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EMBEDDED MINIMAL SURFACES AND TOTAL CURVATURE OF CURVES IN A MANIFOLD

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Abstract. Let M^n be an n-dimensional complete simply connected Riemannian manifold with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. It is proved that every branched minimal surface in M bounded by a smooth Jordan curve Γ with total curvature $\leq 4\pi + \kappa^2 \inf_{p\in M} \text{Area}(p⨉\Gamma)$ is embedded. $p⨉\Gamma$ denotes the geodesic cone over Γ with vertex p. It follows that a Jordan curve Γ in M^3 with total curvature $\leq 4\pi + \kappa^2 \inf_{p\in M} \text{Area}(p⨉\Gamma)$ is unknotted. In the hemisphere $\mathbf{S}^n_+$, we prove the embeddedness of any minimal surface whose boundary curve has total curvature $\leq 4\pi - \sup_{p\in \mathbf{S}^n_+} \text{Area}(p⨉\Gamma)$.

1. Introduction

After the formidable problem of Plateau in Euclidean $\mathbf{R}^n$ was settled by Douglas and Radó, mathematicians' attention was drawn to the uniqueness and embeddedness of their solutions (see [D] and [R1].). The first uniqueness result was obtained by Radó ([R2], p. 100). He proved that if a simple closed curve $\Gamma \subset \mathbf{R}^3$ has a one-to-one projection onto the boundary of a convex region $R \subset \mathbf{R}^2$, then Γ bounds a unique minimal disk. In fact any minimal surface bounded by Γ is a graph over R, and hence is simply connected and embedded. Later Nitsche [N2] showed that if Γ is analytic with total curvature $\leq 4\pi$, then Γ bounds exactly one minimal disk.

The embeddedness of the minimal disk bounded by a Jordan curve Γ was first obtained by Gulliver and Spruck [GS] under the assumption that Γ has total curvature $\leq 4\pi$ and is extreme (that is, it lies on the boundary of a convex set). In the same paper, they conjectured that either condition alone would be sufficient for the embeddedness of an area-minimizing disk. Moreover Nitsche himself asked whether his unique solution is free of self-intersection ([N3], esp. p. 463). Indeed Tomi-Tromba [TT], Almgren-Simon [AS], and Meeks-Yau [MY] derived the embeddedness of a minimal disk bounded by an extreme Γ; [MY] proved embeddedness of any area-minimizing disk. But the sufficiency of the total curvature condition alone, when Γ is not assumed to be extreme, remained open for 25 years.

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Finally, in a very recent paper, Ekholm, White, and Wienholtz [EWW] ingeniously proved the embeddedness of any minimal surface bounded by a curve Γ in $\mathbf{R}^n$ with total curvature $\leq 4\pi$. Their idea is based on the following observations.

(i) The logarithm of the distance function $\rho(x) = d(x,p)$ in $\mathbf{R}^n$ is a fundamental solution of the Laplacian on a two-dimensional plane through p. Similarly, $\log \rho(x)$ is harmonic on a cone $p⋇\Gamma$ over Γ with vertex p. By contrast, $\log \rho(x)$ is strictly subharmonic on a nonplanar (branched) minimal surface in $\mathbf{R}^n$. This part of their proof is intimately related to the well-known monotonicity formula.

(ii) By the Gauss-Bonnet theorem, 2π times the density at p of the cone $p⋇\Gamma$ is at most the total curvature of Γ.

In this paper we extend the Ekholm-White-Wienholtz result to minimal surfaces in an n-dimensional Riemannian manifold M with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. The two observations above can be appropriately generalized for these purposes. Thus, it is proved that if Γ is a Jordan curve in M^n with total curvature

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2 \inf_{p\in M} \mathrm{Area}(p⋇\Gamma),$$

then every branched minimal surface bounded by Γ is embedded (Theorem 3.) More precisely, the infimum of area is taken only over geodesic cones with vertex lying in the convex hull $\mathcal{H}_{\mathrm{cvx}}(\Gamma)$ of Γ. In the presence of variable ambient curvatures, a key point is the introduction of a new metric of constant Gauss curvature on $p⋇\Gamma$.

A similar theorem is also proved for minimal surfaces in the hemisphere $\mathbf{S}^n_+$, using $\kappa^2 = -1$ (see Theorem 1.) This case is simpler, since only one metric is needed on $p⋇\Gamma$, and will be demonstrated first. In this paper, we have not carried out the extension of our results to continuous Jordan curves, as was done in [M] and in [EWW].

As in [EWW], our theorem has a topological implication: any Jordan curve in M^3 with total curvature $\leq 4\pi + \kappa^2 \inf_{p\in M}$ Area$(p⋇\Gamma)$ is unknotted. This appears to be a new extension of the Fáry-Milnor theorem, which showed that any knotted curve in $\mathbf{R}^3$ has total curvature greater than 4π [F], [M]. Brickell and Hsiung proved our unknotting result for the case when M^3 is the hyperbolic space of constant sectional curvature $-\kappa^2$ (see Theorem 4 of [BH].) It should also be mentioned that Schmitz [S] and Alexander-Bishop [AB] obtained the unknottedness of a Jordan curve with total curvature $\leq 4\pi$ in a simply connected Riemannian 3-manifold of nonpositive sectional curvature, which is the case $\kappa = 0$ of our Theorem 4. Alexander and Bishop also noted that the minimum total curvature among knotted curves in any non-positively curved 3-manifold is exactly 4π. But for the case of a manifold M^3 with sectional curvature $\leq -\kappa^2 < 0$ our hypothesis on the total curvature of Γ is weaker, and more natural, since there are no homotheties, and thus no scaling, in M^3.

One indication of the naturalness of our hypothesis, that a curve $\Gamma \subset M^n$ have total curvature $\leq 4\pi + \kappa^2 \inf_{p \in M}$ Area$(p \divideontimes \Gamma)$, is the fact that every closed curve in M has total curvature at least $2\pi + \kappa^2 \inf_{p\in M}$ Area$(p \divideontimes \Gamma)$.

2. Embeddedness of Minimal Surfaces in the Hemisphere

Recall that in the open hemisphere $\mathbf{S}^n_+ := \{x \in \mathbf{R}^{n+1} : |x| = 1, x_{n+1} > 0\}$, any two points p, q may be connected by a unique geodesic, namely the shorter arc of the unique great circle of $\mathbf{S}^n$ passing through p and q. In particular, for any point $p \in \mathbf{S}^n_+$ and any immersed curve Γ in $\mathbf{S}^n_+$, we may define the *geodesic cone* $p \divideontimes \Gamma$ to be the union of the geodesic segments from p to q, over all $q \in \Gamma$. The smallest closed subset of $\mathbf{S}^n_+$ which contains a set $S \subset \mathbf{S}^n_+$ and contains the geodesic segment between any two of its points is the *convex hull* of S, and will be written as $\mathcal{H}_{\mathrm{cvx}}(S)$. Observe that, since $\mathbf{S}^n_+$ is a space form, $\mathcal{H}_{\mathrm{cvx}}(S)$ may also be described as the intersection of all closed hemispheres containing S. It follows that if Σ is an immersed minimal surface in $\mathbf{S}^n_+$ with compact closure, whose boundary $\partial\Sigma \subset S$, then $\Sigma \subset \mathcal{H}_{\mathrm{cvx}}(S)$.

Definition 1. Define the *maximum cone area* of a curve $\Gamma \subset \mathbf{S}^n_+$ as

$$\overline{\mathcal{A}}(\Gamma) := \sup_{p \in \mathcal{H}_{\mathrm{cvx}}(\Gamma)} \mathrm{Area}(p \divideontimes \Gamma).$$

Theorem 1. *Let Γ be a C^2 Jordan curve in the n-dimensional hemisphere $\mathbf{S}^n_+$. Suppose Σ^2 is a branched minimal surface, having compact closure in $\mathbf{S}^n_+$ and boundary $\Gamma = \partial\Sigma$. If the total curvature of Γ satisfies*

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi - \overline{\mathcal{A}}(\Gamma), \tag{1}$$

then $\overline{\Sigma}$ is an embedding.

In the definition of $\mathcal{C}_{\mathrm{tot}}$, $\vec{k}$ denotes the curvature vector of Γ. If a point traverses Γ with unit speed, then its acceleration vector in $\mathbf{S}^n_+$ coincides with $\vec{k}$. A branched minimal surface is one which may fail to be immersed at a discrete set of singularities, which are all branch points; see Definition 2 below.

We shall give the proof of Theorem 1 at the end of this section.

Theorem 1 has an interesting topological consequence: a new extension of the Fáry-Milnor Theorem. The Fáry-Milnor Theorem showed that a knotted curve in Euclidean $\mathbf{R}^3$ has total curvature at least 4π ([F], [M].) The next theorem is what we feel is an appropriate analogue of the Fáry-Milnor Theorem, when $\mathbf{R}^3$ is replaced by $\mathbf{S}^3_+$. We are not aware of any previous results on total curvature of knots in $\mathbf{S}^3_+$. Note that the bound required from above on total curvature in this theorem may be zero or even negative, in which case the theorem fails. However, in Example 1 below, we shall show that the bound is sharp, in the sense that there are knotted curves in $\mathbf{S}^3_+$ for which the total curvature is close to zero and the maximum cone area is close to 4π.

Theorem 2. *If* Γ *is a* C^2 *Jordan curve in* $\mathbf{S}^3_+$, *with total curvature*

$$\int_\Gamma |\vec{k}|\, ds \le 4\pi - \overline{\mathcal{A}}(\Gamma),$$

then Γ *is unknotted.*

Proof. It follows from a theorem of Morrey that there is a smooth branched immersion of the disk into $\mathbf{S}^3_+$ with boundary Γ, having smallest area among surfaces of the type of the disk. Morrey's result [Mo] requires the ambient manifold M^3 to be complete and homogeneously regular. Recall that homogeneous regularity is an appropriately weak version of bounded geometry; see [Mo]. In order to apply Morrey's result to our case, we first need to construct a complete and homogeneously regular manifold M^3 in place of $\mathbf{S}^3_+$. Since Γ is compact, it lies in a closed geodesic ball $B_R \subset \mathbf{S}^3_+$ of radius $R < \pi/2$, with center the point of rotational symmetry $p_0 \in \mathbf{S}^3_+$. We extend B_R isometrically to a Riemannian manifold M diffeomorphic to $\mathbf{R}^3$, with a rotationally symmetric metric, so that M is complete and homogeneously regular, and the distance balls B_r of M from p_0 are convex, $0 < r < \infty$. To make M homogeneously regular, we may choose the metric to have e. g. the cylindrical form $\mathbf{S}^2_b \times [r_1, \infty)$ outside a compact set. Morrey's result shows that there is a smooth branched immersion of the disk into M with boundary Γ, having smallest area among surfaces of the type of the disk. Write its closed image as Σ. Since Σ is compact, it lies in B_{r_0} for some r_0, and since each B_r is convex, $R \le r < \infty$, we see by the maximum principle that $\Sigma \subset B_R$. Therefore $\Sigma \subset \mathbf{S}^3_+$.

According to Theorem 1, this area-minimizing disk must be an embedding of the disk into $\mathbf{S}^3_+$ with boundary Γ; this shows that Γ is unknotted. □

An alternative proof of Theorem 2 may be given for a real-analytic curve Γ, and by approximation for a C^2 curve which satisfies $\mathcal{C}_{\text{tot}}(\Gamma) < 4\pi - \overline{\mathcal{A}}(\Gamma)$. The alternate proof requires Theorem 1 only for an immersed minimal surface Σ, and cites the result that the area-minimizing branched immersion from the disk into $\mathbf{S}^3_+$ with boundary Γ must be an immersion up to the boundary (see [A], [G] and [GL].)

Example 1. *With this example, we shall show that the hypothesis*

$$\mathcal{C}_{\text{tot}}(\Gamma) \le 4\pi - \overline{\mathcal{A}}(\Gamma)$$

of Theorems 1 and 2 (which may appear very strong from a certain point of view) is actually sharp.

Let Γ_0 be the double cover of the circle of some radius $R < \pi/2$ in a totally geodesic $\mathbf{S}^2_+ \subset \mathbf{S}^3_+$, with center at p_0. This example is a family of $(2, 2m+1)$-torus knots Γ_η in $\mathbf{S}^3_+$, $\eta > 0$, for any fixed positive integer m, such that the C^2 distance between Γ_η and Γ_0 as parameterized curves approaches zero as $\eta \to 0$, and such that

$$\mathcal{C}_{\text{tot}}(\Gamma_\eta) < 4\pi - \overline{\mathcal{A}}(\Gamma_\eta) + \eta.$$

To be specific, we might choose Γ_η to lie on the boundary of the tubular neighborhood of Γ_0 at a radius which tends to 0 as $\eta \to 0$.

We first compute the geometric invariants of Γ_0. Its length is $4\pi \sin R$, and its curvature is constant: $|\vec{k}| \equiv \cot R$. Thus, $\mathcal{C}_{\mathrm{tot}}(\Gamma_0) = 4\pi \cos R$. The maximum cone area $\overline{\mathcal{A}}(\Gamma_0) = 2 \cdot 2\pi \int_0^R \sin r\, dr = 4\pi(1 - \cos R)$ is achieved by the double cover of the totally geodesic disk of radius R, since this disk is the convex hull $\mathcal{H}_{\mathrm{cvx}}(\Gamma_0)$ of Γ_0. Thus, equality holds in hypothesis (1) for Γ_0: $\mathcal{C}_{\mathrm{tot}}(\Gamma_0) = 4\pi - \overline{\mathcal{A}}(\Gamma_0)$. But both of the geometric invariants $\overline{\mathcal{A}}(\Gamma)$ and $\mathcal{C}_{\mathrm{tot}}(\Gamma)$ are continuous as Γ varies in C^2. We find therefore $\mathcal{C}_{\mathrm{tot}}(\Gamma_\eta) < 4\pi - \overline{\mathcal{A}}(\Gamma_\eta) + \eta$, as claimed, showing that hypothesis (1) is sharp.

If we choose $R = R(\eta) \to \pi/2$, then we may obtain further that $\mathcal{C}_{\mathrm{tot}}(\Gamma_\eta) \to 0$. □

Propositions 1 and 2 below will form the core of the proof of Theorem 1.

For the rest of this section, we shall write $G(r) := \log \tan(r/2)$ for the Green's function of the two-dimensional sphere. Choose a point $p \in \mathbf{S}^n_+$, and for all $x \in \mathbf{S}^n_+$, define $\rho(x) := d(x, p)$, the distance measured in $\mathbf{S}^n_+$.

Lemma 1. *Let N^2 be a two-dimensional manifold immersed in $\mathbf{S}^n_+$. Then except at p,*

$$\triangle_N G(\rho) = 2\,\frac{\cos\rho}{\sin^2\rho}\left(1 - |\nabla_N \rho|^2\right) + \frac{d\rho(\vec{H})}{\sin\rho},$$

where $\vec{H}$ denotes the mean curvature vector of N.

Proof. In $\mathbf{S}^n_+$, the Hessian of the distance function is $\overline{\nabla}^2 \rho = \cot\rho\,(g - \overline{\nabla}\rho \otimes \overline{\nabla}\rho)$, where g is the metric tensor of $\mathbf{S}^n_+$. The trace formula states that

$$\triangle_N G = \sum_{\alpha=1}^{2} \overline{\nabla}^2 G(e_\alpha, e_\alpha) + dG(\vec{H}),$$

where $\{e_1, e_2\}$ is an orthonormal basis for the tangent plane to N. These formulas are well known (see e. g. [CG2], pp. 172, 174.) Choosing $\{e_1, e_2\}$ with $d\rho(e_2) = 0$ and $d\rho(e_1) = |\nabla_N \rho|$, we have

$$\overline{\nabla}^2 G(e_1, e_1) = \frac{\cos\rho}{\sin^2\rho}(1 - 2\,d\rho(e_1)^2)$$

and

$$\overline{\nabla}^2 G(e_2, e_2) = \frac{\cos\rho}{\sin^2\rho}.$$

The conclusion follows. □

Definition 2. Let Ω be a Riemann surface, k a positive integer. A mapping $f : \Omega \to M^n$ has a branch point of order k at $w_0 \in \Omega$ if its complex first partial derivative $f_w := \frac{1}{2}(f_u - i f_v)$ satisfies $\lim_{w \to w_0}[f_w(w)(w - w_0)^{-k}] = \vec{a} \in \mathbf{C}^n \backslash \{0\}$. Here u and v are the real and imaginary parts of the local complex variable $w \in \Omega$, and $i = \sqrt{-1}$.

A branched minimal surface $f : \Omega \to M^n$ is a conformally parameterized harmonic mapping. By abuse of language, we shall also refer to the image $\Sigma = f(\Omega)$ of f as a branched minimal surface.

It may be shown that each point of a branched minimal surface either is a branch point or has an immersed neighborhood; moreover, the real and imaginary parts of the complex vector $\vec{a}$ in the definition of a branch point are orthogonal and have equal length (see [HH]). The importance of branched minimal surfaces stems from the fact that the solution of Plateau's problem for a minimal surface of a given topological type in $\mathbf{R}^n$ or in M^n is not an immersion in general, but only a branched immersion. Solutions to this variational problem are necessarily immersions only when $n = 3$ ([A], [G], [GL]), or when the boundary curve meets hyperplanes of $\mathbf{R}^n$ in at most five points ([R2], pp. 34–35), or when the topological type is not prescribed ([Fed], [HS].)

The following lemma describes the effect of branch points on area and divergence-theorem computations on a branched minimal surface. Part **(b)** shows that if $p \notin \Sigma$, then there is no effect on the integral of $\triangle_\Sigma G$. The conclusion of part **(a)** may be interpreted to say that for some purposes, Σ acts like the $(k+1)$-fold cover of a smooth surface near a branch point of order k.

Lemma 2. *Let $\Sigma = f(\Omega)$ be a branched minimal surface in a Riemannian manifold M.*

(a) *Let $p = f(w_0)$, $w_0 \in \overline{\Omega}$, be a branch point of Σ of order k. If ν_Σ is the unit normal vector to $\Sigma \cap \partial B_\varepsilon(p)$ tangent to Σ and pointing towards p, then as $\varepsilon \to 0$, $\nu_\Sigma \to -\overline{\nabla}\rho$ uniformly on $\Sigma \cap \partial B_\varepsilon(p)$. After rescaling to unit radius, the curve $\Sigma \cap \partial B_\varepsilon(p)$ converges in C^1 norm to the constant-speed $(k+1)$-fold cover (resp. half of the constant-speed $(k+1)$-fold cover) of a great circle in the unit sphere of $T_p(M)$, if $w_0 \in \Omega$ (resp. $w_0 \in \partial\Omega$). Moreover, if $w_0 \in \partial\Omega$ and f maps $\partial\Omega$ monotonically to a C^2 curve Γ, then k is even.*

(b) *If $p \notin \Sigma$, then*
$$\int_\Sigma \triangle_\Sigma G\, dA = \int_{\partial\Sigma} \nu_\Sigma \cdot \overline{\nabla} G\, ds,$$
where ν_Σ is the outward unit normal vector to $\partial\Sigma$ tangent to Σ.

Proof. Choose local conformal coordinates for Ω near w_0 and Riemannian normal coordinates for M at p. Write $\vec{a} = \lim_{w\to w_0} \big((w-w_0)^{-k} f_w(w)\big) =: \vec{b} + i\vec{c}$, where the real vectors $\vec{b}$ and $\vec{c}$ are orthogonal and have the same length (see [HH].) Then as $w \to w_0$, the tangent plane to Σ at $f(w)$ converges to the plane in $T_p(M)$ spanned by $\vec{b}$ and $\vec{c}$. Integration shows that $f(w) - f(w_0)$ is the real part of $\frac{2}{k+1}\vec{a}(w-w_0)^{k+1}$, modulo a term which tends to zero faster than $|w-w_0|^{k+1}$. The parity of k at a boundary branch point was shown in [N1], p. 332. The conclusions of part **(a)** follow.

To prove part **(b)**, we apply part **(a)** to each branch point $q_i = f(w_i)$ of Σ, $1 \le i \le m$. The divergence theorem on $\Sigma \setminus \cup_{i=1}^m B_\varepsilon(q_i)$ leads to the m additional boundary terms

$$\int_{\Sigma\cap\partial B_\varepsilon(q_i)} \nu_\Sigma \cdot \overline{\nabla} G \, ds.$$

Since $p \notin \Sigma$, $\nu_\Sigma \cdot \overline{\nabla} G$ is uniformly bounded in a neighborhood of q_i, while the length of $\Sigma \cap \partial B_\varepsilon(q_i)$ approaches 0 by part **(a)**, so these additional boundary terms tend to 0 as $\varepsilon \to 0$. $\square$

Corollary 1. *If Σ^2 is a branched minimal surface in $\mathbf{S}^n_+$, then $G(\rho(x)) = \log\tan(\rho(x)/2)$ is subharmonic on Σ. If C is the cone $p \circledast \partial\Sigma$ over the pole p of the distance function ρ, then $G(\rho)$ is harmonic on C, except at p.*

Proof. Since ρ is a distance function in $\mathbf{S}^n_+$, $|\nabla_\Sigma \rho| \leq 1$, while on the cone, since the $\mathbf{S}^n_+$-gradient $\overline{\nabla}\rho$ is tangent to C, $|\nabla_C \rho| \equiv 1$. The mean curvature vector of Σ vanishes, and the mean curvature vector of C is orthogonal to the gradient $\overline{\nabla}\rho$. Lemmas 1 and 2**(a)** now imply that $\triangle_\Sigma G(\rho) \geq 0$ and $\triangle_C G(\rho) \equiv 0$, except at p.

If $p \in \Sigma$, then the outward normal derivative of $G(\rho)$ on $\partial B_\varepsilon(p) \cap \Sigma$ approaches $+\infty$ as $\varepsilon \to 0$ (if p is a branch point of Σ, Lemma 2**(a)** will be useful here), which implies that G is subharmonic everywhere on Σ. $\square$

For a 2-dimensional immersed Lipschitz surface, or a branched surface, $N^2 \subset \mathbf{S}^n_+$, we define the *density of N at q* to be the limit

$$\Theta_N(q) := \lim_{\varepsilon\to 0} \frac{\mathrm{Area}(N \cap B_\varepsilon(q))}{\pi\varepsilon^2}. \tag{2}$$

Here, $B_\varepsilon(q)$ is the geodesic ball of $\mathbf{S}^n_+$ with spherical radius ε, centered at q. Note that when N is smooth or a cone, we may also compute the density in terms of lengths:

$$\Theta_N(q) = \lim_{\varepsilon\to 0} \frac{L(N \cap \partial B_\varepsilon(q))}{2\pi\varepsilon}.$$

Of course, the same limit is also obtained if the denominators in these two quotients are replaced by the spherical area $2\pi(1-\cos\varepsilon)$ and spherical length $2\pi\sin\varepsilon$, respectively. Observe that if N is a smoothly immersed submanifold and has a self-intersection at $p \in \mathbf{S}^n_+$, then $\Theta_N(p) \geq 2$. Also note that if p is a branch point of N of order k, then $\Theta_N(p) \geq k+1$ (see the proof of Lemma 2**(a)**.)

Proposition 1 (Density Comparison)**.** *Let Γ be a C^2 immersed closed curve in $\mathbf{S}^n_+$. Choose $p \in \mathbf{S}^n_+ \backslash \Gamma$. If Σ^2 is a branched minimal surface in $\mathbf{S}^n_+$ with boundary $\partial\Sigma = \Gamma$, and C is the cone $p \circledast \Gamma$ over p, then their densities at p satisfy the inequality*

$$\Theta_\Sigma(p) < \Theta_C(p),$$

unless Σ is totally geodesic.

Proof. By Corollary 1, we have $\triangle_\Sigma G(\rho) \geq 0$ and $\triangle_C G(\rho) \equiv 0$, where $G(\rho(x)) := \log\tan(\rho(x)/2)$ and $\rho(x) := d(x,p)$. For small $\varepsilon > 0$, write $C_\varepsilon := C \backslash B_\varepsilon(p)$, and similarly Σ_ε. Then the boundary of Σ_ε is $\Gamma \cup (\Sigma \cap \partial B_\varepsilon(p))$. Let ν_Σ (ν_C, respectively) be the outward unit normal vector tangent to Σ_ε at $\partial\Sigma_\varepsilon$ (to C_ε at ∂C_ε, resp.). Then

$$0 \leq \int_{\Sigma_\varepsilon} \triangle_\Sigma G(\rho)\, dA = \int_{\partial\Sigma_\varepsilon} \nu_\Sigma \cdot \overline{\nabla} G\, ds = \int_{\Sigma\cap\partial B_\varepsilon(p)} \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sin\varepsilon}\, ds + \int_\Gamma \frac{\nu_\Sigma \cdot \overline{\nabla}\rho}{\sin\rho}\, ds.$$

Along the small boundary component $\Sigma \cap \partial B_\varepsilon(p)$, as $\varepsilon \to 0$, $\nu_\Sigma \cdot \overline{\nabla}\rho \to -1$ uniformly, and

$$\frac{L(\Sigma \cap \partial B_\varepsilon(p))}{2\pi\sin\varepsilon} \to \Theta_\Sigma(p).$$

Along Γ, $\nu_\Sigma \cdot \overline{\nabla}\rho \leq \nu_C \cdot \overline{\nabla}\rho$. Hence as $\varepsilon \to 0$, we find

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \frac{\nu_C \cdot \overline{\nabla}\rho}{\sin\rho}\, ds.$$

Similarly, along $C \cap \partial B_\varepsilon(p)$, we have $\nu_C \equiv -\overline{\nabla}\rho$. After applying the divergence theorem to the vector field $\nabla_C G(\rho)$ on C_ε, we find

$$2\pi\Theta_C(p) = \int_\Gamma \frac{\nu_C \cdot \overline{\nabla}\rho}{\sin\rho}\, ds. \tag{3}$$

This implies $\Theta_\Sigma(p) \leq \Theta_C(p)$. If equality holds, then $\triangle_\Sigma G \equiv 0$, which requires $|\nabla_\Sigma\rho| \equiv 1$ according to Lemma 1. This can only happen when Σ is totally geodesic.

We have tacitly assumed that $C\backslash\{p\}$ is immersed in M. Equation (3) may be proved in the general case either by analysis in singular coordinates or by approximation; we shall carry out an appropriate approximation argument at the end of the proof of the next proposition. ☐

Proposition 2 (Gauss-Bonnet)**.** *Consider the geodesic cone $C = p \divideontimes \Gamma$ over an immersed C^2 curve Γ in $\mathbf{S}^n_+$, $n \geq 2$.*

(a) *If $p \notin \Gamma$, then*

$$2\pi\Theta_C(p) = \text{Area}(C) - \int_\Gamma \vec{k} \cdot \nu_C\, ds.$$

(b) *If $p \in \Gamma$, then*

$$2\pi\Theta_C(p) = \text{Area}(C) - \int_\Gamma \vec{k} \cdot \nu_C\, ds - \pi.$$

Proof. We first assume that $C\backslash\{p\}$ is immersed in $\mathbf{S}^n_+$.

Consider case **(a)**, where $p \notin \Gamma$. By the Gauss-Bonnet formula on C_ε, for ε less than the distance from p to Γ,

$$\int_{C_\varepsilon} K\, dA - \int_\Gamma \vec{k} \cdot \nu_C\, ds - \int_{C\cap\partial B_\varepsilon(p)} \vec{k} \cdot \nu_C\, ds = 2\pi\chi(C_\varepsilon). \tag{4}$$

where χ is the Euler number and K is the intrinsic Gauss curvature of C. Since C_ε is an immersed annulus, we have $\chi(C_\varepsilon) = 0$. Now C has principal curvature zero in the $\overline{\nabla}\rho$ direction, so the determinant of its second fundamental form vanishes, and by the Gauss equation, K equals the sectional curvature $\overline{K} = 1$ of the ambient $\mathbf{S}^n_+$.

Along $C \cap \partial B_\varepsilon(p)$, $\nu_C = -\overline{\nabla}\rho$ and $\vec{k} \cdot \nu_C \equiv \cot \varepsilon$. Thus, we may compute

$$\lim_{\varepsilon \to 0} \int_{C \cap \partial B_\varepsilon(p)} \vec{k} \cdot \nu_C \, ds = \lim_{\varepsilon \to 0} (\cot \varepsilon)\, L(C \cap \partial B_\varepsilon(p)) = 2\pi \Theta_C(p), \tag{5}$$

so that formula (4) implies

$$\operatorname{Area}(C) - \int_\Gamma \vec{k} \cdot \nu_C \, ds - 2\pi \Theta_C(p) = 0, \tag{6}$$

which proves Proposition 2**(a)** when $C \backslash \{p\}$ is an immersion.

The proof of part **(b)** is analogous. However, when $p \in \Gamma$, for small ε, C_ε is a topological disk, so that $\chi(C_\varepsilon) = 1$. Also, the boundary of C_ε consists of the arc $C \cap \partial B_\varepsilon(p)$ and the arc $\Gamma_\varepsilon := \Gamma \backslash B_\varepsilon(p)$. For small $\varepsilon > 0$, these arcs meet at two points forming exterior angles $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. Equation (4) becomes

$$\int_{C_\varepsilon} K \, dA - \int_{\Gamma_\varepsilon} \vec{k} \cdot \nu_C \, ds - \int_{C \cap \partial B_\varepsilon(p)} \vec{k} \cdot \nu_C \, ds + \alpha(\varepsilon) + \beta(\varepsilon) = 2\pi.$$

Since Γ is smooth, $\alpha(\varepsilon) \to \pi/2$ and $\beta(\varepsilon) \to \pi/2$ as $\varepsilon \to 0$, which yields

$$\operatorname{Area}(C) - \int_\Gamma \vec{k} \cdot \nu_C \, ds - 2\pi \Theta_C(p) = \pi, \tag{7}$$

and Proposition 2**(b)** follows.

In general, the cone $C = p \times \Gamma$ need not be an immersion away from p. The problem arises exactly on the set $A \subset \Gamma$ where Γ is tangent to the radial geodesic from p, that is, the unit tangent vector $\vec{T}$ coincides with $\pm\overline{\nabla}\rho$. Let us choose a C^1 mapping $\vec{T_\delta}$ from Γ into the unit tangent bundle of $\mathbf{S}^n_+ \backslash \{p\}$, which is C^1-close to $\vec{T}$ and transverse to the two sections $\pm\overline{\nabla}\rho$. The two sections $\pm\overline{\nabla}\rho$ define a codimension-$(n-1)$ submanifold of the total space of the unit tangent bundle. If $n \geq 3$, transversality means that $\vec{T_\delta}$ is disjoint from this submanifold. If $n = 2$, we first embed $\mathbf{S}^2_+$ as a totally geodesic surface in $\mathbf{S}^3_+$, and then require transversality for $\vec{T_\delta}$. In order to ensure that $\vec{T_\delta}$ is the tangent vector to a *closed* curve Γ_δ, we adjust $\vec{T_\delta}$ to satisfy the $n-1$ closure conditions, for small δ. In the case $p \in \Gamma$, we may require $p \in \Gamma_\delta$. Then $p \times \Gamma_\delta$ satisfies formula (3), and formula (6) or (7), if $p \notin \Gamma$ or $p \in \Gamma$, respectively.

We claim that, since $\Gamma_\delta \to \Gamma$ in the C^2 norm, each term of equation (3), (6) or (7) is the limit, as $\delta \to 0$, of the corresponding quantity for Γ_δ. To be precise, it should be observed that in general, the cone $C = p \times \Gamma$ is only $C^{1,1}$ up to the boundary Γ. Namely, the *outward* unit normal vector ν_C satisfies $\nu_C \cdot \overline{\nabla}\rho \geq 0$. For q in the set $A \subset \Gamma$, $\nu_C(q)$ is nonunique; clearly, for $q_\delta \to q$, $q_\delta \in \Gamma_\delta$, the normal vectors $\nu_{C_\delta}(q_\delta)$ need not converge. Nonetheless, the inward geodesic curvature $k = -\vec{k} \cdot \nu_C$ is well defined almost everywhere on Γ, since $\vec{k} = 0$ almost everywhere on the problematic set A. Similarly, $\nu_C \cdot \overline{\nabla}\rho$ is well defined almost everywhere on Γ. Both k and $\nu_C \cdot \overline{\nabla}\rho$ are pointwide limits almost everywhere of the corresponding quantities for Γ_δ, which are uniformly bounded. The dominated convergence theorem now implies that formula (3), and either formula (6) or (7), hold for any C^2 curve $\Gamma \subset \mathbf{S}^n_+$. $\square$

Proof of Embedding Theorem 1. Let Σ^2 be a branched minimal surface in $\mathbf{S}^n_+$ whose boundary $\partial\Sigma = \Gamma$ is a C^2 Jordan curve satisfying the hypothesis (1):

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi - \overline{\mathcal{A}}(\Gamma).$$

Note that $\Sigma \subset \mathcal{H}_{\mathrm{cvx}}(\Gamma)$ by the maximum principle. To show that Σ has no interior branch points and is embedded, it suffices to show that $\Theta_\Sigma(p) < 2$ for all $p \in \Sigma$ ($p \notin \Gamma$).

Choose $p \in \Sigma$, and let $C = p \divideontimes \Gamma$ be the geodesic cone over Γ with vertex p. If Σ is totally geodesic, then it is the subset of a totally geodesic $\mathbf{S}^2_+$ bounded by the embedded curve $\Gamma \subset \mathbf{S}^2_+$, so Σ is embedded. Otherwise, by Propositions 1 and 2**(a)**, we have

$$2\pi\Theta_\Sigma(p) < 2\pi\Theta_C(p) = -\int_\Gamma \vec{k} \cdot \nu_C\, ds + \mathrm{Area}(C).$$

Since $p \in \mathcal{H}_{\mathrm{cvx}}(\Gamma)$, $\mathrm{Area}(C)$ is less than or equal to the maximum cone area $\overline{\mathcal{A}}(\Gamma)$. But $-\vec{k} \cdot \nu_C \leq |\vec{k}|$, so hypothesis (1) implies $\Theta_\Sigma(p) < 2$, as required.

It remains to rule out boundary branch points (in the case $n = 3$ of Theorem 2, this would follow by well-known arguments from embeddedness in the interior, e.g. [GL].) If $p \in \Gamma$, then by Propositions 1 and 2**(b)**, unless Σ is totally geodesic, we have

$$2\pi\Theta_\Sigma(p) < 2\pi\Theta_C(p) = -\int_\Gamma \vec{k} \cdot \nu_C\, ds + \mathrm{Area}(C) - \pi.$$

Using hypothesis (1) as before, we find that $\Theta_\Sigma(p) < 3/2$. For a boundary branch point p of order k, the density $\Theta_\Sigma(p) \geq (k+1)/2$, and k is even, by Lemma 2**(a)**. This would imply that $\Theta_\Sigma(p) \geq 3/2$, which is impossible. We have shown that $\overline{\Sigma}$ is embedded. $\square$

3. Embeddedness of Minimal Surfaces in Negatively Curved Spaces

We now turn our attention to the case of nonpositive ambient sectional curvature. For a minimal surface in hyperbolic space, embeddedness may be proved in complete analogy to section 2 above, with $-\overline{\mathcal{A}}(\Gamma)$ replaced in hypothesis (1) by the infimum of areas of cones. However, unlike the case of $\mathbf{S}^n_+$, the nonpositively curved case can be significantly improved to permit variable sectional curvature, and the inequalities require only a nonpositive upper bound $-\kappa^2$ on ambient sectional curvature.

Thus, throughout this section we assume that M is an n-dimensional complete, simply connected Riemannian manifold with sectional curvature bounded above by a nonpositive constant $-\kappa^2$. Let Γ be a C^2 immersed curve in M. We define the *(geodesic) cone* $C = p{\divideontimes}\Gamma$ *over* Γ *with vertex* p as the union of the geodesic segments from p to q, over all $q \in \Gamma$. Since the geodesic joining any two points of M is unique and depends smoothly on its endpoints, $C\backslash\{p\}$ is the image of a C^2 mapping.

The main tool which will be added to the methods employed in Section 2 above is comparison with a metric $\widehat{g}$ of constant Gauss curvature $-\kappa^2$ on the geodesic cone C; see Definition 4 below. This metric was introduced by the first author in his study of isoperimetric inequalities on minimal surfaces ([C].)

Definition 3. Define the *minimum cone area* of Γ as

$$\mathcal{A}(\Gamma) := \inf_{p\in\mathcal{H}_{\mathrm{cvx}}(\Gamma)} \mathrm{Area}(p{\divideontimes}\Gamma).$$

Remark 1. A refinement of the methods of this paper would be to replace the convex hull of Γ in Definitions 1 and 3 with the (usually) smaller mean-curvature hull of Γ. This would allow Theorems 1 and 3 to be proved with slightly weaker hypotheses. The *mean-curvature hull* of a subset $S \subset M$ is defined as the intersection of the closures of C^2 open subsets of M which contain S, have boundaries of nonnegative mean curvature (with respect to the inward unit normal), and which are members of a continuous exhaustion of M by open subsets whose boundaries have nonnegative mean curvature. It follows that if Σ is a branched minimal surface in M with compact closure, then Σ lies inside the mean-curvature hull of $\partial\Sigma$.

In this regard, it should be noted that Brickell and Hsiung actually proved the unknotting Theorem 4 for the special case when M^3 is the hyperbolic space of constant sectional curvature $-\kappa^2$, and the infimum of area is taken only over cones whose vertices lie on Γ itself (see [BH].)

Theorem 3. *Let Σ^2 be a branched minimal surface (of arbitrary topological type) in an n-dimensional complete, simply connected Riemannian manifold M whose sectional curvature is bounded above by a nonpositive constant $-\kappa^2$. Write $\Gamma = \partial\Sigma$, which we assume to be a C^2 Jordan curve, i.e. a C^2 embedding of the circle S^1. If the total curvature of Γ satisfies*

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \leq 4\pi + \kappa^2\,\mathcal{A}(\Gamma), \tag{8}$$

then $\overline{\Sigma}$ is an embedding.

We shall give the proof of Theorem 3 at the end of this section.

Theorem 3 implies a substantial extension of the Fáry-Milnor Theorem, which was proved for $\kappa = 0$ in [AB] and [S]. The proof of the following theorem is similar to the proof of Theorem 2 above.

Theorem 4. *Let Γ be a C^2 Jordan curve in a complete, simply connected Riemannian 3-manifold M with sectional curvature $\leq -\kappa^2$. If the total curvature* *of* Γ *satisfies*

$$\int_\Gamma |\vec{k}|\, ds \le 4\pi + \kappa^2 \mathcal{A}(\Gamma),$$

then Γ *is unknotted.*

Example 2. *This example shows that the hypothesis*

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) \le 4\pi + \kappa^2 \mathcal{A}(\Gamma)$$

of Theorems 3 and 4 is sharp.

Let Γ_0 be the double cover of the circle of radius R in a totally geodesic $\mathbf{H}^2 \subset \mathbf{H}^3$. Here $\mathbf{H}^n$ is the n-dimensional hyperbolic space of constant sectional curvature $-\kappa^2 = -1$. In a similar fashion to Example 1, given any choice of positive integer m, the example is a one-parameter family of $(2, 2m+1)$-torus knots Γ_η in $\mathbf{H}^3$, $\eta > 0$, with $\Gamma_\eta \to \Gamma_0$ and with

$$\mathcal{C}_{\mathrm{tot}}(\Gamma_\eta) < 4\pi + \mathcal{A}(\Gamma_\eta) + \eta.$$

In fact, Γ_0 has length $4\pi \sinh R$, curvature $|\vec{k}| \equiv \coth R$, $\mathcal{C}_{\mathrm{tot}}(\Gamma_0) = 4\pi \cosh R$, and $\mathcal{A}(\Gamma_0) = 4\pi(\cosh R - 1)$. □

We shall now present six results, in preparation for the proof of Theorem 3.

Write $G(r) := \log \tanh(\kappa r/2)$ for the Green's function of the two-dimensional hyperbolic plane $\mathbf{H}^2(-\kappa^2)$ with Gauss curvature $\equiv -\kappa^2 < 0$, and $G(r) := \log r$ for $\mathbf{R}^2$, if $\kappa = 0$. We compute $dG/dr = \kappa/\sinh \kappa r$ or $dG/dr = 1/r$, respectively. Choose a point $p \in M$, and define $\rho(x) := d(x, p)$, using the distance function $d(\cdot, \cdot)$ of M.

Lemma 3. *Let* N^2 *be a two-dimensional manifold immersed in a complete, simply connected Riemannian manifold* M *whose sectional curvature is bounded above by* $-\kappa^2$, $\kappa \ge 0$. *Then*

(a) *except at* p,

$$\triangle_N G(\rho) \ge 2\kappa^2 \frac{\cosh \kappa\rho}{\sinh^2 \kappa\rho} \left(1 - |\nabla_N \rho|^2\right) + \kappa \frac{d\rho(\vec{H})}{\sinh \kappa\rho} \text{ in case } \kappa > 0,$$

and

$$\triangle_N G(\rho) \ge \frac{2}{\rho^2} \left(1 - |\nabla_N \rho|^2\right) + \frac{d\rho(\vec{H})}{\rho} \text{ in case } \kappa = 0,$$

where $\vec{H}$ *is the mean curvature vector of* N.

(b)

$$\triangle_N \log(1 + \cosh \kappa\rho) \ge \kappa^2 + \kappa \tanh(\kappa\rho/2)\, d\rho(\vec{H}) \text{ in case } \kappa > 0,$$

and

$$\triangle_N \rho^2 \ge 4 + 2\rho\, d\rho(\vec{H}) \text{ in case } \kappa = 0.$$

Proof. By the Hessian comparison theorem, the Hessian of the distance function ρ of M satisfies

$$\overline{\nabla}^2\rho \geq \kappa\coth\kappa\rho(g - \overline{\nabla}\rho\otimes\overline{\nabla}\rho) \text{ for } \kappa > 0, \text{ and } \overline{\nabla}^2\rho^2 \geq 2g \text{ for } \kappa = 0,$$

where g is the metric tensor of M (see [SY], p. 4).

As in the proof of Lemma 1, after applying the trace formula, this inequality leads us to the conclusion of part **(a)**.

For the proof of part **(b)**, we again use the trace formula and note that

$$\overline{\nabla}^2\log(1+\cosh\kappa\rho) \geq \frac{\kappa^2}{1+\cosh\kappa\rho}\left[\cosh\kappa\rho\cdot g + (1-\cosh\kappa\rho)\overline{\nabla}\rho\otimes\overline{\nabla}\rho\right] \text{ for } \kappa > 0.$$

□

For a 2-dimensional immersed Lipschitz submanifold, or a branched surface, $N \subset M$ and a point $q \in M$, we define the *density of* N *at* q to be the limit

$$\Theta_N(q) := \lim_{\varepsilon\to 0}\frac{\text{Area}(N\cap B_\varepsilon(q))}{\pi\varepsilon^2} \tag{9}$$

as in definition (2) above. As observed in section 2 above, if N is a smoothly immersed submanifold of M and has a self-intersection at $p \in M$, then $\Theta_N(p) \geq 2$. Further, if p is an interior branch point of N of order k, then $\Theta_N(p) \geq k+1$; at a boundary branch point, $\Theta_N(p) \geq (k+1)/2$.

Let Γ be a C^2 immersed closed curve in M. Choose $p \in M$. If Σ^2 is a branched minimal surface in M with boundary $\partial\Sigma = \Gamma$, and C is the cone $p{\times\!\!\!\times}\Gamma$ over p, then the key ingredient in the proof of Theorem 3 is to give an upper bound of $\Theta_\Sigma(p)$ by $\Theta_C(p)$. Unfortunately this is impossible unless M is rotationally symmetric about p. To get around this difficulty we need to define a constant-curvature metric $\widehat{g}$ on C as follows.

Definition 4. Let $\widehat{g}$ be a new metric on C with constant Gauss curvature $-\kappa^2$ such that the distance from p remains the same as in the original metric g, and so does the arclength element of Γ. More precisely, every geodesic from p under g remains a geodesic of equal length under $\widehat{g}$, the length of any arc of Γ remains the same, and the angles between the tangent vector to Γ and the geodesic from p remain unchanged.

We shall write $\widehat{C}$ for the two-dimensional Riemannian manifold $(C, \widehat{g})$, which is singular at p. In order to construct $\widehat{C}$, we may start with an arc-length parameter s along Γ. Let $r(s)$ be the distance in C from the corresponding point of Γ to p. Then choose a point $\widehat{p} \in \mathbf{H}^2(-\kappa^2)$, and let a curve $\widehat{\Gamma}$ locally isometric to Γ be traced out in $\mathbf{H}^2(-\kappa^2)$ so that the distance from $\widehat{p}$ equals $r(s)$. Let $\widehat{C} = \widehat{p}{\times\!\!\!\times}\widehat{\Gamma}$, which may be in a covering of $\mathbf{H}^2(-\kappa^2)$ branched over $\widehat{p}$, and finally glue $\widehat{C}$ along the geodesic segments from $\widehat{p}$ to the initial and final points (cf. [C], p. 211.) Note that the angle between two geodesics at p becomes larger under $\widehat{g}$, as we shall see in Proposition 5 below.

Corollary 2.

(a) *If Σ^2 is a branched minimal surface in M, then $G(\rho)$ is subharmonic on Σ.*

(b) *If $\widehat{C}$ is the cone $p \divideontimes \partial\Sigma$ over the pole p of the distance function ρ in M with the metric $\widehat{g}$ of Gauss curvature $\equiv -\kappa^2$, then $G(\rho)$ is harmonic on $\widehat{C}$, except at p.*

(c) *Further, on $\widehat{C}$*

$$\triangle_{\hat{C}} \log(1 + \cosh \kappa\rho) = \kappa^2 \text{ for } \kappa > 0, \text{ and}$$
$$\triangle_{\hat{C}} \rho^2 = 4 \text{ for } \kappa = 0.$$

Proof.

(a) On Σ, the mean curvature vector of Σ vanishes and $|\nabla_\Sigma \rho| \le 1$, hence $\triangle_\Sigma G(\rho) \ge 0$, except at p, according to Lemma 3**(a)**. Near p, we argue as in the proof of Corollary 1.

(b) On the cone $\widehat{C}$, however, we apply Lemma 3**(a)** with $M = N = \widehat{C}$, so that $\vec{H} \equiv 0$ and $|\nabla_{\hat{C}} \rho| \equiv 1$. Moreover constancy of the Gauss curvature on $\widehat{C}$ forces all the inequalities in the proof of Lemma 3**(a)** to become equality and consequently $\triangle_{\hat{C}} G(\rho) \equiv 0$.

(c) Similarly for part **(c)**.

$\square$

Remark 2. The following four propositions treat the cone $C = p \divideontimes \Gamma$. In the proof of each, it is convenient to assume that the cone is immersed except at p. This implies that $\widehat{C} \backslash \{p\}$ is a smooth two-dimensional manifold with Gauss curvature $\widehat{K} \equiv -\kappa^2$. This assumption entails no loss of generality, since, as a curve in M, Γ is the C^2 limit of closed curves Γ_δ with the property that $p \divideontimes \Gamma_\delta$ is immersed except at p. Specifically, the geodesic curvatures k and $\widehat{k}$ considered below, and the normal derivative $\nu_C \cdot \overline{\nabla} \rho$ of ρ, are the pointwise limits almost everywhere of the corresponding quantities for Γ_δ. This may be proven as at the end of the proof of Proposition 2 above.

Proposition 3 (Density Comparison)**.** *Let Σ^2 be a branched minimal surface in an n-dimensional simply connected Riemannian manifold M with sectional curvature $\le -\kappa^2$. If $\widehat{C}$ is as in Definition 4 above, then $\Theta_\Sigma(p) < \Theta_{\hat{C}}(p)$ unless Σ is totally geodesic with constant Gauss curvature $-\kappa^2$.*

Proof. By Corollary 2, we have $\triangle_\Sigma G(\rho) \ge 0$ and $\triangle_{\hat{C}} G(\rho) \equiv 0$, where, as above, $G(\rho(x)) := \log \tanh(\kappa\rho(x)/2)$ and $\rho(x) := d_M(x,p)$ or $d_{\hat{C}}(x,p)$ respectively. For small $\varepsilon > 0$, write $\widehat{C}_\varepsilon := \widehat{C} \backslash B_\varepsilon(p)$ and $\Sigma_\varepsilon := \Sigma \backslash B_\varepsilon(p)$, where $B_\varepsilon(p)$ denotes the geodesic ball in M of radius ε and center p. Then the boundary of Σ_ε is $\Gamma \cup (\Sigma \cap \partial B_\varepsilon(p))$. (The component $\Sigma \cap \partial B_\varepsilon(p)$ may be empty.) Let ν_Σ be the outward unit normal vector tangent to Σ_ε at $\partial \Sigma_\varepsilon$. Then

$$0 \le \int_{\Sigma_\varepsilon} \triangle_\Sigma G(\rho)\, dA = \int_{\partial \Sigma_\varepsilon} \nu_\Sigma \cdot \overline{\nabla} G\, ds = \int_{\Sigma \cap \partial B_\varepsilon(p)} \kappa \frac{\nu_\Sigma \cdot \overline{\nabla} \rho}{\sinh \kappa\varepsilon}\, ds + \int_\Gamma \kappa \frac{\nu_\Sigma \cdot \overline{\nabla} \rho}{\sinh \kappa\rho}\, ds.$$

Along the small boundary component $\Sigma \cap \partial B_\varepsilon(p)$, as $\varepsilon \to 0$, $\nu_\Sigma \cdot \overline{\nabla}\rho \to -1$ uniformly, and

$$\kappa \frac{L(\Sigma \cap \partial B_\varepsilon(p))}{2\pi \sinh \kappa\varepsilon} \to \Theta_\Sigma(p).$$

Let ν_C be the outward unit normal vector tangent to C along its boundary. Then it should be noted that

$$\nu_\Sigma \cdot \overline{\nabla}\rho \leq \nu_C \cdot \overline{\nabla}\rho \text{ along } \Gamma.$$

Thus, we find that the inequality above implies

$$2\pi\Theta_\Sigma(p) \leq \int_\Gamma \kappa \frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh \kappa\rho}\, ds. \tag{10}$$

Note here that ν_C, considered as a tangent vector to C, is also the outward unit normal vector in the metric $\widehat{g}$. Along the intrinsic distance sphere $\partial\widehat{B}_\varepsilon(p) \subset \widehat{C}$, $-\nabla\rho$ is the outward unit normal vector tangent to $\widehat{C}_\varepsilon$. Hence by Corollary 2(b), assuming $C\backslash\{p\}$ is immersed, as $\varepsilon \to 0$,

$$0 = \int_{\widehat{C}_\varepsilon} \triangle_{\hat{C}} G(\rho)\, dA \to -2\pi\Theta_{\hat{C}}(p) + \int_\Gamma \kappa \frac{\nu_C \cdot \nabla\rho}{\sinh \kappa\rho}\, ds.$$

See Remark 2 for the non-immersed case. Therefore, by inequality (10),

$$2\pi\Theta_{\hat{C}}(p) = \int_\Gamma \kappa \frac{\nu_C \cdot \overline{\nabla}\rho}{\sinh \kappa\rho}\, ds \geq 2\pi\Theta_\Sigma(p),$$

which is the desired estimate.

If equality holds, then $\triangle_\Sigma G \equiv 0$, which requires $|\nabla_\Sigma \rho| \equiv 1$ according to Lemma 3. But this means that Σ is a cone over p, as well as being minimal, which can only occur when Σ is totally geodesic. Moreover, $\triangle_\Sigma G \equiv 0$ now implies that $\triangle_\Sigma \rho \equiv \kappa \coth \kappa\rho$, which, along with $K_\Sigma \leq -\kappa^2$, implies that Σ has constant Gauss curvature $K_\Sigma \equiv -\kappa^2$. □

Proposition 4 (Geodesic Curvature Comparison)**.** *Let Γ be a C^2 curve in M^n, a manifold with sectional curvatures $\leq -\kappa^2$, and let C be the cone $p \times\!\!\!\!\times \Gamma$. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as in Definition 4 above, then $k(q) \geq \widehat{k}(q)$ for almost all $q \in \Gamma$, where k and $\widehat{k}$ denote the inward geodesic curvatures of Γ in C and $\widehat{C}$, respectively.*

Proof. We first assume that $C\backslash\{p\}$ is immersed. For $\rho_0 > 0$, let $\Gamma_0 = C \cap \partial B_{\rho_0}(p)$, and let k_0 be the geodesic curvature of Γ_0 in C. Also, let $\widehat{k}_0$ be the geodesic curvature of Γ_0 in $\widehat{C}$. To estimate k_0 and $\widehat{k}_0$ let us define V ($\widehat{V}$, respectively) to be a Jacobi field in C ($\widehat{C}$, respectively) along the unit-speed geodesic γ from p to $q \in \Gamma$, satisfying

$$V(p) = \widehat{V}(p) = 0 \text{ and } V \perp \dot{\gamma},\ \widehat{V} \perp \dot{\gamma}. \tag{11}$$

For each $q \in \Gamma$, since $g = \widehat{g}$ along Γ, we may also impose the boundary conditions

$$V(q) = \widehat{V}(q), |V(q)| = |\widehat{V}(q)| = 1, \tag{12}$$

thereby determining V and $\widehat{V}$ uniquely, since K and $\widehat{K}$, the Gauss curvatures of C and $\widehat{C}$ respectively, are nonpositive. In fact, $V = \widehat{V}$ as vector fields on $C\backslash\{p\}$. V and $\widehat{V}$ satisfy the Jacobi equations

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V = R(\dot\gamma, V)\dot\gamma \quad\text{and}\quad \widehat{\nabla}_{\dot\gamma}\widehat{\nabla}_{\dot\gamma}\widehat{V} = \widehat{R}(\dot\gamma, \widehat{V})\dot\gamma, \tag{13}$$

where $\nabla, \widehat{\nabla}$ denote the connections for the metrics $g, \widehat{g}$ respectively, while $R, \widehat{R}$ denote the Riemann curvature tensors of g and $\widehat{g}$, respectively. Write $f(t) = ||V(\gamma(t))||$, and similarly $\widehat{f}(t) = ||\widehat{V}(\gamma(t))||$, where the norms are measured using g and $\widehat{g}$, respectively. Since C and $\widehat{C}$ have dimension 2, equations (13) are equivalent to the scalar Jacobi equations

$$f''(t) + K(\gamma(t))f(t) = 0, \quad \widehat{f}''(t) + \widehat{K}(\gamma(t))\widehat{f}(t) = 0. \tag{14}$$

By the Gauss equation we have

$$K = R_M(\dot\gamma, V, V, \dot\gamma)/||V||^2 + \det(B),$$

where R_M is the Riemann curvature tensor of M and B is the second fundamental form of C in M. Since C is a cone, we have $\det(B) = 0$, and it follows that C has Gauss curvature

$$K \leq -\kappa^2.$$

We next compute k_0 and $\widehat{k}_0$. Extend V and $\widehat{V}$ as normal Jacobi fields along all radial geodesics from p. Also, let W be the unit vector field which is tangent to the radial geodesics. Then $[V, W] \equiv 0$ and $\langle V, W\rangle \equiv 0$. Similarly, $[\widehat{V}, W] \equiv 0$ and $\langle \widehat{V}, W\rangle \equiv 0$. Then

$$||V||^2 k_0 = -\langle \overline{\nabla}_V V, W\rangle = \langle V, \overline{\nabla}_V W\rangle = \langle V, \overline{\nabla}_{\dot\gamma} V\rangle = \dot\gamma(||V||^2)/2 = f'(t)f(t).$$

Thus $k_0(\gamma(t)) = f'(t)/f(t)$. Similarly, we compute $\widehat{k}_0(\gamma(t)) = \widehat{f}'(t)/\widehat{f}(t)$. As is well known, the scalar Jacobi equations (14) are equivalent to the Riccati equations

$$k_0'(\gamma(t)) + k_0(\gamma(t))^2 = -K(\gamma(t)) \geq \kappa^2,$$

and

$$\widehat{k}_0'(\gamma(t)) + \widehat{k}_0(\gamma(t))^2 = -\widehat{K}(\gamma(t)) = \kappa^2.$$

It follows that the difference satisfies a homogeneous linear differential inequality

$$(k_0 - \widehat{k}_0)' + (k_0 + \widehat{k}_0)(k_0 - \widehat{k}_0) = -K + \widehat{K} \geq 0.$$

Meanwhile, $k_0 - \widehat{k}_0 = (f'\widehat{f} - \widehat{f}'f)/(\widehat{f}f) \to 0$ as $t \to 0$, as follows from L'Hospital's rule using the equations (14). Therefore

$$f'/f - \widehat{f}'/\widehat{f} = k_0 - \widehat{k}_0 \geq 0. \tag{15}$$

We are now in a position to compare the respective inward geodesic curvatures k and $\widehat{k}$ of Γ. Write $T = (V/f)\cos\varphi - W\sin\varphi$ for the unit tangent vector to Γ: T has unit length with respect to either metric g or $\widehat{g}$. Then $\nabla_T T = -k\,\nu_C$ and $\widehat{\nabla}_T T = -\widehat{k}\,\nu_C$, where $\nu_C = (V/f)\sin\varphi + W\cos\varphi$ is the outward unit normal vector to Γ, with respect to either metric, and $\cos\varphi \geq 0$. We compute $\nabla_W W = \nabla_W(V/f) = 0$, $\nabla_{V/f}(V/f) = -k_0 W$ and $\nabla_V W = k_0 V$. It follows in a straightforward fashion that $-k\,\nu_C = \nabla_T T = -k_0\,\nu_C \cos\varphi - \nu_C T(\varphi)$. Thus $k = k_0 \cos\varphi + T(\varphi)$, and similarly $\widehat{k} = \widehat{k}_0 \cos\varphi + T(\varphi)$. Hence

$$k - \widehat{k} = (k_0 - \widehat{k}_0)\cos\varphi \geq 0.$$

Remark 2 now implies that $k \geq \widehat{k}$ almost everywhere in the general case where $C\backslash\{p\}$ need not be immersed. $\square$

Remark 3. The proof of Proposition 4 holds more generally, for any two metrics g, $\widehat{g}$ on a cone which have the same unit-speed geodesics from the vertex, agree at the boundary, and whose Gaussian curvatures satisfy $K \leq \widehat{K}$.

Proposition 5 (Density and Area Comparison)**.** *Let Γ be a C^2 curve in M^n, and let $C = p⋇\Gamma$, as in Proposition 4. If $\widehat{C}$ is the cone C with the constant curvature metric $\widehat{g}$, as in Definition 4 above, then the densities $\Theta_C(p) \leq \Theta_{\hat{C}}(p)$ and the areas* $\mathrm{Area}(C) \leq \mathrm{Area}(\widehat{C})$.

Proof. The inequality (15) above implies that $f(t)/\widehat{f}(t)$ is increasing. Recalling the normalization $f = \widehat{f}$ at each $q \in \Gamma$ and $f = \widehat{f} = 0$ at p, we see that $f(t) \leq \widehat{f}(t)$ along γ, $f' \geq \widehat{f}'$ at q, and $f' \leq \widehat{f}'$ at p. Note that $\mathrm{Area}(C)$ and $\mathrm{Area}(\widehat{C})$ may be written as the same double integral with respective integrands f and $\widehat{f}$. $\square$

Remark 4. We note here an interesting inequality, related to Proposition 5 above, although we will not need it in this paper:

$$\mathrm{Area}(\Sigma) \leq \mathrm{Area}(\widehat{C}).$$

The proof follows analogously to Proposition 3, using Lemma 3(b) and Corollary 2.

Proposition 6 (Gauss-Bonnet)**.**

(a) *For any geodesic cone $\widehat{C} = p⋇\Gamma, p \notin \Gamma$, with constant curvature $-\kappa^2$ over an immersed C^2 curve Γ in M^n, $n \geq 2$,*

$$2\pi\Theta_{\hat{C}}(p) + \kappa^2\,\mathrm{Area}(\widehat{C}) = \int_\Gamma \widehat{k}\; ds,$$

where $\widehat{k}$ is the geodesic curvature of Γ in $\widehat{C}$.

(b) *If $p \in \Gamma$, then*

$$2\pi\Theta_{\hat{C}}(p) + \kappa^2\,\mathrm{Area}(\widehat{C}) = \int_\Gamma \widehat{k}\; ds - \pi.$$

Proof. **(a)** Consider $p \notin \Gamma$. By the Gauss-Bonnet formula on $\widehat{C}_\varepsilon := \widehat{C}\backslash B_\varepsilon(p)$,

$$\int_{\hat{C}_\varepsilon} \widehat{K}\,dA + \int_\Gamma \widehat{k}\,ds + \int_{\hat{C}\cap\partial B_\varepsilon(p)} \widehat{k}\,ds = 2\pi\chi(\widehat{C}_\varepsilon) = 0, \tag{16}$$

where $\widehat{K} \equiv -\kappa^2$ is the intrinsic Gauss curvature of $\widehat{C}_\varepsilon$. Since $\widehat{C}_\varepsilon$ is an immersed annulus, the Euler number $\chi(\widehat{C}_\varepsilon) = 0$.

The geodesic curvature of $\widehat{C} \cap \partial B_\varepsilon(p)$ is the negative of the curvature of $\partial B_\varepsilon(p)$ as a curve in $\mathbf{H}^2(-\kappa^2)$, namely, $-\kappa \coth \kappa\varepsilon$. Thus,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\hat{C}\cap\partial B_\varepsilon(p)} \widehat{k}\, ds &= -\lim_{\varepsilon\to 0} (\kappa \coth \kappa\varepsilon) L(\widehat{C} \cap \partial B_\varepsilon(p)) \\ &= -\lim_{\varepsilon\to 0} (\cosh \kappa\varepsilon) 2\pi\Theta_{\hat{C}}(p) = -2\pi\Theta_{\hat{C}}(p). \end{aligned}$$

Since $\mathrm{Area}(\widehat{C}_\varepsilon) \to \mathrm{Area}(\widehat{C})$, the Gauss-Bonnet formula (16) now implies

$$-\kappa^2\, \mathrm{Area}(\widehat{C}) + \int_\Gamma \widehat{k}\, ds - 2\pi\Theta_{\hat{C}}(p) = 0, \tag{17}$$

which proves Proposition 6**(a)** when $C\backslash\{p\}$ is an immersion. The general case follows from Remark 2. The proof of **(b)** is analogous to **(a)** and Proposition 2**(b)**. $\square$

Proof of Embedding Theorem 3. Let Σ^2 be a branched minimal surface in M whose boundary $\partial\Sigma = \Gamma$ is a C^2 Jordan curve satisfying the hypothesis (8):

$$\mathcal{C}_{\mathrm{tot}}(\Gamma) := \int_\Gamma |\vec{k}|\, ds \le 4\pi + \kappa^2 \mathcal{A}(\Gamma),$$

where $-\kappa^2$ is an upper bound on sectional curvatures of the ambient manifold M. We need to show that $\overline{\Sigma}$ has no branch points and is embedded. Thus, it will suffice to show that $\Theta_\Sigma(p) < 2$ at all $p \in M\backslash\Gamma$ and that $\Theta_\Sigma(p) < 3/2$ at $p \in \Gamma$.

Consider any $p \in \Sigma\backslash\Gamma$, and let $C = p \divideontimes \Gamma$ be the geodesic cone over Γ with vertex p. If Σ is totally geodesic, then Σ is embedded, since there are no compact totally geodesic surfaces and no geodesic loops in M. Otherwise, by Proposition 3 and Proposition 6**(a)**, we have

$$2\pi\Theta_\Sigma(p) < 2\pi\Theta_{\hat{C}}(p) = \int_\Gamma \widehat{k}\, ds - \kappa^2\, \mathrm{Area}(\widehat{C}).$$

Recall that $\Sigma \subset \mathcal{H}_{\mathrm{cvx}}(\Gamma)$. Hence Proposition 5 implies that $\mathrm{Area}(\widehat{C})$ is at least equal to the minimum cone area $\mathcal{A}(\Gamma)$, and since $\widehat{k} \le k \le |\vec{k}|$ almost everywhere along Γ by Proposition 4, we find

$$2\pi\Theta_\Sigma(p) < \mathcal{C}_{\mathrm{tot}}(\Gamma) - \kappa^2 \mathcal{A}(\Gamma).$$

Therefore, hypothesis (8) implies $\Theta_\Sigma(p) < 2$. If $p \in \Gamma$, apply Proposition 6**(b)** to show $\Theta_\Sigma(p) < 3/2$. Then, as in the proof of Theorem 1, the embedded character of $\overline{\Sigma}$ follows. $\square$

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