QUANTIZATION OF ORDINARY AND HIGHER TEICHMÜLLER SPACES OF SURFACES

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Abstract. Fock-Goncharov's cluster varieties have gained much interest of mathematicians and physicists since they appeared in the early 2000's. In this talk, I will focus on certain moduli spaces of local systems on Riemann surfaces, especially some versions of ordinary and higher Teichmüller spaces. I will formulate some problems about these objects, present ideas from physicists, discuss recent results and open problems. I will try to avoid spending too much time on rigorous definitions, and instead try to convey the main ideas.

Contents

Note. This is a lecture note for my two consecutive talks to be given at KIAS, Korea (April 2, 2021), under the same title and abstract, for an occasion of "A mathematician's lecture for the physicists." This note and the talks will be somewhat informal instead of being rigorous, for the main concern is to convey the ideas. As a disclaimer, I only present a particular viewpoint on the subject, and do not try to give a comprehensive overview.

1. Classical aspect

Here we review some basic constructions of Fock and Goncharov [FG06].

Let $\mathfrak{S}$ be a non-compact smooth surface, given as a compact oriented surface of genus g minus n punctures. We will call such a surface a *punctured surface*. Let G be a split reductive algebraic group over $\mathbb{Q}$. In this talk,

$$\mathrm{G} = \mathrm{SL}_m \quad \text{or} \quad \mathrm{PGL}_m \qquad (\text{with } m \geq 2)$$

Here G is being viewed as a scheme, which means that for any chosen field $\mathbf{k}$, one obtains a group $\mathrm{G}(\mathbf{k})$, which is the set of $\mathbf{k}$ points of G. Given a surface $\mathfrak{S}$ and a group G, one can define some versions of the so-called *higher Teichmüller spaces*, which we will review now. Some versions of the ordinary Teichmüller spaces will be prototypical examples.

A G-*local system* $\mathcal{L}$ *on* $\mathfrak{S}$ means a (right) principal G-bundle on $\mathfrak{S}$ together with a flat G-connection. To $\mathcal{L}$ is associated its *monodromy (or, holonomy) representation*

$$\rho_{\mathcal{L}} : \pi_1(\mathfrak{S}) \to \mathrm{G}$$

which is a group homomorphism defined up to conjugation by an element of G. If we let

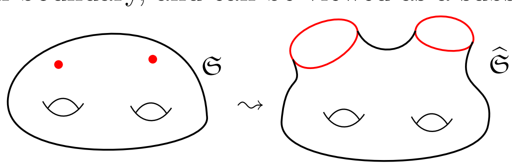
$$\mathscr{L}_{\mathrm{G},\mathfrak{S}} := \text{the moduli space of G-local systems on } \mathfrak{S},$$

we have the natural identification

$$\mathscr{L}_{\mathrm{G},\mathfrak{S}} \ \leftrightarrow \ \mathrm{Hom}(\pi_1(\mathfrak{S}), \mathrm{G})/\mathrm{G}$$

with the set of all group homomorphisms $\pi_1(\mathfrak{S}) \to \mathrm{G}$ defined up to conjugation action by G. It turns out that it is a good idea to consider some modifications of $\mathscr{L}_{\mathrm{G},\mathfrak{S}}$ obtained by adding certain data at boundary.

Let $\widehat{\mathfrak{S}}$ be obtained from the punctured surface $\mathfrak{S}$ by removing some open neighborhood of each puncture that is homeomorphic to the open unit disc minus the origin, i.e. a punctured disc. So $\widehat{\mathfrak{S}}$ is a surface with boundary, and can be viewed as a subspace of $\mathfrak{S}$.



Let B be a Borel subgroup of G, e.g. the subgroup of all upper triangular matrices in G. Then the flag variety $\mathcal{B}$ of G can be expressed as G/B. For a G-local system $\mathcal{L}$ on $\mathfrak{S}$, one can consider the *associated flag bundle* (by modding out each fiber of $\mathcal{L}$ by B from the right)

$$\mathcal{L}_{\mathcal{B}} := \mathcal{L} \times_{\mathrm{G}} (\mathrm{G}/\mathrm{B}),$$

whose fiber at each point of $\mathfrak{S}$ can be identified with the flag variety $\mathcal{B} = \mathrm{G}/\mathrm{B}$. A *framing* on $\mathcal{L}$ is a choice of a flat section β of $\mathscr{L}_{\mathcal{B}}$ restricted to the boundary $\partial\widehat{\mathfrak{S}}$. We call such a pair $(\mathcal{L}, \beta)$ a *framed* G-local system on $\mathfrak{S}$. Let

$$\mathscr{X}_{\mathrm{G},\mathfrak{S}} := \text{the moduli space of framed G-local systems on } \mathfrak{S}.$$

A different version of enhancement uses the maximal unipotent subgroup $\mathrm{U} := [\mathrm{B}, \mathrm{B}]$ of G, e.g. the subgroup of all upper triangular matrices with 1's on the diagonal. When G is of type A_{2k}, E_6, E_8, F_4 or G_2, a *decoration* on a G-local system $\mathcal{L}$ on $\mathfrak{S}$ is a choice of a flat section α of the associated principal affine bundle $\mathcal{L}_{\mathcal{A}} := \mathcal{L} \times_{\mathrm{G}} (\mathrm{G}/\mathrm{U})$ restricted to $\partial\widehat{\mathfrak{S}}$; a pair $(\mathcal{L}, \alpha)$ is called a *decorated* G-local system on $\mathfrak{S}$. Let

$$\mathscr{A}_{\mathrm{G},\mathfrak{S}} := \text{the moduli space of decorated G-local systems on } \mathfrak{S}.$$

For more general G, the definition of a decorated G-local system is slightly more complicated, and involves bundles on the punctured tangent bundle or the unit tangle bundle of $\mathfrak{S}$.

A good thing about these boundary-enhanced moduli spaces $\mathscr{X}_{\mathrm{G},\mathfrak{S}}$ and $\mathscr{A}_{\mathrm{G},\mathfrak{S}}$ is that they admit structures of *cluster varieties*, which we describe now. More precisely, we will review that each of

$$\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}} \quad \text{and} \quad \mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$$

has a structure of a cluster $\mathscr{A}$-variety and that of a cluster $\mathscr{X}$-variety, respectively, where $m \geq 2$.

We first recall the notion of a cluster variety. Let Q be a *quiver* with n nodes; that is, it is a collection of oriented edges between these n nodes, where we do not allow cycles of length 1 or 2. Call these oriented edges *arrows*. Denote

$$\mathcal{V}(Q) = \text{the set of all nodes of the quiver } Q.$$

A *seed* $\mathscr{A}$*-torus* is the pair $(Q,(\mathbb{G}_m)^{\mathcal{V}(Q)})$, i.e. a quiver Q, together with the split algebraic torus $(\mathbb{G}_m)^{\mathcal{V}(Q)} \cong (\mathbb{G}_m)^n$ of dimension n, where the coordinate functions A_i are enumerated by the nodes i of the quiver Q; so it is an affine scheme whose ring of regular functions is the ring of Laurent polynomials in these coordinate variables A_i, $i \in \mathcal{V}(Q)$. At a node k of Q, one can *mutate* the quiver Q at k, to get a new quiver $\mu_k(Q)$ as follows. For $i,j \in \mathcal{V}(Q)$, let

$$\varepsilon_{ij} = \#(\text{arrows in } Q \text{ from } i \text{ to } j) - \#(\text{arrows in } Q \text{ from } j \text{ to } i)$$

i.e. $\varepsilon = (\varepsilon_{ij})$ is the *signed adjacency matrix* of Q. If we write ε' the signed adjacency matrix for the mutated quiver $Q' = \mu_k(Q)$, the formula for ε' is

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i,j\}, \\ \varepsilon_{ij} + \frac{1}{2}(\varepsilon_{ik}|\varepsilon_{kj}| + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } k \notin \{i,j\}. \end{cases}$$

We say two quivers are *mutation-equivalent* if they are related by a finite sequence of quiver mutations. When $Q' = \mu_k(Q)$, one defines a rational map, again denoted by μ_k, between the corresponding tori

$$\mu_k : (\mathbb{G}_m)^{\mathcal{V}(Q)} \dashrightarrow (\mathbb{G}_m)^{\mathcal{V}(Q')}$$

defined as

$$\mu_k^* : A'_i \mapsto \begin{cases} A_i & \text{if } i \neq k, \\ A_k^{-1}(\prod_{j|\varepsilon_{kj}>0} A_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj}<0} A_j^{-\varepsilon_{kj}}) & \text{if } i = k, \end{cases}$$

called the *cluster* $\mathscr{A}$*-mutation* formula. In this setting, we say that the seed $\mathscr{A}$-torus $(Q',(\mathbb{G}_m)^{\mathcal{V}(Q')})$ is obtained from $(Q,(\mathbb{G}_m)^{\mathcal{V}(Q)})$ by the process of mutation μ_k. Start from one seed $\mathscr{A}$-torus (which is determined by a quiver Q), mutate to get more seed $\mathscr{A}$-tori repeatedly. Glue all the resulting seed $\mathscr{A}$-tori together along the maps μ_k. The resulting scheme is called the *cluster $\mathscr{A}$-variety*. Since it essentially depends only on the mutation-equivalence class $|Q|$ of a quiver Q, we denote this scheme by $\mathscr{A}_{|Q|}$.

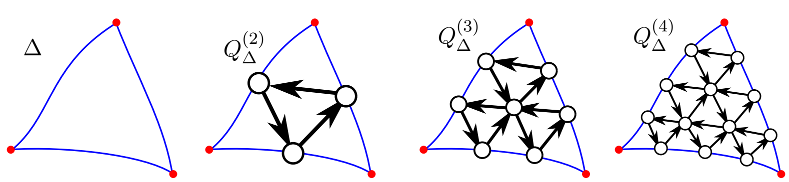
A cluster $\mathscr{X}$-variety is defined similarly. A *seed $\mathscr{X}$-torus* is a pair $(Q, (\mathbb{G}_m)^{\mathcal{V}(Q)})$, where the i-th coordinate is denoted by X_i, for each node i of Q. The mutation μ_k at a node k is defined as before for the quiver Q, and as follows for the coordinate variables

$$\mu_k^* : X_i' \mapsto \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} & \text{if } i \neq k, \end{cases}$$

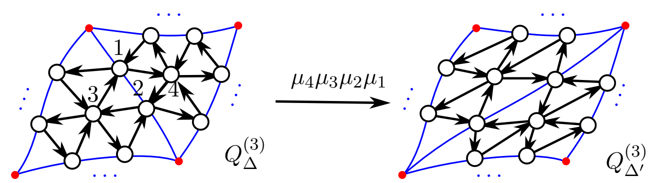
where $\mathrm{sgn}(a) = 1$ if $a > 0$ and $\mathrm{sgn}(a) = -1$ if $a < 0$; this is called the *cluster $\mathscr{X}$-mutation* formula. The scheme obtained by gluing all seed $\mathscr{X}$-tori related to an initial one by sequences of mutations is called the *cluster $\mathscr{X}$-variety*, which we denote by $\mathscr{X}_{|Q|}$. Note that the cluster $\mathscr{X}$-variety is equipped with a Poisson structure, which is given on each seed $\mathscr{X}$-chart as

$$\{X_i, X_j\} = \varepsilon_{ij} X_i X_j.$$

We now describe the relevant quivers for the moduli stacks $\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}}$ and $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$. First, consider an *ideal triangulation* of a punctured surface $\mathfrak{S}$, i.e. a collection Δ of disjoint unoriented paths in $\mathfrak{S}$ running between punctures of $\mathfrak{S}$ dividing the surface $\mathfrak{S}$ into (ideal) triangles. The constituents of Δ are called *(ideal) arcs*. Given an ideal triangulation Δ of $\mathfrak{S}$, we consider a special quiver $Q_\Delta^{(m)}$ called the *m-triangulation quiver*, embedded in the surface $\mathfrak{S}$, constructed by gluing the quivers associated to the ideal triangles of Δ as in the following pictures.



Then, when $m = 2$, the set of nodes $\mathcal{V}(Q_\Delta^{(2)})$ is in a natural bijection with Δ: one node of $Q_\Delta^{(2)}$ per each arc of Δ. When $m = 3$, $Q_\Delta^{(3)}$ has two nodes on each arc of Δ and one node in the interior of each ideal triangle of Δ. If two triangulations Δ and Δ' differ by exactly one edge, say k, we say that they are related by a *flip* at k. In this case, it is observed in [FG06] that the quivers $Q_\Delta^{(m)}$ and $Q_{\Delta'}^{(m)}$ are related by a certain sequence of $(m-1)^2$ quiver mutations, e.g.



Further, it is proved in [FG06] that, for each Δ, there are birational isomorphisms

$$\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}} \dashrightarrow (\mathbb{G}_m)^{\mathcal{V}(Q^{(m)}_\Delta)} \quad \text{and} \quad \mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}} \dashrightarrow (\mathbb{G}_m)^{\mathcal{V}(Q^{(m)}_\Delta)},$$

forming charts on the moduli spaces, and for two charts for Δ and Δ' that are related by a flip at k, the transition maps between these charts are exactly the sequences of cluster $\mathscr{A}$-mutations and cluster $\mathscr{X}$-mutations respectively, associated to the sequence of quiver mutations relating $Q^{(m)}_\Delta$ and $Q^{(m)}_{\Delta'}$. So, one can say that there are birational isomorphisms

$$\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}} \dashrightarrow \mathscr{A}_{|Q^{(m)}_\Delta|} \quad \text{and} \quad \mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}} \dashrightarrow \mathscr{X}_{|Q^{(m)}_\Delta|}$$

between the moduli spaces and the cluster $\mathscr{A}$- and $\mathscr{X}$-varieties.

For any cluster varieties $\mathscr{A}_{|Q|}$ and $\mathscr{X}_{|Q|}$, since the transition maps do not involve subtractions, one can evaluate not only at a field, but also at a *semi-field* $(\mathbb{P}, \oplus, \odot)$, which is like a field without subtraction. As a set, $\mathscr{A}_{|Q|}(\mathbb{P})$ is obtained by gluing $\mathbb{P}^{\mathcal{V}(Q)}$ by the tropical versions of the cluster $\mathscr{A}$-mutations, and likewise for $\mathscr{X}$. Prominent examples of semi-fields are the positive-real semi-field $(\mathbb{R}_{>0}, +, \cdot)$, and the tropical integer semi-field $\mathbb{Z}^t = (\mathbb{Z}, \max(\cdot,\cdot), +)$. The sets

$$\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}}(\mathbb{R}_{>0}) = \mathscr{A}_{|Q^{(m)}_\Delta|}(\mathbb{R}_{>0}) \quad \text{and} \quad \mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}(\mathbb{R}_{>0}) = \mathscr{X}_{|Q^{(m)}_\Delta|}(\mathbb{R}_{>0})$$

are what are called the *higher Teichmüller spaces* by Fock and Goncharov [FG06]. They are in bijection with $(\mathbb{R}_{>0})^{\mathcal{V}(Q^{(m)}_\Delta)}$, and can be viewed as smooth manifolds this way. Note $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}(\mathbb{R}_{>0})$ is then a Poisson manifold. When $m = 2$, these spaces recover some versions of classical Teichmüller spaces.

$\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{R}_{>0}) \leftrightarrow$ decorated Teichmüller space of $\mathfrak{S}$ (of Penner),

$\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}(\mathbb{R}_{>0}) \leftrightarrow$ enhanced (or, holed) Teichmüller space (of Bonahon and collaborators).

As shall be seen at least partly, the tropical integer sets $\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}}(\mathbb{Z}^t)$ and $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}(\mathbb{Z}^t)$ are interesting objects of study, and play crucial roles in the quantization too.

2. Quantization: what to quantize?

Let $\mathfrak{S}$ be a triangulable punctured surface. Let $m \geq 2$. Per each ideal triangulation Δ of $\mathfrak{S}$, one has the cluster $\mathscr{X}$-coordinate functions X_i on the moduli space $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$, enumerated by the nodes i of the m-triangulation quiver $Q^{(m)}_\Delta$. These coordinate functions X_i together with their inverses X_i^{-1} form a Poisson algebra, with the Poisson bracket given by

$$\{X_i, X_j\} = \varepsilon_{ij} X_i X_j$$

where $\varepsilon = (\varepsilon_{ij})_{i,j}$ is the signed adjacency matrix for the quiver $Q^{(m)}_\Delta$. A standard choice of a non-commutative quantum algebra that deforms this Poisson algebra is the algebra generated by the symbols $\widehat{X}_i^{\pm 1}$ mod out by the relations

$$\widehat{X}_i \widehat{X}_j = q^{2\varepsilon_{ij}} \widehat{X}_j \widehat{X}_i, \qquad \forall i, j \in \mathcal{V}(Q_\Delta^{(m)}),$$

where q is the quantum parameter.

In the classical setting, the Poisson algebra for Δ generated by $X_i^{\pm 1}$, $i \in \mathcal{V}(Q_\Delta^{(m)})$, is related to the Poisson algebra for Δ' generated by $(X_i')^{\pm 1}$, $i \in \mathcal{V}(Q_{\Delta'}^{(m)})$, via certain sequence of mutation maps μ_k^* which are rational, as seen before. For example, if $m = 2$ and Δ and Δ' are related by the flip at an edge corresponding to the node k, then $\mu_k^* X_k' = X_k^{-1}$, $\mu_k^* X_i' = X_i(1 + X_k)$ if $\varepsilon_{ik} = -1$, $\mu_k^* X_i' = X_i(1 + X_k^{-1})^{-1}$ if $\varepsilon_{ik} = 1$, etc.

One would seek for quantum versions μ_k^q of these mutation maps μ_k^*, relating the quantum variables $\widehat{X}_i^{\pm 1}$, $i \in \mathcal{V}(Q_\Delta^{(m)})$, to the quantum variables $(\widehat{X}_i')^{\pm 1}$, $i \in \mathcal{V}(Q_{\Delta'}^{(m)})$. We would require that

(1) μ_k^q should recover the classical map μ_k^* if one puts $q = 1$ and remove hats from the generators; and
(2) μ_k^q should satisfy the consistency relations satisfied by its classical counterparts, e.g. $\mu_k^q \mu_k^q = \mathrm{id}$ should hold always, $\mu_i^q \mu_j^q \mu_i^q \mu_j^q = \mathrm{id}$ should hold whenever $\varepsilon_{ij} = 0$, and $\mu_i^q \mu_j^q \mu_i^q \mu_j^q \mu_i^q = (ij)$ should hold whenever $\varepsilon_{ij} = \pm 1$ (where (ij) stands for the index exchange $i \leftrightarrow j$ of two nodes)

Such an answer is found by Chekhov-Fock [CF99], Liu [L09] and Fock-Goncharov [FG09]; it reads e.g. $\mu_k^q \widehat{X}_k' = \widehat{X}_k^{-1}$, $\mu_k^q \widehat{X}_i' = \widehat{X}_i(1 + q\widehat{X}_k)$ if $\varepsilon_{ik} = -1$, $\mu_k^q \widehat{X}_i' = \widehat{X}_i(1 + q\hat{X}_k^{-1})^{-1}$ if $\varepsilon_{ik} = 1$, etc. So these maps μ_k^q provide a consistent system of quantum observable algebras.

We consider the problem of constructing a *deformation quantization* map, which is a map

$$\text{the classical observable algebra} \;\to\; \text{the quantum observable algebra},$$

satisfying certain favorable conditions. One first needs to pin down precisely which classical observable functions to be quantized. For several reasons, in this case, a good choice is the class of functions on $\mathscr{X}_{\mathrm{PGL}_m, \mathfrak{S}}$ that are *universally Laurent*, i.e. those that can be written as Laurent polynomials in the cluster $\mathscr{X}$-variables X_i, $i \in \mathcal{V}(Q_\Delta^{(m)})$, for *every* ideal triangulation Δ. In fact, what are better understood are those functions that are *regular* functions on the moduli stack $\mathscr{X}_{\mathrm{PGL}_m, \mathfrak{S}}$, or that are Laurent polynomial functions for *every* cluster $\mathscr{X}$-chart, not just for every cluster $\mathscr{X}$-chart associated to an ideal triangulation. It is not known in general whether these two notions of universally Laurent coincide, but let us not go into this issue in this note.

Even before considering the quantization problem, it is quite a nontrivial and interesting problem to identify all possible universally Laurent functions, and moreover to construct a basis of the ring of all universally Laurent functions such that this basis satisfies certain nice properties. Once one constructs a basis, then one can try to quantize each basic universally Laurent function. A particular form of this problem of finding basic universally Laurent functions in the classical setting is dubbed the *Fock-Goncharov duality conjectures* for cluster varieties, originally proposed in [FG06].

A prototypical example is about the moduli space $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$, which is birational to the cluster $\mathscr{X}$-variety associated to the m-triangulation quivers $Q^{(m)}_\Delta$ of the ideal triangulations Δ of a punctured surface $\mathfrak{S}$. An answer in the case $m = 2$ is already given in the original paper by Fock and Goncharov [FG06]. This answer is geometric, and at the same time provides a straightforward algorithm to compute the basic universally Laurent functions. An analog of such a geometric answer is extended to $m = 3$ only recently in the work [K20] of myself, building up on some previous works of physicists and mathematicians. In fact, Gross, Hacking, Keel and Kontsevich, in their seminal paper [GHKK18], constructed an answer to this duality conjecture for much more general class of cluster varieties, which is found in [GS18] to include these moduli spaces $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$. However, we note that the construction of [GHKK18] applied to $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$ would not involve geometry of surfaces, and the computation of Gross-Hacking-Keel-Kontsevich's basic universally Laurent functions are in general enormously complicated. A natural and important unsolved problem is the equality (or a precise relationship) between the Gross-Hacking-Keel-Kontsevich's answer [GHKK18] and those of Fock-Goncharov [FG06] for $m = 2$ and myself [K20] for $m = 3$.

What is most relevant to the present note is the case for $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$. The basis of the ring $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}})$ of regular (i.e. universally Laurent) functions on $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$ constructed by Fock and Goncharov [FG06] is enumerated by $\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{Z}^t)$, i.e. the set of tropical integer points of the moduli space $\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}$. Here $\mathbb{Z}^t$ is the semi-field of tropical integers, which is $\mathbb{Z}$ as a set, equipped with the tropical addition $\oplus$ given by $a \oplus b := \max(a, b)$ and the tropical multiplication $\odot$ given by $a \odot b := a + b$. Note that tropical division is possible, but not tropical subtraction. As a set, $\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{Z}^t)$ is obtained by gluing together the sets $\mathbb{Z}^{Q^{(2)}_\Delta}$, for ideal triangulations Δ of the surface $\mathfrak{S}$, along the tropical versions of the cluster $\mathscr{A}$-mutation formulas.

$$\mu_k^* a_i' = \begin{cases} a_i & \text{if } i = k, \\ -a_k + \max(\sum_{j|\varepsilon_{kj}>0} \varepsilon_{kj} a_j, \, \sum_{j|\varepsilon_{kj}<0} (-\varepsilon_{kj} a_j)) & \text{if } i \neq k, \end{cases}$$

where $\varepsilon = (\varepsilon_{ij})$ is the signed adjacency matrix for the quiver $Q^{(2)}_\Delta$. The Fock-Goncharov's duality map [FG06] in this setting is an injective map

$$\mathbb{I} : \mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{Z}^t) \to \mathscr{O}(\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}),$$

such that the image of $\mathbb{I}$ forms a basis of $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}})$, the ring of all universally Laurent functions on $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$.

What is nice about the moduli space $\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}$ or the relevant cluster $\mathscr{A}$-variety $\mathscr{A}_{|Q_\Delta^{(2)}|}$ is that this set $\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{Z}^t)$ has a geometric realization [FG06]

$$\mathscr{A}_{\mathrm{SL}_2,\mathfrak{S}}(\mathbb{Z}^t) \leftrightarrow \{\textit{even integral } \mathscr{A}\textit{-laminations} \text{ in } \mathfrak{S}\}$$

An *integral $\mathscr{A}$-lamination* ℓ in $\mathfrak{S}$ is represented as a finite collection of mutually disjoint simple loops in $\mathfrak{S}$ with integer weights on the loops, such that the weight on a loop is non-negative unless the loop is a *peripheral loop*, i.e. a small loop surrounding the puncture; zero-weight loop can be removed, and homotopic loops can be combined with the weights added. An integral $\mathscr{A}$-lamination ℓ is said to be *even* if it satisfies a certain parity condition with respect to special coordinate systems of the space of all integral $\mathscr{A}$-laminations, but let us not go into this detail now.

We partially recall Fock-Goncharov's construction of $\mathbb{I}$. Let ℓ be an integral $\mathscr{A}$-lamination represented by a single simple loop γ with weight 1, where γ is not a peripheral loop. The corresponding function $\mathbb{I}(\ell)$ is essentially the *trace-of-holonomy* (or *trace-of-monodromy*) function on $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$, which has a natural geometric meaning. Moreover, for each triangulation Δ, $\mathbb{I}(\ell)$ can be computed explicitly as a Laurent polynomial in the square-roots $X_i^{1/2}$ of the cluster $\mathscr{X}$-variables for the nodes i of the 2-triangulation quiver $Q_\Delta^{(2)}$, i.e. corresponding to edges i of Δ. We will denote by $\mathbb{I}_\Delta(\ell)$ this Laurent polynomial expression for Δ. First, homotope γ so that it meets Δ at a minimal number of intersection points, and give an arbitrary orientation on γ; then break γ into alternating concatenation $\gamma_1.\gamma_2.\gamma_3.\cdots.\gamma_N$ of two kinds of small pieces γ_j, one kind being small pieces passing through an edge of Δ, and the other kind being small pieces inside triangles, either left turn or right turn. For each small piece γ_j, define a *basic monodromy matrix* M_{γ_j} as

$$M_{\gamma_j} = \begin{cases} \begin{pmatrix} X_i^{1/2} & 0 \\ 0 & X_i^{-1/2} \end{pmatrix} & \text{if } \gamma_j \text{ passes through the edge } i \text{ of } \Delta; \\ \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } \gamma_j \text{ is a left turn piece in a triangle;} \\ \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) & \text{if } \gamma_j \text{ is a right turn piece in a triangle.} \end{cases}$$

Multiply these matrices while traveling along γ once around, and then take trace to get $\mathbb{I}_\Delta(\ell)$:

$$\mathbb{I}_\Delta(\ell) = \mathrm{tr}\,(M_{\gamma_1} M_{\gamma_2} \cdots M_{\gamma_N})$$

For convenience, denote the square-root variables by

$$Z_i := X_i^{1/2}, \qquad i \in \Delta \quad (i \in \mathcal{V}(Q_\Delta^{(2)})).$$

It is clear that $\mathbb{I}_\Delta(\ell)$ computed this way is a Laurent polynomial in $Z_i = X_i^{1/2}$, $i \in \Delta$, with non-negative integral coefficients. What is not so clear is the compatibility under flips

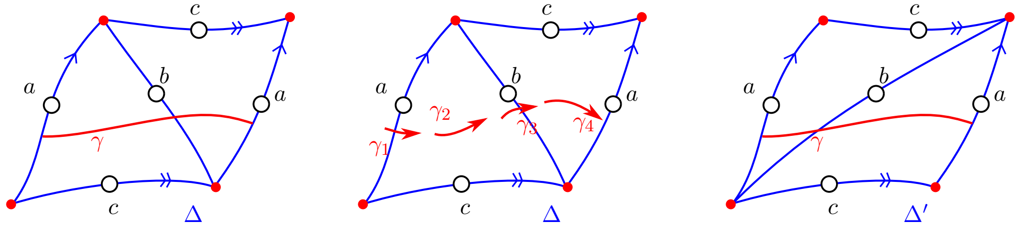
$$\mu_k^*(\mathbb{I}_{\Delta'}(\ell)) = \mathbb{I}_\Delta(\ell),$$

which nevertheless indeed holds. When one write as

$$\mathbb{I}_\Delta(\ell) = \sum_{(a_1,\ldots a_n)\in\mathbb{Z}^n} \overline{\Omega}_\Delta(\ell; a_1,\ldots,a_n)\cdot \textstyle\prod_{i=1}^n Z_i^{a_i}$$

with n = number of edges of Δ and $\overline{\Omega}_\Delta(\ell;a_1,\ldots,a_n)\in\mathbb{Z}_{\geq 0}$, these coefficients $\overline{\Omega}_\Delta(\ell;a_1,\ldots,a_n)$ seem to correspond to the counting of the 'framed BPS states' in physics.

Here is an example when ℓ is a single loop γ in a once-punctured torus $\mathfrak{S}$ with weight 1, with some ideal triangulations Δ and Δ' of $\mathfrak{S}$ related by a flip:



The white circles are the nodes of $Q^{(2)}_\Delta$, with their labels a, b, c indicated as above.

$$\begin{aligned}\mathbb{I}_\Delta(\ell) &= \mathrm{tr}\,(M_{\gamma_1}M_{\gamma_2}M_{\gamma_3}M_{\gamma_4}) \\ &= \mathrm{tr}\left(\begin{pmatrix} Z_a & 0 \\ 0 & Z_a^{-1}\end{pmatrix}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\begin{pmatrix} Z_b & 0 \\ 0 & Z_b^{-1}\end{pmatrix}\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\right) \\ &= Z_aZ_b + Z_aZ_b^{-1} + Z_a^{-1}Z_b^{-1}\end{aligned}$$

Note that Δ' is obtained from Δ by flipping at the edge of Δ corresponding to b. A similar computation for Δ' gives

$$\mathbb{I}_{\Delta'}(\ell) = Z'_aZ'_b + (Z'_a)^{-1}Z'_b + (Z'_a)^{-1}(Z'_b)^{-1}$$

Then $\mathbb{I}_\Delta(\ell) = X_a^{1/2}X_b^{1/2} + X_a^{1/2}X_b^{-1/2} + X_a^{-1/2}X_b^{-1/2}$ and $\mathbb{I}_{\Delta'}(\ell) = (X'_a)^{1/2}(X'_b)^{1/2} + (X'_a)^{-1/2}(X'_b)^{1/2} + (X'_a)^{-1/2}(X'_b)^{-1/2}$ are related by the cluster $\mathscr{X}$-mutation map μ_k^*, which sends X'_a to $X_a(1+X_b)^2$ and X'_b to X_b^{-1}.

3. Deformation quantization problem for cluster $\mathscr{X}$-varieties

Coming back to the quantization problem, say for $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$, the question is now to figure out how to quantize each basic universally (square-root) Laurent function $\mathbb{I}(\ell)$. For each triangulation Δ, $\mathbb{I}(\ell) = \mathbb{I}_\Delta(\ell)$ is a Laurent polynomial in Z_i, $i \in \Delta$, with integer coefficients. We would seek to construct its quantum version $\mathbb{I}^q_\Delta(\ell)$, as a Laurent polynomial in the non-commutative (square-root) quantum variables $\widehat{Z}_i$, $i\in\Delta$, with coefficients in $\mathbb{Z}[q^{\pm 1/4}]$, which satisfy

$$\widehat{Z}_i\widehat{Z}_j = (q^{1/4})^{2\varepsilon_{ij}}\widehat{Z}_j\widehat{Z}_i, \quad \forall i,j\in\Delta\ (\leftrightarrow \mathcal{V}(Q^{(2)}_\Delta)).$$

The first stipulation we impose on the sought-for quantized function $\mathbb{I}^q_\Delta(\ell)$ is of course

(1) the <u>*classical limit*</u>, i.e.

$$\mathbb{I}^1_\Delta(\ell) = \mathbb{I}_\Delta(\ell).$$

As usual, one also would require the self-adjointness, or

(2) the $*$-invariance
$$*\mathbb{I}^q_\Delta(\ell) = \mathbb{I}^q_\Delta(\ell),$$
where $*$ stands for the algebra anti-isomorphism s.t. $*\widehat{Z}_i = \widehat{Z}_i$, $\forall i$.

But these two conditions are too weak, and there are lots of solutions to $\mathbb{I}^q_\Delta(\ell)$, i.e. the quantum ordering problem has many solutions. One of the standard answer for $\mathbb{I}^q_\Delta(\ell)$ is the term-by-term *Weyl-ordering*, e.g. for the example from the last section when ℓ is a loop γ in a once-punctured torus $\mathfrak{S}$ with weight 1, we would consider
$$\mathbb{I}^q_\Delta(\ell) = q^{1/2}\widehat{Z}_a\widehat{Z}_b + q^{-1/2}\widehat{Z}_a\widehat{Z}_b^{-1} + q^{1/2}\widehat{Z}_a^{-1}\widehat{Z}_b^{-1}.$$
In a more general situation, this means we consider $\mathbb{I}^q_\Delta(\ell)$ being in the form of a sum of the following Weyl-ordered Laurent monomials
$$[\widehat{Z}_1^{a_1}\widehat{Z}_2^{a_2}\cdots\widehat{Z}_n^{a_n}]_{\mathrm{Weyl}} = q^{-\sum_{i<j}\varepsilon_{ij}a_ia_j/4}\widehat{Z}_1^{a_1}\widehat{Z}_2^{a_2}\cdots\widehat{Z}_n^{a_n}$$
In fact, this term-by-term Weyl-ordered Laurent polynomial does not provide a correct answer to the deformation quantization problem of cluster $\mathscr{X}$-varieties in general. This is because of yet another condition that we require, namely

(3) the compatibility under the quantum coordinate change maps, i.e. the quantum mutation maps μ^q_k
$$\mu^q_k(\mathbb{I}^q_{\Delta'}(\ell)) = \mathbb{I}^q_\Delta(\ell).$$
Indeed, the term-by-term quantum Laurent polynomials do not necessarily satisfy the above compatibility equation.

In physics, there seems to be a solution of the form
$$\mathbb{I}^q_\Delta(\ell) = \sum_{(a_1,\ldots a_n)\in\mathbb{Z}^n} \underline{\overline{\Omega}}_\Delta(\ell; a_1,\ldots,a_n;q)\cdot[\textstyle\prod_{i=1}^n \widehat{Z}_i^{a_i}]_{\mathrm{Weyl}}$$
where the coefficient $\underline{\overline{\Omega}}_\Delta(\ell; a_1,\ldots,a_n;q) \in \mathbb{Z}[q^{\pm 1}]$ is the quantity called the 'framed protected spin character'. It is not clear to me whether this $\underline{\overline{\Omega}}_\Delta(\ell; a_1,\ldots,a_n;q)$ is a well-defined determined quantity in physics which we just have to compute the value of, or is some sought-for undetermined quantity to be constructed in order for it to satisfy some conditions. In this talk, we will regard them as undetermined quantity that we seek to construct, so that $\mathbb{I}^q_\Delta(\ell)$ satisfies the three conditions (1), (2) and (3). We will review a solution by mathematicians, and another by physicists.

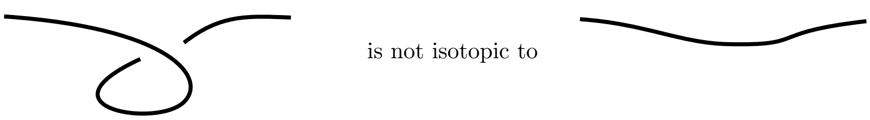
4. SL_2 quantum trace of holonomy

We first partially describe the solution by mathematicians. This solution for the quantum functions $\mathbb{I}^q_\Delta(\ell)$ uses the 3-dimensional space

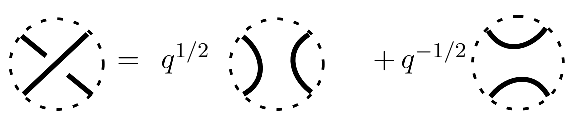
$$\mathfrak{S} \times (-1,1),$$

called the *thickened surface*. For a point in $\mathfrak{S} \times (-1,1)$, the second coordinate value in $(-1,1)$ is called the *elevation* of this point.

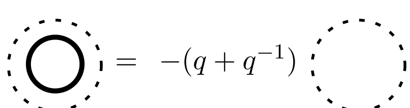
A basic object is a *framed link* L in $\mathfrak{S}\times(-1,1)$, which is a disjoint union of circles S^1 embedded in $\mathfrak{S} \times (-1,1)$, equipped with a *framing*, i.e. a nowhere tangent continuous vector field on L (i.e. per each $x \in L$, a vector in $T_x(\mathfrak{S} \times (-1,1)) \setminus T_x L$ is chosen, so that these choices are continuous over L). So L can be visualized by something like a ribbon. Often, a framed link is considered up to *isotopy*, i.e. homotopy within the class of framed links. A framed link L in $\mathfrak{S}\times(-1,1)$ is usually depicted by its projection onto the surface $\mathfrak{S}$, with the *blackboard framing*. That is, before projecting to $\mathfrak{S}$, one isotope L so that the framing at every point of L is *upward vertical*, i.e. parallel to the $(-1,1)$-factor and pointing toward 1, and the only singularities of the projection of L are transverse double intersections, called *crossings*, where one indicates the strands in lower elevations by broken lines:



Given a punctured surface $\mathfrak{S}$ and a quantum parameter $q \in \mathbb{C}^\times$, one defines the *(Kauffman bracket) (*SL_2*) skein algebra* $\mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$ as the $\mathbb{C}$-vector space freely generated by the set of all isotopy classes of framed links in $\mathfrak{S} \times (-1,1)$, mod out by the *(Kauffman bracket) (*SL_2*) skein relations*



and the *trivial loop relations*



Denote by $[L] \in \mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$ the equivalence class represented by a framed link L. The product structure on $\mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$ is given by superposition, i.e.

$$[L_1] \cdot [L_2] = [L_1 \cup L_2]$$

where L_1, L_2 are framed links in $\mathfrak{S}\times(-1,1)$ such that $L_1 \subset \mathfrak{S}\times(-1,0)$ and $L_2 \subset \mathfrak{S}\times(0,1)$; that is, you stack $[L_2]$ over $[L_1]$. This skein algebra $\mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$ was first considered by Turaev [T91], Przytycki [P91], and others, since 1980's, and was shown to be a quantum algebra deforming the ring of functions on the SL_2-character variety of $\mathfrak{S}$ [PS00], which is closely related to the Teichmüller space of $\mathfrak{S}$.

The main ingredient in the first solution to the quantum functions $\mathbb{I}^q_\Delta(\ell)$ is the following map connecting the skein algebra $\mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$ and the algebra of (square-roots of) quantum cluster $\mathscr{X}$-variables, which was constructed by Bonahon and Wong.

Theorem 4.1 (Bonahon-Wong [BW11]; SL_2 quantum trace map)**.** *There exists a family of algebra homomorphisms*

$$\mathrm{Tr}^q_\Delta = \mathrm{Tr}^q_{\Delta;\mathfrak{S}} : \mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S}) \to \{\textit{(quantum) Laurent polynomials in } \widehat{Z}_i,\ i \in \Delta\}$$

for each triangulable punctured surface $\mathfrak{S}$ and each ideal triangulation Δ of $\mathfrak{S}$, satisfying favorable properties, including:

(1) *if ℓ is a lamination in the surface $\mathfrak{S}$ consisting of a single non-peripheral simple loop γ in $\mathfrak{S}$ with weight 1, and L_γ is a framed link in $\mathfrak{S}\times(-1,1)$ obtained as a constant-elevation upward-vertical lift of γ, then when $q=1$ we have*

$$\mathrm{Tr}^1_\Delta([L_\gamma]) = \mathbb{I}_\Delta(\ell)$$

(with $\widehat{Z}_i$ corresponding to Z_i);

(2) *if Δ and Δ' are ideal triangulations of a surface $\mathfrak{S}$ related by a flip at the edge k, and if L is a framed link in $\mathfrak{S}\times(-1,1)$, then the values of the quantum trace maps $\mathrm{Tr}^q_\Delta([L])$ and $\mathrm{Tr}^q_{\Delta'}([L])$ are related by the quantum mutation maps*

$$\mu^q_k(\mathrm{Tr}^q_{\Delta'}([L])) = \mathrm{Tr}^q_\Delta([L]).$$

So, for a lamination ℓ in $\mathfrak{S}$ consisting of a single non-peripheral simple loop γ with weight 1, we set

$$\mathbb{I}^q_\Delta(\ell) := \mathrm{Tr}^q_\Delta([L_\gamma]),$$

which is the sought-for quantum function satisfying all three conditions (1), (2) and (3) of the previous section. Building on this basic case coming from the Bonahon-Wong quantum trace, Allegretti and myself [AK17] were able to construct quantum functions $\mathbb{I}^q_\Delta(\ell)$ for all (even) integral $\mathcal{A}$-laminations ℓ in $\mathfrak{S}$ that satisfy several desirable properties.

In fact, the key point of Bonahon-Wong's construction in [BW11] lies in the special property of their maps Tr^q_Δ, namely the compatibility under cutting and gluing of the surface $\mathfrak{S}$ along an ideal arc of Δ. Cutting a punctured surface along an ideal arcs does not yield a punctured surface, but a surface with boundary with 'punctures' on the boundary, called *marked points*. Such a surface is called a *generalized marked surface*, or a *decorated surface*. One first needs to define a suitable version of a skein algebra for a generalized marked surface based on *tangles* which can have endpoints over boundary arcs, in order to formulate the cutting/gluing axiom for the quantum trace maps. When one cuts the surface $\mathfrak{S}$ along all the edges of a triangulation Δ, the cutting/gluing property yields a state-sum type formula for Tr^q_Δ, and the value of Tr^q_Δ are determined by those of Tr^q_t for triangles t. The computation and the structure of Bonahon-Wong's quantum trace are thus quite algebraic and combinatorial in nature.

Anyhow, this solution $\mathbb{I}^q_\Delta(\ell)$ to quantum functions enjoy several nice properties, as required partially by the quantum Fock-Goncharov duality conjecture. Note first that $\mathbb{I}^q_\Delta(\ell)$ is a Laurent polynomial in the non-commutative variables $\widehat{Z}_i$, $i \in \Delta$, with coefficients in $\mathbb{Z}[q^{\pm 1/4}]$. There are two kinds of positivity properties, which are important both in mathematics and physics.

Theorem 4.2 (Cho-Kim-K.-Oh [CKKO20]; Laurent coefficient positivity)**.** *The coefficients of* $\mathbb{I}^q_\Delta(\ell)$ *are non-negative, i.e. lie in* $\mathbb{Z}_{\geq 0}[q^{\pm 1/4}]$.

A core of the proof of this theorem is a certain statement about combinatorics related to a simple loop in $\mathfrak{S}$ and an ideal triangulation Δ.

For the other positivity, first note the following property of $\mathbb{I}^q_\Delta = \mathbb{I}^q$.

Proposition 4.3 (Allegretti-K. [AK17]; product-to-sum)**.** *One has*

$$\mathbb{I}^q(\ell)\mathbb{I}^q(\ell') = \sum_{\ell''} c^q(\ell,\ell';\ell'')\,\mathbb{I}^q(\ell'')$$

where the sum is over all laminations ℓ''*, while the coefficients* $c^q(\ell,\ell';\ell'') \in \mathbb{Z}[q^{\pm 1/4}]$ *are zero for all but finitely many* ℓ''.

The coefficients $c^q(\ell,\ell';\ell'')$ are called the *structure constants* of this quantized 'basis' consisting of $\mathbb{I}^q(\ell)$.

Conjecture 4.4 (structure-constant positivity)**.** $c^q(\ell,\ell';\ell'')$ *are non-negative, i.e. lie in* $\mathbb{Z}_{\geq 0}[q^{\pm 1/4}]$.

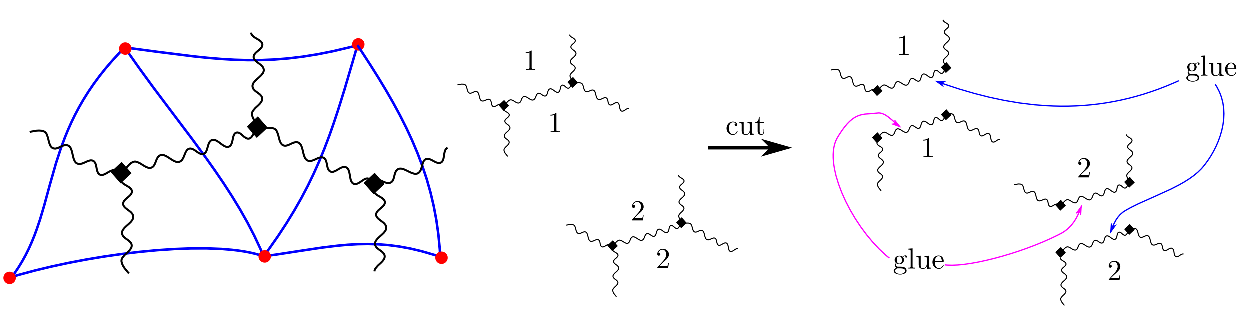
This conjecture has been known for about 10 years at least, and for the quantum functions $\mathbb{I}^q(\ell)$ currently in discussion, this conjecture is essentially equivalent to an analogous statement about a corresponding basis of the skein algebra $\mathcal{S}^q_{\mathrm{SL}_2}(\mathfrak{S})$. The classical version $q = 1$ is proved in [T14]. The quantum version was not known even for a single surface (other than the three-punctured sphere, for which the problem is trivial), until it was proved just for the once-punctured torus and the four-punctured sphere in [B20]. Recently, Mandel and Qin announced that they proved the quantum version for general surfaces (perhaps not for all possible punctured surfaces) in their upcoming joint work, by showing that the Allegretti-Kim quantum functions $\mathbb{I}^q(\ell)$ [AK17] which quantize the Fock-Goncharov functions $\mathbb{I}(\ell)$ [FG06] coincide with the Davison-Mandel's quantum theta functions [DM19] which quantize the Gross-Hacking-Keel-Kontsevich's theta functions [GHKK18]. By the way, as mentioned already, even in the classical setting, the equality of the Fock-Goncharov's functions and Gross-Hacking-Keel-Kontsevich's functions has not been proved, again even for a single surface (maybe other than the three-punctured sphere). Even with [B20] and the upcoming work of Mandel and Qin, their proof are somewhat indirect, and so it would still be an interesting problem to seek for a more direct and elementary skein-theoretic proof of this structure-constant positivity.

5. Ideas from physics

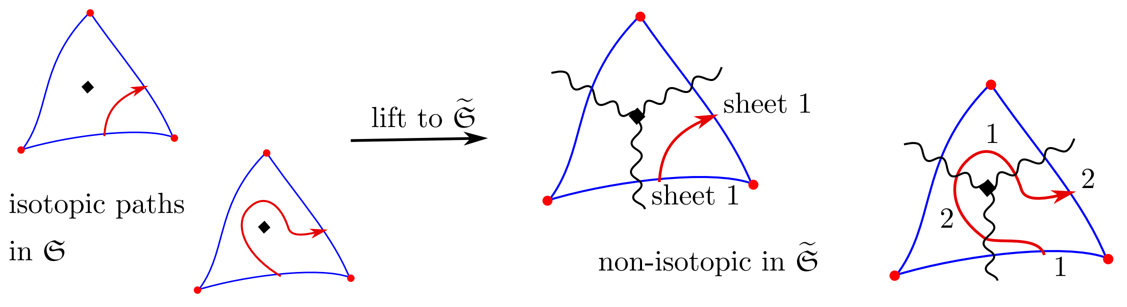
Although Bonahon-Wong's quantum trace maps essentially provided one good answer to the quantum functions $\mathbb{I}^q_\Delta(\ell)$, the actual computation of the quantum Laurent polynomials $\mathbb{I}^q_\Delta(\ell)$ somewhat lacks geometric intuition, while being purely algebraic and combinatorial.

We now review another solution to the quantum functions $\mathbb{I}^q_\Delta(\ell)$, constructed by physicists. The basis of this solution is Gaiotto-Moore-Neitzke's work [GMN13] on spectral networks and the non-abelianization process, which gives a correspondence between GL_K-bundles on a surface $\mathfrak{S}$ with GL_1-bundles on some K-fold branched cover of $\mathfrak{S}$, referred to as a Seiberg-Witten curve. Building on [GMN13], Galakhov, Longhi and Moore [GLM15] proposed how to quantize the parallel transport maps along open paths, using the notion of 'writhe'. Then Gabella [G17] incorporated Bonahon-Wong's idea [BW11] in order to close up the endpoints of an open path, providing a candidate for a quantum function $\mathbb{I}^q_\Delta(\ell)$ for closed curves ℓ, i.e. loops ℓ. Here we briefly review these constructions for $K = 2$, slightly reformulated.

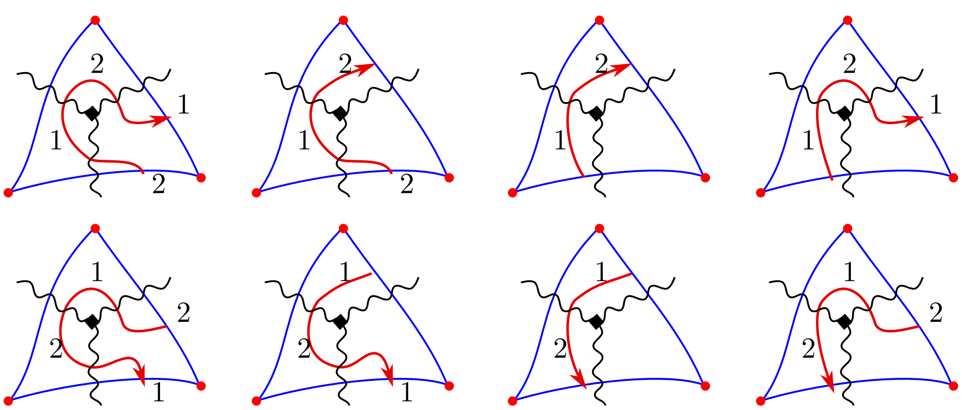
Let $\mathfrak{S}$ be a punctured surface, with an ideal triangulation Δ. For each triangle t of Δ, choose a point v_t in the interior of t, which would be the branch points (denoted by ◆ in the pictures below). For each edge e of Δ, denote by t and s the two ideal triangles having e as a side. Choose a simple path connecting v_t and v_s that meets Δ exactly once transversally on e. These paths will be the branch cuts (denoted by wiggly lines in the pictures below). Take two copies of $\mathfrak{S}$, one called the sheet 1 and the other the sheet 2. Cut along all branch cuts, and glue the two sheets along the cuts interchangeably, to obtain a branched double cover surface $\widetilde{\mathfrak{S}}$; so, for each branch cut, on one side of this cut there is sheet 1 and on the other side is sheet 2.



Take an *oriented* simple loop γ in the original surface $\mathfrak{S}$, considered up to isotopy in $\mathfrak{S}$. We will consider lifts $\widetilde{\gamma}$ in the branched double cover surface $\widetilde{\mathfrak{S}}$, considered up to isotopy in $\widetilde{\mathfrak{S}}$. Note that isotopic loops in $\mathfrak{S}$ yield non-isotopic lifts in $\widetilde{\mathfrak{S}}$ if their relative positions with respect to the branch points are different, as seen in the picture below.



Suppose γ is presented as concatenation of left turn or right turn segments in triangles, each of which is a simple path connecting two distinct sides of a triangle, not passing through the branch point. A lift $\widetilde{\gamma}$ is *non-admissible* if there is an occurrence of one of the following 8 cases (where 1 and 2 are sheet numbers), and *admissible* otherwise.



non-admissible lifs in $\widetilde{\mathfrak{S}}$

The non-abelianization process gives the correspondence (roughly)

$$\mathrm{GL}_2 \text{ (or } \mathrm{SL}_2\text{) parallel transport along } \gamma \;\leftrightarrow \sum_{\substack{\text{admissible lifts} \\ \widetilde{\gamma} \text{ of } \gamma}} \mathrm{GL}_1 \text{ parallel transport along } \widetilde{\gamma}$$

For the case when γ is a (simple) loop, each admissible lift $\widetilde{\gamma}$ corresponds to a Laurent monomial

$$Z_\Delta^{\widetilde{\gamma}} = Z_1^{a_1} Z_2^{a_2} \cdots Z_n^{a_n}$$

where $a_1, \ldots, a_n \in \mathbb{Z}$ are certain coordinates of $\widetilde{\gamma}$ associated to edges of Δ, defined as follows. Each intersection of $\widetilde{\gamma}$ and a lift in $\widetilde{\mathfrak{S}}$ of an edge of Δ can be assigned a sign$\in \{+,-\}$, indicating which sheet it is located at; $+$ means $\widetilde{\gamma}$ is entering the lifted edge of Δ on its left at sheet 1, or on its right at sheet 2, and $-$ means otherwise. Then, for each edge i of Δ, a_i is the net sum of these signs of the intersections of $\widetilde{\gamma}$ and the lifts of i, where the signs $+,-$ are counted as the numbers $+1,-1$. Note that, different lifts $\widetilde{\gamma}$ may yield a same Laurent monomial. Anyhow, the classical setting is

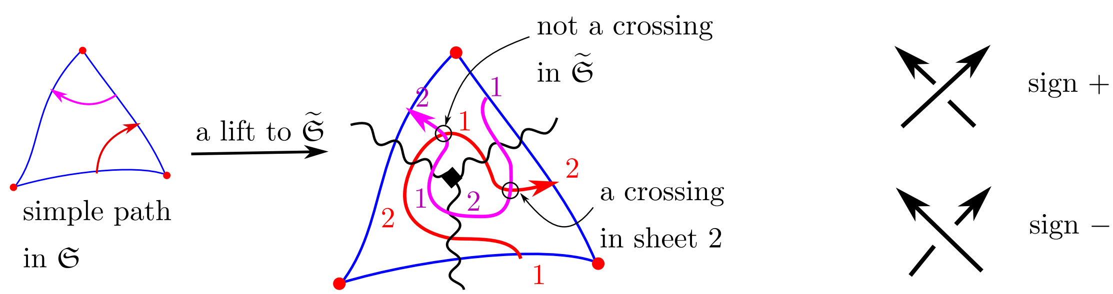
$$\mathbb{I}_\Delta(\gamma) = \sum_{\widetilde{\gamma}} Z_\Delta^{\widetilde{\gamma}},$$

where the sum is over all admissible lifts $\widetilde{\gamma}$ in $\widetilde{\mathfrak{S}}$ of γ, considered up to isotopy in $\widetilde{\mathfrak{S}}$.

An interesting story begins in the quantization of each of this Laurent monomial $Z_\Delta^{\widetilde{\gamma}}$. Galakhov, Longhi and Moore [GLM15] proposed that the classical Laurent monomial $Z_\Delta^{\widetilde{\gamma}}$ be quantized by

$$(\text{coefficient for } \widetilde{\gamma}) \cdot [\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\text{Weyl}}$$

where $[\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\text{Weyl}} = [\widehat{Z}_1^{a_1} \cdots \widehat{Z}_n^{a_n}]_{\text{Weyl}}$, and the $(\text{coefficient for } \widetilde{\gamma}) \in \mathbb{Z}[q^{\pm 1}]$ is defined as follows. Note first that, even though γ is a simple loop in $\mathfrak{S}$, i.e. has no self-intersection, its lift $\widetilde{\gamma}$ can have self-intersections, which we call crossings.



To each crossing we associate a sign$\in \{+,-\}$ as above, at the right; the natural way to distinguish these two kinds of crossings is by putting the curve $\widetilde{\gamma}$ into the 3d space $\widetilde{\mathfrak{S}} \times (-1,1)$, so that the picture as above indicate the over or under-crossing. In the original paper [GLM15], the authors used the 'always going up' elevations for the curve in $\widetilde{\mathfrak{S}} \times (-1,1)$, in which case one does not need to refer to the third dimension and just use the ordering coming from the parameter values of points of the oriented path $\widetilde{\gamma}$; the smaller parameter means the earlier point of $\widetilde{\gamma}$ and hence the 'lower' strand. Anyhow, define

$$\text{wr}(\widetilde{\gamma}) \in \mathbb{Z}$$

be the *writhe* of $\widetilde{\gamma}$, i.e. the net sum of signs of its crossings. The quantization of $Z_\Delta^{\widetilde{\gamma}}$ suggested in [GLM15] is

$$q^{\text{wr}(\widetilde{\gamma})} \cdot [\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\text{Weyl}}$$

This construction in [GLM15] using the always-going-up elevations works fine for an open path γ. In case when γ is a closed path, the resulting quantum Laurent polynomial

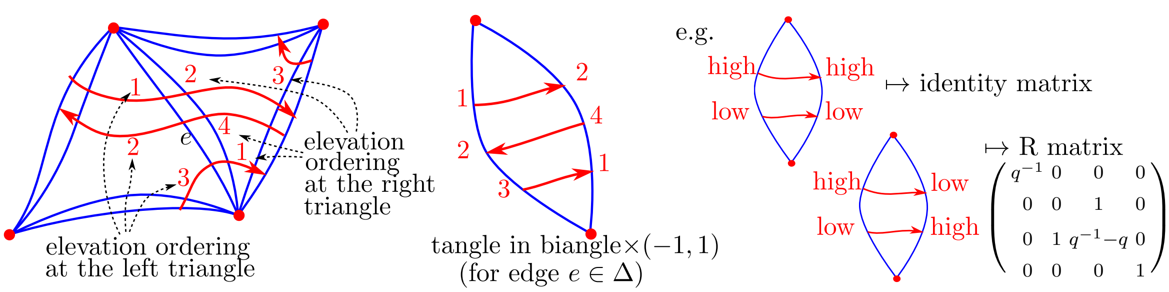
$$\sum_{\widetilde{\gamma}} q^{\text{wr}(\widetilde{\gamma})} \cdot [\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\text{Weyl}}$$

may depend on the choice of a basepoint of γ. In the 3d space $\widetilde{\mathfrak{S}} \times (-1,1)$, the always-going-up path cannot close up!

To remedy this problem, Gabella proposed [G17] a quantization of $Z_\Delta^{\widetilde{\gamma}}$ in the form

$$(\text{correction factor for } \widetilde{\gamma}) \cdot q^{\mathrm{wr}(\widetilde{\gamma})} \cdot [\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\mathrm{Weyl}}$$

where the (correction factor for $\widetilde{\gamma}$) $\in \mathbb{Z}[q^{\pm 1}]$ is defined as follows, inspired by [BW11]. Suppose that we chose a basepoint of γ among the intersection points of γ and the edges of Δ. Say we chose the basepoint x of γ lying in the edge e of Δ. Consider a lift $\widetilde{\gamma}$ in $\widetilde{\mathfrak{S}}$, and embed it into the 3d space $\widetilde{\mathfrak{S}}$ by giving it the always-going-up elevations; so, as one travels from the starting basepoint of $\widetilde{\gamma}$ til the terminal endpoint, the elevation is strictly increasing. In order to close up to get a (framed) knot in $\widetilde{\mathfrak{S}} \times (-1,1)$, one needs to add a 'going-down' path in the end. Following [BW11], Gabella arranges this situation as follows. First, fatten each edge of Δ to an *ideal biangle*, by adding one more ideal arc per each edge of Δ, so that the added arc is isotopic to the relevant original arc. When choosing the always-going-up elevations of $\widetilde{\gamma}$ in $\widetilde{\mathfrak{S}} \times (-1,1)$, we require that the elevations strictly increase along the segments of $\widetilde{\gamma}$ lying over biangles, while they stay constant along the segments lying over triangles. Now, over the biangle corresponding to the above special edge e of Δ over which the initial basepoint of $\widetilde{\gamma}$ lies, we have some r segments of $\widetilde{\gamma}$, where $r-1$ of them are 'going up' and the remaining one of them is 'going down'.



To such an r-segment diagram data in a thickened biangle, Gabella considers certain composition of $r-1$ number of 'R-matrices' of (the standard K-dimensional representation of) $\mathcal{U}_q(\mathfrak{gl}_K)$, which is a certain linear transformation between (certain) vector spaces. To $r=1$ is associated the identity transformation, to $r=2$ is associated a certain twisted version of the R-matrix of $\mathcal{U}_q(\mathfrak{gl}_K)$, and for $r \geq 2$ one cuts the r-segment diagram into 'composition' of $r-1$ number of 2-segment diagrams (joined with parallel $r-2$ segments), and associate the corresponding composition of R-matrices. Gabella's correction factor is then defined as the matrix elements of this composition of R-matrices. It is partially proved in [G17] that the sum

$$\sum_{\widetilde{\gamma}} (\text{correction factor for } \widetilde{\gamma}) \cdot q^{\mathrm{wr}(\widetilde{\gamma})} \cdot [\widehat{Z}_\Delta^{\widetilde{\gamma}}]_{\mathrm{Weyl}}$$

does not depend on the choice of a basepoint of γ. This is Gabella's *quantum holonomy*, providing a candidate for the sought-for quantum function $\mathbb{I}_\Delta^q(\gamma)$.

Theorem 5.1 (K.-Lê-Son; [KLS18]; the equality of the two solutions to quantum regular function)**.** *Gabella's solution to the quantum function $\mathbb{I}^q_\Delta(\ell)$ using his quantum holonomy coincides with Bonahon-Wong(-Allegretti-Kim)'s solution for $\mathbb{I}^q_\Delta(\ell)$ using the Bonahon-Wong quantum trace.*

To be more precise, the equality does not hold on the nose; we had to modify and slightly correct Gabella's construction. For example, in his original proposal Gabella avoided using the (quantum) square-root variables, and in his answer, the factor $[\widehat{Z}^{\widetilde{\gamma}}_\Delta]_{\rm Weyl}$ is replaced by a certain relevant Laurent monomial in the usual quantum variables $\widehat{X}_i = \widehat{Z}_i^2$. We note that the quantum function originally proposed by Gabella this way is not compatible under the quantum mutation maps for flips. Another important aspect of [KLS18] is its full treatment on what happens over biangles, which Gabella explained only partially. In particular, we established an explicit connection to the Reshetikhin-Turaev operator invariants of (framed) tangles over biangles [RT90], associated to the standard 2d representation of $\mathcal{U}_q(\mathfrak{sl}_2)$, which also explains why the R-matrices appear. This complete treatment for biangles is a key step toward the proof of the main theorem of [KLS18], namely the above theorem. In addition, we tried to interpret the language used in [G17] into a mathematical one as much as possible.

6. Future directions : higher rank

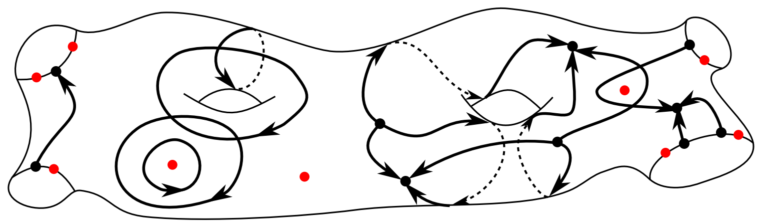
So far we only discussed the quantization problem for the moduli space $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$, i.e. how to quantize the Poisson algebra $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}})$. To recall again, a basic ingredient in the solution of this problem is a canonical basis of $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}})$, built essentially out of the trace-of-holonomy (or, trace-of-monodromy) functions along simple loops in $\mathfrak{S}$.

A natural and important future research direction is a higher rank generalization, i.e. the quantization problem for the space $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$, or the Poisson algebra $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}})$. One might try approaching this problem as in the case of $m=2$, i.e. to begin with searching for a canonical basis of $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}})$ to be quantized. As mentioned before, such a nice basis is found in [GHKK18], called the *theta basis*, enumerated by the set $\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}}(\mathbb{Z}^t)$, and its quantized version is constructed in [DM19]. However, the theta bases and their quantization are almost purely combinatorial, involving only the combinatorics related to the relevant quivers $Q^{(m)}_\Delta$, i.e. the m-triangulation quivers of the ideal trianglations of $\mathfrak{S}$. That is, they do not use the geometry of the surface $\mathfrak{S}$ and the moduli space $\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}$ almost at all, and the computation of each individual basic regular function, i.e. the theta function, is notoriously heavy, since it involves counting of combinatorial objects in high dimensional Euclidean spaces (roughly of dimension being the number of nodes of $Q^{(m)}_\Delta$, or twice that). So, what had been missing for $m \ge 3$ is the geometric model for the duality map

$$\mathscr{A}_{\mathrm{SL}_m,\mathfrak{S}}(\mathbb{Z}^t) \to \mathscr{O}(\mathscr{X}_{\mathrm{PGL}_m,\mathfrak{S}}),$$

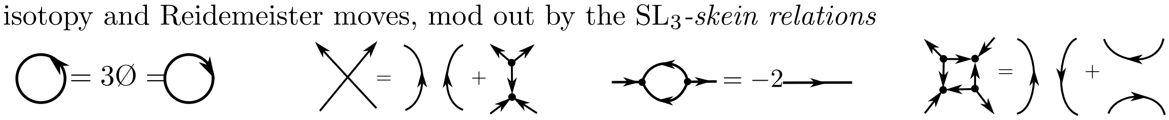
as well as that for its quantization, which are as geometric and computationally straightforward as Fock-Goncharov's [FG06] and Allegretti-Kim's [AK17] answers for $m = 2$.

This task is accomplished for $m = 3$ by a recent work of myself [K20], building on the works of other people on the subject and the theory of SL_3 skein algebras. The most crucial ingredient is the notion of an SL_3-*web* in a surface $\mathfrak{S}$, which is a union of oriented loops and oriented tri-valent graphs on $\mathfrak{S}$, so that each vertex is either a source or a sink; besides these vertices, we also allow transverse double self-intersections, called crossings. When $\mathfrak{S}$ has a boundary, we allow a component graph of an SL_3-web to have 1-valent endpoints lying in the boundary. The SL_3-web living in an n-gon (i.e. a closed disc with n marked points on the boundary) first appeared in Kuperberg's work [K96] on the theory of invariants of representations of SL_3 as well as those of other Lie groups and algebras. Later it was generalized to a surface version by Sikora [S01].



example of an SL_3-web in a surface $\mathfrak{S}$ with boundary
(red points are the punctures and marked points on the boundary)

Sikora defined the *(commutative)* SL_3-*skein algebra* $\mathcal{S}_{\mathrm{SL}_3}(\mathfrak{S}; \mathcal{R})$ of the surface $\mathfrak{S}$, where $\mathcal{R}$ is any ring with unity, as the free $\mathcal{R}$-module generated by all SL_3-webs in $\mathfrak{S}$ considered up to isotopy and Reidemeister moves, mod out by the SL_3-*skein relations*



which are local relations which already appeared in [K96], where the product is given by superposition, i.e. stacking, i.e. union. Sikora [S01] proved the isomorphism

$$\mathcal{S}_{\mathrm{SL}_3}(\mathfrak{S}; \mathbb{Q}) \to \mathscr{O}(\mathscr{L}_{\mathrm{SL}_3, \mathfrak{S}})$$

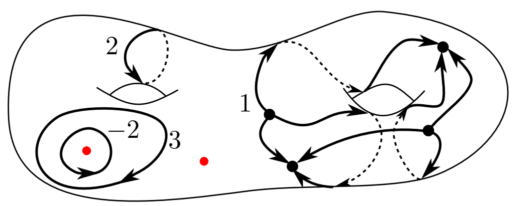
between the SL_3-skein algebra and the SL_3-character variety of $\mathfrak{S}$, sending an oriented loop to the trace-of-holonomy function along that loop. This isomorphism is the first bridge between the world of topology and the world of moduli spaces, and hence is a starting point of work [K20]. Sikora and Westbury [SW07] proved that $\mathcal{S}_{\mathrm{SL}_3}(\mathfrak{S}; \mathbb{Q})$ has a basis consisting of *non-elliptic* SL_3-webs, i.e. SL_3-webs without crossing and without a contractible region bounded either by a loop, a 2-gon, or a 4-gon (as appearing in the SL_3-skein relations). So, the works [S01] and [SW07] together yield the following baby version of the 'duality' map

$$\{\text{non-elliptic } \mathrm{SL}_3\text{-webs in } \mathfrak{S}\} \to \mathscr{O}(\mathscr{L}_{\mathrm{SL}_3, \mathfrak{S}})$$

which is an injective map whose image forms a basis of $\mathscr{O}(\mathscr{L}_{\mathrm{SL}_3,\mathfrak{S}})$.

In [K20] I introduced the notion of an SL_3-*lamination* in $\mathfrak{S}$, which is a non-elliptic SL_3-web in $\mathfrak{S}$ with integer weights on its components, subject to the following condition and equivalence relation: assuming that $\mathfrak{S}$ is a punctured surface (i.e. without boundary),

(1) the weight of a component is non-negative, unless the component is a peripheral loop (i.e. a small loop surrounding a puncture);
(2) the weight of a component involving a 3-valent vertex is 1;
(3) the component of zero weight can be removed;
(4) homotopic components can be combined by summing the weights.



example of an SL_3-lamination in a punctured surface (red dots are punctures)

Extending Sikora and Westbury's 'duality' map, I constructed another 'duality' map

$$\{\mathrm{SL}_3\text{-laminations in } \mathfrak{S}\} \to \mathscr{O}(\mathscr{X}_{\mathrm{SL}_3,\mathfrak{S}})$$

which is an injective map whose image forms a basis of $\mathscr{O}(\mathscr{X}_{\mathrm{SL}_3,\mathfrak{S}})$. The peripheral loops should be treated a bit differently than before.

For a triangulable punctured surface $\mathfrak{S}$, Frohman and Sikora [FS20] constructed a certain coordinate system on the set of all non-elliptic SL_3-webs in $\mathfrak{S}$, where the coordinate system depends on the choice of an ideal triangulation Δ of $\mathfrak{S}$; in fact, they also worked on generalized marked surfaces (i.e. surfaces with boundary). Douglas and Sun [DS20a] considered a modified version, inspired by the work of a physicist Xie [X13] on the Fock-Goncharov duality map:

$$\{\text{non-elliptic } \mathrm{SL}_3\text{-webs in } \mathfrak{S}\} \to (\tfrac{1}{3}\mathbb{Z})^{\mathcal{V}(Q^{(3)}_\Delta)}, \quad \ell \mapsto (a_{v;\Delta}(\ell))_{v:\text{nodes of } \mathrm{Q}^{(3)}_\Delta};$$

in particular, there is a $\frac{1}{3}\mathbb{Z}$-valued coordinate $a_{v;\Delta}(\ell)$ per each node v of the 3-triangulation quiver $Q^{(3)}_\Delta$ of Δ. Under the change of ideal triangulations $\Delta \rightsquigarrow \Delta'$, it is shown [DS20b] that the Douglas-Sun coordinates transform precisely by the tropical version of the sequence of cluster $\mathscr{A}$-mutations as mentioned in §1, in particular corresponding to the sequence (of length 4) of quiver mutations relating $Q^{(3)}_\Delta$ and $Q^{(3)}_{\Delta'}$.

I first extended the Douglas-Sun coordinates to SL_3-laminations, called the *tropical coordinates*, and identified the image set as a certain subset of $(\frac{1}{3}\mathbb{Z})^{\mathcal{V}(Q^{(3)}_\Delta)}$. As a consequence, I showed that the SL_3-laminations whose tropical coordinates are all integers (instead of lying in $\frac{1}{3}\mathbb{Z}$), which I called *congruent*, are in bijection with the integral lattice:

$$\{\text{congruent } \mathrm{SL}_3\text{-laminations in } \mathfrak{S}\} \xrightarrow{1:1} \mathbb{Z}^{\mathcal{V}(Q^{(3)}_\Delta)} \subset (\tfrac{1}{3}\mathbb{Z})^{\mathcal{V}(Q^{(3)}_\Delta)}$$

Combining with the result [DS20b] on the transformation formula, this provides a geometric model for the set $\mathscr{A}_{\mathrm{SL}_3,\mathfrak{S}}(\mathbb{Z}^t)$ of tropical integer points of the moduli space $\mathscr{A}_{\mathrm{SL}_3,\mathfrak{S}}$, or of the cluster $\mathscr{A}$-variety $\mathscr{A}_{|Q^{(3)}_\Delta|}$:

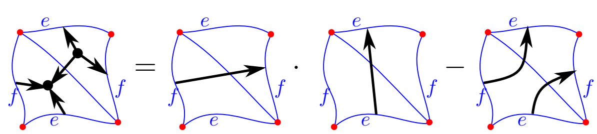
$$\mathscr{A}_{\mathrm{SL}_3,\mathfrak{S}}(\mathbb{Z}^t) = \mathscr{A}_{|Q^{(3)}_\Delta|}(\mathbb{Z}^t) \leftrightarrow \{\text{congruent SL}_3\text{-laminations in } \mathfrak{S}\},$$

which is in the style of Fock-Goncharov's work on $m = 2$.

The first main result of [K20] is the duality map

$$\mathbb{I} : \mathscr{A}_{\mathrm{SL}_3,\mathfrak{S}}(\mathbb{Z}^t) = \{\text{congruent SL}_3\text{-laminations in } \mathfrak{S}\} \longrightarrow \mathscr{O}(\mathscr{X}_{\mathrm{PGL}_3,\mathfrak{S}}).$$

For each congruent SL_3-lamination ℓ on $\mathfrak{S}$, I constructed a regular function $\mathbb{I}(\ell)$ on $\mathscr{X}_{\mathrm{PGL}_3,\mathfrak{S}}$. Per each ideal triangulation Δ of $\mathfrak{S}$, this function $\mathbb{I}(\ell)$ can be written as a Laurent polynomial $\mathbb{I}_\Delta(\ell)$ in the cluster $\mathscr{X}$-coordinates associated to the nodes of $Q^{(3)}_\Delta$ with integer coefficients. The computation of this Laurent polynomial is relatively straightforward. For a loop, it is essentially the trace of holonomy along the loop, where the 3×3 holonomy matrix can be obtained as a product of basic monodromy matrices, like in the case of $\mathscr{X}_{\mathrm{PGL}_2,\mathfrak{S}}$; these basic monodromy matrices for small pieces of loops are essentially already in [FG06], and we just need to suitably normalize them. For an SL_3-web having 3-valent vertices, we either use SL_3-skein relations to express it using only loops; for example, in a once-punctured torus, see below:



Another way to compute the function for an SL_3-web is to first fatten each edge of Δ to a biangle, then push all 3-valent vertices of the SL_3-web to biangles through isotopy, and use the state-sum type formula developed in [K20] which boils the computation down to computations in the triangles and biangles; for each triangle it's just about the basic monodromy matrices mentioned above, and for each biangle the computation is about the Reshetikhin-Turaev invariants for tangles associated to the standard 3d representation of $\mathcal{U}(\mathfrak{sl}_3)$ [RT90].

One of the crucial statements about this duality map $\mathbb{I}$ is that, for each congruent SL_3-lamination ℓ, the corresponding function $\mathbb{I}_\Delta(\ell)$ has a unique highest degree term given by $\prod_{v\in\mathcal{V}(Q^{(3)}_\Delta)} X_v^{a_{v;\Delta}(\ell)}$, where $a_{v;\Delta}(\ell)$ are the tropical coordinates of ℓ at the nodes v of the 3-triangulation quiver $Q^{(3)}_\Delta$. This highest term statement was already predicted in [X13], and is only fully formulated and proven in [K20], with the help of the computational machinery developed in [K20], namely the state-sum type formula. This highest term statement is a major step toward the proof that the image of $\mathbb{I}$ forms a basis of $\mathscr{O}(\mathscr{X}_{\mathrm{PGL}_3,\mathfrak{S}})$.

Moreover, in [K20] a quantum version $\mathbb{I}^q_\Delta$ of this duality map $\mathbb{I}_\Delta$ is also constructed. For this, I first established an SL_3 version of the quantum trace map, which is a map from the non-commutative SL_3-skein algebra $\mathcal{S}^q_{\mathrm{SL}_3}(\mathfrak{S})$ [S05] to a suitable ring of non-commutative Laurent polynomials, generated by the 'cube-roots' of the quantum variables $\widehat{X}_v^{\pm 1}$, $v \in \mathcal{V}(Q^{(3)}_\Delta)$.

We end this note by mentioning several future problems about this SL_3-PGL_3 duality.

(1) (Laurent coefficient positivity) In the Laurent polynomial expression $\mathbb{I}_\Delta(\ell) \in \mathscr{O}(\mathscr{X}_{\mathrm{PGL}_3,\mathfrak{S}})$, the coefficients are integers. Are these coefficients non-negative?

(2) (structure constant positivity) From the first main theorem of [K20], for two congruent SL_3-laminations $\ell, \ell' \in \mathscr{A}_{\mathrm{SL}_3,\mathfrak{S}}(\mathbb{Z}^t)$ we know that the product-to-sum formula

$$\mathbb{I}(\ell)\mathbb{I}(\ell') = \sum_{\ell''} c(\ell, \ell'; \ell'')\mathbb{I}(\ell'')$$

holds for some integer structure constants $c(\ell, \ell'; \ell'') \in \mathbb{Z}$, where the sum is a finite sum. Are these structure constants $c(\ell, \ell'; \ell'')$ positive? Or, is there a natural way to modify $\mathbb{I}$ a little bit (like, replacing bangles by bracelets), so that these constants are positive?

(3) Does this duality map $\mathbb{I}_\Delta$ coincides with Gross-Hacking-Keel-Kontsevich's theta basis duality map for the quiver $Q^{(3)}_\Delta$ [GHKK18]?

(4) Same questions for the quantum duality map $\mathbb{I}^q_\Delta$, with (3) for Davison-Mandel's quantum theta basis duality map [DM19].

(5) Does the value of the quantum duality map $\mathbb{I}^q_\Delta$ (or the SL_3 quantum trace) at a simple oriented loop coincide with Gabella's quantum holonomy (for $K = 3$) [G17], perhaps after some modification?

(6) Construct higher rank generalization, i.e. the SL_m-PGL_m duality, for $m \geq 4$.

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