i on ∂B. Observe that

$$E(u|_{\partial B}) = \int_{\partial B}\sum_i |\overline{\nabla} u_i|^2 = \int_{\partial B}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right).$$

It follows from (14–2) that

$$(n-2)E(u|_B) = E(u|_{\partial B}) - \int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2, \tag{14–3}$$

which gives the desired inequality. If $u(x) = x/|x|$, then $|\partial u/\partial r| = 0$ and hence equality holds. □

Remark 1. (i) Lemma 14.1 and Theorem 14.2 fail to hold for nonstationary harmonic maps. See [**Po**][**Ri**] for such maps.

(ii) We should remark, in relation to Theorem 14.2, J.C.Wood's theorem that any smooth harmonic map u ($n \geq 2$) which is constant on ∂B is constant [**Wo**] (see also [**KW**]); the case for weakly harmonic maps is still open (see [**Sc**] for the definition of weakly harmonic maps).

(iii) When $n = 3, 4, 5, 6$, there is a sequence $\{\phi_i\}$ of C^2 harmonic maps $\phi_i : \bar{B} \to S^n \subset \mathbb{R}^{n+1}$ (see [**SY2**]) such that $\phi_i(x) = (x, 0)$ for $x \in \partial B$, $E(\phi_i) < E(\phi_{i+1})$, and

$$n - 2 = \inf_i \frac{E(\phi_i|_{\partial B})}{E(\phi_i)}.$$

Now we prove the isoenergy inequality for a harmonic map from $\bar{B}$ into $\mathbb{R}^k$. Although it is a special case of the isoenergy inequality for harmonic maps into a nonpositively curved space (Theorem 14.4), we state it independently because the proof of the Euclidean case is different and interesting in its own right.

THEOREM 14.3. [**C3**] *Suppose that u is a smooth harmonic map from $\bar{B} \subset \mathbb{R}^n, n \geq 2$, into $\mathbb{R}^k$. Then we have the isoenergy inequality*

$$(n-1)E(u) \leq E(u|_{\partial B}),$$

where equality holds if and only if u is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$.

PROOF. (14–1) implies

$$\Delta u_i = 0, \qquad i = 1, \ldots, k.$$

Hence

$$E(u) = \frac{1}{2}\int_B \Delta \sum_i u_i^2 = \int_{\partial B} \sum_i u_i \frac{\partial u_i}{\partial r} \leq \left(\int_{\partial B} \sum_i u_i^2\right)^{1/2} \left(\int_{\partial B} \left|\frac{\partial u}{\partial r}\right|^2\right)^{1/2},$$

where, without loss of generality, we assume $\int_{\partial B} u_i = 0$ for $i = 1, \ldots, k$. Using (14–3) and the fact that $n-1$ is the first eigenvalue of the Laplacian on ∂B, one sees that the right-hand side of the preceding display is at most

$$\left(\frac{1}{n-1}\int_{\partial B} \sum_i |\overline{\nabla} u_i|^2\right)^{1/2} \left(E(u|_{\partial B}) - (n-2)E(u)\right)^{1/2}.$$

Hence by combining the inequalities above one gets

$$E(u)^2 \leq \frac{1}{n-1} E(u|_{\partial B})\big(E(u|_{\partial B}) - (n-2)E(u)\big),$$

which gives the desired isoenergy inequality. Moreover equality holds if and only if u_i is a constant multiple of $\partial u_i / \partial r$ and

$$\Delta_{\partial B}\, u_i + (n-1)u_i = 0, \qquad i = 1, \ldots, k,$$

which holds if and only if u is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. □

THEOREM 14.4. [**C3**] *If u is a smooth harmonic map from $\bar{B} \subset \mathbb{R}^n, n \geq 2$, to a k-dimensional Riemannian manifold N of nonpositive sectional curvature, then*

$$(n-1)E(u) \leq E(u|_{\partial B}).$$

PROOF. The Bochner formula [**EL**] says that if $u : M^n \to N^k$ is harmonic then

(14–4)
$$\tfrac{1}{2}\Delta|\nabla u|^2 = ||\nabla' du||^2 - \sum_{\alpha,\beta} R_N(u_*e_\alpha, u_*e_\beta, u_*e_\alpha, u_*e_\beta) + \sum_i \mathrm{Ric}_M(u^*\theta_i, u^*\theta_i),$$

where ∇' is the pullback connection from TN, $e_1, \dots, e_n$ is an orthonormal basis for TM and $\theta_1, \dots, \theta_k$ is orthonormal for T^*N. Hence for $M = \bar{B}$ and N nonpositively curved, $|\nabla u|^2$ is subharmonic. Since the mean value of a subharmonic function on a sphere of radius r centered at the origin is monotonically nondecreasing as a function of r, one can deduce that

$$\frac{E(u)}{\omega_n} \le \frac{1}{n\omega_n}\int_{\partial B} |\nabla u|^2 = \frac{1}{n\omega_n}\left((n-2)E(u) + 2\int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2\right),$$

where equality follows from (14–2). So

$$E(u) \le \int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2. \tag{14–5}$$

Then adding (14–3) to (14–5) gives the isoenergy inequality. □

Remark 2. In case u is a harmonic map from a ball B_ρ of radius ρ into N, one obviously has

$$(n-1)E(u) \le \rho E(u|_{\partial B_\rho}).$$

When the target manifold N is nonpositively curved we have an extension theorem by Eells–Sampson [**ES**] and Hamilton [**Ha**]: Given $\phi \in C^3(\bar{B}, N)$, there is a harmonic map $u \in C^2(\bar{B}, N)$ such that $u = \phi$ on ∂B, and u is homotopic to ϕ. Since this theorem allows us to impose a condition on $u|_{\partial B}$, e.g. conformality, one can obtain a mixture of the isoenergy inequality and the isoperimetric inequality as follows.

COROLLARY 14.5. [**C3**] *Suppose N^k is nonpositively curved and let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$, $n = 2, 3$.*

(a) *If $u : \bar{B}^2 \to N$ is harmonic and $u|_{\partial B}$ is a constant speed map, then*

$$4\pi \mathrm{Area}(u(B)) \le 2\pi E(u) \le \mathrm{Length}(u(\partial B))^2.$$

(b) *If $u : \bar{B}^3 \to N$ is harmonic and $u|_{\partial B}$ is conformal, then*

$$E(u) \le \mathrm{Area}(u(\partial B)).$$

(c) *If $u : \bar{B}^n \to N$ is harmonic and $u|_{\partial B}$ is conformal, then*

$$(n\omega_n)^{3-n}E(u)^{n-1} \le \mathrm{Vol}(u(\partial B))^2.$$

PROOF. The first inequality in (a) is well known. For the second, use the constant speed condition and Theorem 14.4. Part (b) is a special case of (c). For (c), let k^2 be the conformal factor of $u|_{\partial B}$. Then

$$\begin{aligned} E(u) &\le \frac{1}{n-1}E(u|_{\partial B}) = \int_{\partial B} k^2 \le \left(\int_{\partial B} 1\right)^{(n-3)/(n-1)} \left(\int_{\partial B} k^{n-1}\right)^{2/(n-1)} \\ &= (n\omega_n)^{(n-3)/(n-1)}\, \mathrm{Vol}(u(\partial B))^{2/(n-1)}. \end{aligned}$$ □

14.2. Eigenvalue Estimate. Given an $(n-1)$-dimensional minimal submanifold Σ in $S^l \subset \mathbb{R}^{l+1}$, $O \Join \Sigma$ is the cone from the origin O of $\mathbb{R}^{l+1}$ over Σ, that is, the union of the unit line segments from O to the points of Σ. It is well known that $O \Join \Sigma$ is an n-dimensional minimal submanifold of $\mathbb{R}^{l+1}$. In this section we want to consider the isoenergy inequality of a harmonic map u from $O \Join \Sigma$ into $\mathbb{R}^k$. In the proof of the isoenergy inequality of Theorem 14.3 we used the fact that $n-1$ is the first eigenvalue of the Laplacian on S^{n-1}. However, we do not know the exact value of the first eigenvalue $\lambda_1(\Sigma)$ of the minimal submanifold $\Sigma \subset S^l$. Therefore, instead of deriving an isoenergy inequality, we obtain, by reversing the argument, an upper bound of the first eigenvalue in terms of the energy of the harmonic map u and its boundary energy. To do this, we need the following monotonicity on $O \Join \Sigma$.

LEMMA 14.6. *Let Σ be an $(n-1)$-dimensional submanifold of $S^l \subset \mathbb{R}^{l+1}$. If u is a harmonic map from $O \Join \Sigma$ into $\mathbb{R}^k$ which is C^2 up to and including the boundary Σ, then*

$$(n-2)\int_{O \Join \Sigma} |\nabla u|^2 = \int_\Sigma \left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r}\right|^2 \right).$$

PROOF. Let $\Sigma_\rho = \{x \in O \Join \Sigma : |x| < \rho\}$. Note that the quantity

$$\Theta(\rho) = \rho^{2-n} \int_{\Sigma_\rho} |\nabla u|^2 \tag{14–6}$$

is invariant under scaling. More precisely, if we denote $u^\rho(x) = u(\rho x)$, then

$$\Theta(\rho) = \int_{\Sigma_1} |\nabla u^\rho|^2.$$

So at $\rho = 1$ we see that

$$\frac{d}{d\rho}\Theta(\rho) = 2\sum_{i=1}^k \int_{\Sigma_1} \left\langle \nabla u_i, \nabla \frac{du_i^\rho}{d\rho} \right\rangle = 2\sum_{i=1}^k \int_\Sigma \langle x, \nabla u_i \rangle \frac{du_i^\rho}{d\rho} - 2\sum_{i=1}^k \int_{\Sigma_1} \triangle u_i \frac{du_i^\rho}{d\rho}.$$

Since $\langle x, \nabla u_i \rangle = du_i^\rho/d\rho$, $du^\rho/d\rho = \partial u/\partial r$ on Σ and $\triangle u_i = 0$ on Σ_1, we get

$$\left(\frac{d}{d\rho}\Theta(\rho) \right)_{\rho=1} = 2\int_\Sigma \left| \frac{\partial u}{\partial r} \right|^2. \tag{14–7}$$

Thus (14–6) and (14–7) complete the proof. □

THEOREM 14.7. *Let Σ be an $(n-1)$-dimensional submanifold of $S^l \subset \mathbb{R}^{l+1}$ and let $u : O \Join \Sigma \to \mathbb{R}^k$ be a harmonic map which is C^2 up to and including the boundary Σ. Then*

$$\lambda_1(\Sigma) \le \frac{E(u|_\Sigma)}{E(u)} \left(\frac{E(u|_\Sigma)}{E(u)} - n + 2 \right). \tag{14–8}$$

PROOF. We follow the proof of Theorem 14.3 and use Lemma 14.6. So

$$E(u) = \frac{1}{2}\int_{O \Join \Sigma} \Delta \sum_i u_i^2 = \int_\Sigma \sum_i u_i \frac{\partial u_i}{\partial r} \le \left(\int_\Sigma \sum_i u_i^2 \right)^{1/2} \left(\int_\Sigma \left| \frac{\partial u}{\partial r} \right|^2 \right)^{1/2}$$
$$\le \left(\frac{1}{\lambda_1(\Sigma)} \int_\Sigma \sum_i |\overline{\nabla} u_i|^2 \right)^{1/2} \bigl(E(u|_\Sigma) - (n-2)E(u) \bigr)^{1/2}.$$

Hence

$$\lambda_1(\Sigma)E(u)^2 + (n-2)E(u)E(u|_\Sigma) - E(u|_\Sigma)^2 \le 0,$$

which gives (14–8). ☐

Corollary 14.8. *Let* Σ *be an embedded minimal hypersurface in* S^n. *If* $u : O{\times\!\!\!\times}\Sigma \to \mathbb{R}^k$ *is harmonic and* C^2 *up to and including the boundary, then*

$$(n-1)E(u) \le (-n+2+\sqrt{n^2-2n+2})E(u|_\Sigma).$$

Proof. Combine (14–8) with Choi–Wang's estimate [**CW**]:

$$\frac{n-1}{2} \le \lambda_1(\Sigma).$$ ☐

Remark 3. In a sense (14–8) is similar to Chavel's estimate [**Ch**]:

$$\lambda_1(S) \le \frac{(n-1)}{n^2}\frac{A^2}{V^2},$$

where V is the volume of an n-dimensional minimal submanifold S of an m-dimensional complete simply connected nonpositively curved Riemannian manifold M and A is the volume of ∂S. Chavel's estimate, when applied to $S = O{\times\!\!\!\times}\Sigma \subset M = \mathbb{R}^{n+1}$ with Σ^{n-1} minimal in $S^n \subset \mathbb{R}^{n+1}$, implies $\lambda_1(\Sigma) \le n-1$, which is nothing new. Also our estimate (14–8), when applied to u = identity, draws the same conclusion because $(n-1)E(\mathrm{id}) = E(\mathrm{id}|_\Sigma)$. But should there exist a harmonic map $u : O{\times\!\!\!\times}\Sigma \to \mathbb{R}^k$ satisfying

$$(n-1)E(u) \;>\; E(u|_\Sigma), \tag{14–9}$$

then one would be able to conclude from (14–8) that

$$\lambda_1(\Sigma) \;<\; n-1, \tag{14–10}$$

which would disprove Yau's conjecture [**Ya**, Problem 100]. In fact, since $\mathrm{Ric}_{O{\times\!\!\!\times}\Sigma}$ is nonpositive, one could deduce from the Bochner formula (14–4) that $|\nabla u|^2$ is strictly superharmonic provided

$$||\nabla' du||^2 + \sum_i \mathrm{Ric}_{O{\times\!\!\!\times}\Sigma}(u^*\theta_i, u^*\theta_i) \;<\; 0.$$

Then the argument of the proof of Theorem 14.4 would imply (14–9).

Open Problem 14.9. *Does there exist a harmonic map* $u : O{\times\!\!\!\times}\Sigma \to \mathbb{R}^k$ *satisfying* (14–9)*?*

Open Problem 14.10. *Let* Σ *be an* $(n-1)$*-dimensional embedded minimal hypersurface of* S^n. *A map* $u : \Sigma \to S^n \subset \mathbb{R}^{n+1}$ *is said to be* balanced *if* $\int_\Sigma u$ *equals the zero vector in* $\mathbb{R}^{n+1}$. *Does there exist a balanced energy minimizing map of* Σ *into* S^n *which is different from the identity? If it exists, is its energy smaller than that of the identity?*

If the answer to Open Problem 14.10 is affirmative, then one gets (14–10) since

$$\lambda_1(\Sigma)\mathrm{Vol}(\Sigma) = \lambda_1(\Sigma)\int_\Sigma |u|^2 \le \int_\Sigma |\nabla u|^2 < \; E(\mathrm{id}) = (n-1)\mathrm{Vol}(\Sigma).$$

For a minimal surface of codimension ≥ 2, that case really occurs. Let $\psi : S^2(\sqrt{3}) \to S^4$ be the two-to-one locally isometric minimal immersion whose image is the Veronese surface V diffeomorphic to the projective plane. Let $\widetilde{V}$ be the double covering of V and define $u : \widetilde{V} \to S^4$ by $u = \frac{1}{\sqrt{3}}\psi^{-1}$. Then u is a balanced harmonic map satisfying

$$E(u) = \tfrac{1}{3}E(\mathrm{id}).$$

Indeed $\lambda_1(\widetilde{V}) = \frac{2}{3}$.

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