$\int_0^l k ds < 4\pi \Rightarrow \iint_R K d\sigma < 8\pi$. Assume h_v has 3 critical points. $\Rightarrow \exists$ two minima $\Rightarrow \iint_R K d\sigma \geq 8\pi$

Only two critical points of h_{v_0} $\Rightarrow$ maximum at $\alpha(s_1)$, minimum at $\alpha(s_2)$

Planes perpendicular to v_0 and between $\alpha(s_1)$ and $\alpha(s_2)$ intersect C in exactly two points. $\therefore$ Can connect these pairs by line segments. $\therefore C = \partial D$, $D \approx$ disk. $\therefore$ C is unknotted.

* Any knot has a quadrisecant, a straight line which intersects the knot four times (Pannwitz, 1933).

Project C onto $P \supset$ quadrisecant $\Rightarrow 4\pi \leq \int_{Proj(C)} k ds \leq \int_C k ds$.

alternating (Denne, 2004)

* Fary's proof: $\int_C k ds =$ Average(total curvatures of projections of C on all planes in $\mathbb{R}^3$) $\nRightarrow$ new proof of Fenchel's theorem

$\int_C k ds = \frac{1}{4\pi}\iint_{S^2}\int k(proj(C)) ds\, d\sigma \quad \geq 4\pi$

Given a knotted curve C and its orthogonal projection onto a plane Π there exists a point $p \in \Pi$ such that any ray in Π emanating from p intersects the projected curve in at least two points.

* Milnor's proof: Given a knot C, for any vector v there exists a plane Π perpendicular to v which intersects C in at least 4 points.

$\int_C k ds = \pi \cdot \underset{v \in S^2}{\text{Average}}(\#(\text{critical points of } h_v(s) = \langle \alpha(s), v\rangle))$

: $\int_C k ds = 2\pi$

Milnor also showed that there exists a plane which intersects C in at least six points.

* The Crofton formula $\quad C \subset S^2$. $\quad p \in C$. $\quad p \longleftrightarrow p$를 지나는 S^2의 대원들. (oriented)

$C \longleftrightarrow C$를 만나는 대원들의 총 집합 B. $\quad$ C의 길이 $\approx$ B의 measure : ? (counting multiplicity)

B의 measure $=$ Area$\{p_\gamma : \gamma \in B\}$ $= B(C)$ $\quad$ C는 작은 선분들의 연결.

$\therefore$ Length$(C) = a \cdot$ Area$(B(C))$, $\quad a$: constant. $\quad$ What is a?

C: 대원 $\Rightarrow$ $B(C) = S^2$ with multiplicity 2. $\quad$ p_γ: γ의 북극점

$\therefore 2\pi = a \cdot 2 \cdot 4\pi$, $\quad a = \frac{1}{4}$. $\quad$ γ_p: p를 북극점으로 갖는 대원

$\therefore$ Crofton formula : $\iint_{p \in B(C)} n(p)\, d\sigma = 4$ Length(C), $\quad B(C)$: C를 만나는 대원의 북극점들의 집합

$n(p) = \gamma_p$가 C를 만나는 점의 개수 (p를 북극점으로 갖는 대원이)

Another proof of Fary-Milnor: Suppose $\int_C k ds < 4\pi$.

$\bar{C}$: tangent indicatrix of C. $\quad \frac{1}{4}\iint_{S^2} n(p)\, d\sigma = $ Length$(\bar{C}) = \int_C k ds < 4\pi$.

$\therefore$ Average of $n(p) < 4$ on S^2. $\quad \therefore \exists p \in S^2$ s.t. $n(p) \leq 3$.

$\exists q_1, q_2, q_3 \in \gamma_p \cap \bar{C} \Rightarrow \exists \bar{q}_1, \bar{q}_2, \bar{q}_3 \in C$ s.t. their tangent vectors at C // the plane containing γ_p. $\therefore$ $h(s) := \langle p, \alpha(s)\rangle$, the height of $\alpha(s)$ relative to the plane, has three critical points. $\therefore \exists$ 최대, 최소, 변곡점.

Another proof of Fenchel : $\bar{C}$ is not contained in any open hemisphere.

$\therefore n(p) \geq 2$, $\forall p \in S^2$ $\quad \therefore \int_C k ds \geq 2\pi$. The great circle is the only curve on S^2 with $n \equiv 2$?

10/21

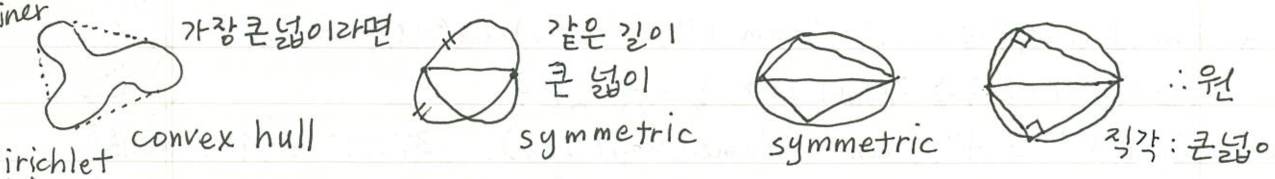
등주부등식 문제 (等周 부등식) isoperimetric problem

일정한 둘레의 곡선 중 가장 큰 넓이를 둘러싸는 것은?

dual problem : 일정한 넓이를 둘러싸는 곡선 중 가장 짧은 것은?

$4\pi A \leq L^2$, $\quad$ "=" $\Leftrightarrow$ 원일 때

*Steiner



Weierstrass : 이의 제기. 가장 큰 넓이의 영역이 존재함을 보였어야.

* Analytic proof :

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, $\quad A = -\int_C y\, dx = -\int_a^b y \frac{dx}{dt} dt$ $\quad$ ($\iint_D 1\, dxdy$: Green's theorem $\int_C P dx + Q dy = \iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy$)

$t := \frac{2\pi}{L} s$. $\quad \int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] dt = \int_0^{2\pi} \left(\frac{ds}{dt}\right)^2 dt = \frac{L^2}{2\pi}$

$\therefore L^2 - 4\pi A = 2\pi \int_0^{2\pi} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + 2y\frac{dx}{dt}\right] dt$

$= 2\pi \int_0^{2\pi} \left(\frac{dx}{dt} + y\right)^2 dt + 2\pi \int_0^{2\pi} \left[\left(\frac{dy}{dt}\right)^2 - y^2\right] dt \geq 0$. $\quad$ Why?

Lemma (Wirtinger, Poincaré) If $y(t)$ is a smooth function with period 2π and $\int_0^{2\pi} y(t)\, dt = 0$, then $\int_0^{2\pi} \left(\frac{dy}{dt}\right)^2 dt \geq \int_0^{2\pi} y^2 dt$, with equality if and only if $y = a\cos t + b\sin t$.

$$(*) \le \left(\int_0^L\int_0^\pi r^2\,d\theta\,ds\right)^{\frac12}\left(\int_0^L\int_0^\pi \sin^2\theta\,d\theta\,ds\right)^{\frac12} \le (2AL)^{\frac12}\left(\frac{\pi}{2}L\right)^{\frac12}. \quad \therefore 4\pi A \le L^2.$$

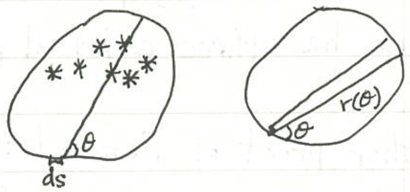
pf) $f(\theta) = \sum_{k=1}^{\infty}(a_k\cos k\theta + b_k\sin k\theta)$, $a_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta\,d\theta$, $b_m = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta\,d\theta$

Parseval : $\int_0^{2\pi} f^2\,d\theta = \pi\sum(a_k^2+b_k^2)$.

$f'(\theta) = \sum(-k a_k \sin k\theta + k b_k\cos k\theta)$, Bessel : $\pi\sum k^2(a_k^2+b_k^2) \le \int_0^{2\pi}(f')^2 d\theta$

$\therefore \int_0^{2\pi}(f')^2\,d\theta - \int_0^{2\pi} f^2\,d\theta \ge \pi\sum_{k=2}^{\infty}(k^2-1)(a_k^2+b_k^2) \ge 0$

"=" $\Rightarrow f = a\cos\theta + b\sin\theta$



* Knothe's proof. $A \ge \frac12\int_0^\pi r(\theta)^2 d\theta$

$2\pi A = \int_0^\pi\int_0^L r(s,\theta)\sin\theta\,ds\,d\theta. = (*)$

Integrate on $C\times C\times[0,\pi]\times[0,\pi]$:

$0 \le \int_C\int_C\int_0^\pi\int_0^\pi (r_1\sin\theta_2 - r_2\sin\theta_1)^2\,d\theta_1 d\theta_2 ds_1 ds_2$

$= 2\int_C\int_0^\pi\int_0^\pi\int_C r_1^2\sin^2\theta_2\,ds_1\,d\theta_1\,d\theta_2\,ds_2 - 2\int_C\int_C\int_0^\pi\int_0^\pi r_1\sin\theta_1\,r_2\sin\theta_2\,d\theta_1 d\theta_2 ds_1 ds_2$

$\le 4A\int_C\int_0^\pi\int_C \sin^2\theta_2\,ds_1\,d\theta_2\,ds_2 - 2\left(\int_C\int_0^\pi r_1\sin\theta_1\,d\theta_1\,ds_1\right)^2 = 2\pi A\,(L^2-4\pi A).$

$4\pi A = L^2 \Rightarrow r_1\sin\theta_2 - r_2\sin\theta_1 \equiv 0$ $\therefore$ For $s_1=s_2=0$, $\theta_2=\frac{\pi}{2}$ we have

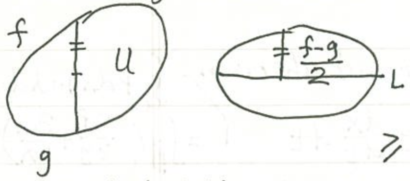
$r_1(0,\theta_1) = r_2(0,\frac{\pi}{2})\sin\theta_1$ $\therefore$ C is a circle with diameter $r_2(0,\frac{\pi}{2})$.

* 고차원 등주부등식 : $V = Volume(U)$, $A = Area(\partial U)$, $U\subset\mathbb{R}^n$.

같은 부피의 영역 중 겉넓이가 가장 작은 것은?

$n^n\omega_n V^{n-1} \le A^n$, $\omega_n = Volume(\text{unit ball})$. $36\pi V^2 \le A^3$: $U\subset\mathbb{R}^3$

Steiner's symmetrization



$Volume(U) = \int_D (f-g) = Volume(U_L)$

$Area(\partial U) = \int_D\sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2} + \int_D\sqrt{1+(\frac{\partial g}{\partial x})^2+(\frac{\partial g}{\partial y})^2}$

$\ge 2\int_D\sqrt{1+\left(\frac{\partial f-\partial g}{2\partial x}\right)^2+\left(\frac{\partial(f-g)}{2\partial y}\right)^2} = Area(\partial U_L)$

$\because |\vec{u}|+|\vec{v}| \ge |\vec{u}+\vec{v}|$

Ω가 가장 작은 겉넓이를 갖는다면 Ω는 L에 대칭. L은 임의. $\therefore$ Ω는 ball 이다.

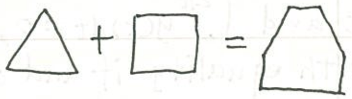
이것도 존재증명이 있어야: U를 평면 L_i에 symmetrization $\Rightarrow U_{L_i}\to$ ball as $i\to\infty$.

10/26

* Brunn-Minkowski inequality: $A+B = \{x+y : x\in A, y\in B\}$ (sum), $A,B\subset\mathbb{R}^n$

$[Volume(A+B)]^{\frac1n} \ge [Volume(A)]^{\frac1n} + [Volume(B)]^{\frac1n}$

B_r : ball of radius r in $\mathbb{R}^n$, $U_r = U + B_r$

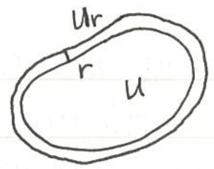


"=" $\Leftrightarrow$ A와 B는 닮은꼴. $B = tA + r$

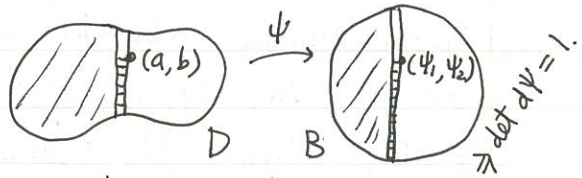
$Volume(U_r) \ge \left([Volume(U)]^{\frac1n} + [Volume(B_r)]^{\frac1n}\right)^n$

$= \left(V^{\frac1n} + [\omega_n r^n]^{\frac1n}\right)^n \ge V + nV^{\frac{n-1}{n}}\cdot\omega_n^{\frac1n}\cdot r$

$\therefore \frac{1}{r}\left(Volume(U_r) - V\right) \ge n\,\omega_n^{\frac1n}\,V^{\frac{n-1}{n}}$, LHS = A.



* Gromov's proof: 넓이 보존사상



B는 D와 같은 넓이를 갖는 원

$Area\{x\le a\} = Area\{x\le\psi_1\}$

$Length\{x=a, y\le b\}\cdot da = Length\{x=\psi_1, y\le\psi_2\}\cdot d\psi_1 \Rightarrow \psi$는 넓이를 보존한다.

$D\psi$: Jacobian matrix of ψ is lower triangular. $D\psi = \begin{pmatrix}\frac{\partial\psi_1}{\partial x} & \frac{\partial\psi_1}{\partial y} \\ \frac{\partial\psi_2}{\partial x} & \frac{\partial\psi_2}{\partial y}\end{pmatrix}$, $\frac{\partial\psi_1}{\partial y} = 0$

$\Rightarrow 2(\det D\psi)^{\frac12} = 2\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial y}\right)^{\frac12} \le \frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} = div\,V$

$\therefore 2 \le div\,V$. $V(p) := \psi(p)$의 위치벡터 $\Rightarrow$ V: vector field on D. $V = (\psi_1,\psi_2)$

$\therefore 2\,Area(D) \le \iint_D div\,V = \int_{\partial D}\langle V,\eta\rangle$, η : ∂D의 외향 단위법선 벡터

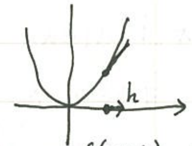
∂D위의 점에서 $|V| \le (\frac{A}{\pi})^{\frac12}$ $\therefore 2A \le |V|\cdot L \le \sqrt{\frac{A}{\pi}}\cdot L$ $\therefore 4\pi A \le L^2$.

"=" $\Leftrightarrow$

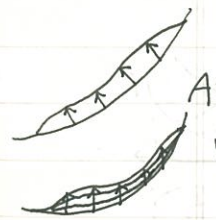
* Minimal Surfaces 극소곡면

K=0 : plane, cylinder, cone

H=0 : minimal surface : locally area minimizing.



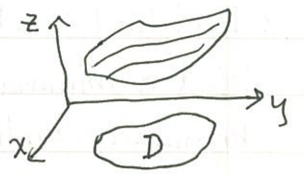
calculus of variations. infinite dimensional derivative $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$



M^2 : minimal graph in $\mathbb{R}^3$, $z = f(x,y)$

$Area(graph(f)) = \iint_D\sqrt{1+|\nabla f|^2}\,dxdy$, $|\nabla f|^2 = (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$.

variation : $f(x,y) + t\,h(x,y)$, $h=0$ on ∂D.



$f^t(x,y)$ ″, $A(t) = Area(graph(f^t))$

$= \iint_D\sqrt{1+|\nabla f + t\nabla h|^2}\,dxdy$

$\frac{d}{dt}A(t)\Big|_{t=0} = \iint_D\frac{\langle\nabla h,\nabla f\rangle}{\sqrt{1+|\nabla f|^2}}\,dxdy = \iint_D div\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\,dxdy = 0,\ \forall h$

$\therefore \frac{\partial}{\partial x}\left(\frac{\frac{\partial f}{\partial x}}{\sqrt{1+|\nabla f|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial f}{\partial y}}{\sqrt{1+|\nabla f|^2}}\right) = 0$, $H = \frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{\frac32}} = 0$

: the minimal surface equation

$\therefore$ graph(f) is a minimal surface $\Leftrightarrow A'(0)=0$ for all variations fixing its boundary.

$\therefore$ 〃 is critical wrt area.

숙제 #6) P.47 6
P.212 11, 12, 13, 14

Theorem If $f(x,y)$ satisfies the minimal surface equation in D and if f is continuous on $\bar{D}$, then $\text{Area}(\text{graph}(f)) \le \text{Area}(\text{graph}(\tilde f))$ for any function $\tilde f(x,y)$ in D having the same values as $f(x,y)$ on ∂D.

10/28

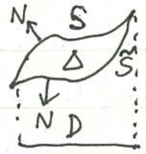
pf) In the domain $D\times\mathbb{R}\subset\mathbb{R}^3$ consider the unit vector field $v(x,y,z)$

$v=\frac{1}{W}(-f_x,-f_y,1)$, $W=\sqrt{1+f_x^2+f_y^2}$. $\Rightarrow$ v: unit normal to graph(f)

$v_z\equiv 0$, $(v_1)_x+(v_2)_y=\frac{1}{W^3}\left[(1+f_y^2)f_{xx}+(1+f_x^2)f_{yy}-2f_xf_yf_{xy}\right]$

$\therefore$ $\text{div}\, v\equiv 0$ in $D\times\mathbb{R}$. $S:=\text{graph}(f)$, $\tilde S:=\text{graph}(\tilde f)$ $\Rightarrow$ $S-\tilde S=\partial\Delta$,

Δ: open set in $D\times\mathbb{R}$.



$0=\iiint_\Delta \text{div}\, v\, dxdydz=\iint_{S-\tilde S}\langle v,N\rangle d\sigma$, N: unit normal to $S-\tilde S$.

$v\equiv N$ on S.

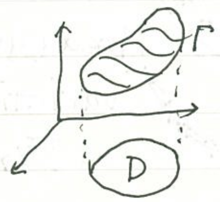
$\text{Area}(S)=\iint_S\langle v,N\rangle d\sigma=\iint_{\tilde S}\langle v,N\rangle d\sigma\le\iint_{\tilde S}1\, d\sigma=\text{Area}(\tilde S)$. "$\le$" $\Rightarrow$ "$<$".

ex)

* Plateau problem (1847)

Given a Jordan curve Γ in $\mathbb{R}^m$, find a surface of least area among all surfaces with Γ as boundary.

Douglas, Radó (1931)



* Dirichlet problem : Γ: $f(x,y)=\varphi$ on ∂D

Find a minimal surface $z=f(x,y)$ satisfying the boundary condition.

Radó (1924): The Dirichlet problem is solvable for all $\varphi\in C^0(\partial D)$ if D is convex.

(P.197~209, do Carmo)

* parametric surfaces : $X(u,v)$. $X^t(u,v)=X(u,v)+th(u,v)N(u,v)$.

$X^t_u=X_u+thN_u+th_uN$, $X^t_v=X_v+thN_v+th_vN$

$\therefore E^t=E+th\cdot 2\langle X_u,N_u\rangle+t^2(h^2\langle N_u,N_u\rangle+h_u^2)$

$F^t=F+th(\langle X_u,N_v\rangle+\langle X_v,N_u\rangle)+t^2(h^2\langle N_u,N_v\rangle+h_uh_v)$

$G^t=G+th\cdot 2\langle X_v,N_v\rangle+t^2(h^2\langle N_v,N_v\rangle+h_vh_v)$

$\langle X_u,N_u\rangle=-e$, $\langle X_u,N_v\rangle=\langle X_v,N_u\rangle=-f$, $\langle X_v,N_v\rangle=-g$

$H=\frac{1}{2}\frac{Eg-2fF+Ge}{EG-F^2}$ $\therefore$ $E^tG^t-(F^t)^2=EG-F^2-2th(Eg-2Ff+Ge)+R$

$=(EG-F^2)(1-4thH)+R$, R: t의 2,3,4 차항들

$\therefore A(t):=\text{Area}(X^t)=\iint_D\sqrt{E^tG^t-(F^t)^2}\,dudv=\iint_D\sqrt{1-4thH+\bar R}\sqrt{EG-F^2}\,dudv$

$A'(0)=-\iint_S 2hH\, d\sigma$ $\therefore$ $2\vec H=-\nabla\text{Area}$, $\vec H:=HN$. $\bar R=R/(EG-F^2)$

$\because hH=\langle hN, HN\rangle$

* minimal surface, soap film, Plateau, Douglas-Radó

* $X(u,v)$, u,v: isothermal if $\langle X_u,X_u\rangle=\langle X_v,X_v\rangle$ and $\langle X_u,X_v\rangle=0$. (*)

Proposition $X(u,v)$: isothermal. $X_{uu}+X_{vv}=2\lambda^2\vec H$, $\lambda^2=\langle X_u,X_u\rangle=\langle X_v,X_v\rangle$

pf) $\frac{\partial}{\partial u}(*)\Rightarrow\langle X_{uu},X_u\rangle=\langle X_{vu},X_v\rangle=-\langle X_u,X_{vv}\rangle$ $\therefore\langle X_{uu}+X_{vv},X_u\rangle=0$

$\frac{\partial}{\partial v}(*)\Rightarrow\langle X_{vv},X_v\rangle=\langle X_{uv},X_u\rangle=-\langle X_{uu},X_v\rangle$ $\therefore\langle X_{uu}+X_{vv},X_v\rangle=0$

$\therefore X_{uu}+X_{vv}\,/\!/\,N$. $X(u,v)$: isothermal $\Rightarrow$ $H=\frac{g+e}{2\lambda^2}$.

$\therefore 2\lambda^2H=e+g=\langle N,X_{uu}+X_{vv}\rangle$ $\therefore X_{uu}+X_{vv}=2\lambda^2\vec H$.

* $X(u,v)$ is minimal. $\Leftrightarrow$ $H\equiv 0$

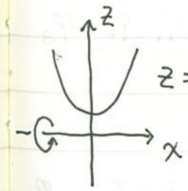
$\Updownarrow$ $\Updownarrow$

$A'(0)=0$, $\forall$variations of X $\Leftrightarrow$ x,y,z are harmonic on X.

11/2

* 1744. Euler showed that the rotation of the catenary catenoid is a minimal surface.

1860. Bonnet proved that the catenoid is the only minimal surface of revolution.



$z=f(x)$. rotate around the x-axis $\Rightarrow$ $\sqrt{z^2+y^2}=f(x)$.

$z=\sqrt{f^2-y^2}$. $z_x=\frac{ff'}{\sqrt{f^2-y^2}}$, $z_y=\frac{-y}{\sqrt{f^2-y^2}}$

$z_{xx}=\frac{f^3f''-f'^2y^2-ff''y^2}{\sqrt{f^2-y^2}^3}$, $z_{xy}=\frac{ff'y}{\sqrt{f^2-y^2}^3}$, $z_{yy}=\frac{-f^2}{\sqrt{f^2-y^2}^3}$

$\therefore (1+z_y^2)z_{xx}-2z_xz_yz_{xy}+(1+z_x^2)z_{yy}=$

$$= \frac{1}{\sqrt{f^2-y^2}^5}\left(f^5f''-f^2f'^2y^2-f^3f''y^2+2f^2f'^2y^2-f^4+f^2y^2-f^4f'^2\right)$$
$$= \frac{1}{\sqrt{f^2-y^2}^5}(f^2-y^2)f^2(ff''-f'^2-1)=0$$
$$(ff''-f'^2-1)' = f'f''+ff'''-2f'f''=0 \quad \therefore ff'''-f'f''=0$$
$$\therefore \left(\frac{f''}{f}\right)' = \frac{f'''f-f''f'}{f^2}=0 \quad \therefore \frac{f''}{f}=\text{const} \quad \therefore f=\cosh x$$

$f=\sinh x$ cannot be a solution because $ff''-f'^2=1$.
$\therefore X(u,v)=(\cosh u\cos v, \cosh u \sin v, u)$

* helicoid : $X(u,v)=(\sinh u\cos v, \sinh u\sin v, v)$
Meusnier constructed in 1776, Catalan showed it is the only ruled minimal surface in 1842.
l : line in the helicoid S. ρ_l: rotation of $\mathbb{R}^3$ about l by $180°$
$\Rightarrow \rho_l$ is an isometry. Let $\vec{H}(p)$ be the mean curvature vector of S at $p\in l$. Since ρ_l is an isometry of $\mathbb{R}^3$ $\rho_l(\vec{H}(p))$ is also the mean curvature vector of $\rho_l(S)$ at $\rho_l(p)$. But $\rho_l(S)=S$ and $\rho_l(p)=p$. Hence $\vec{H}(p)=\rho_l(\vec{H}(p))$ and so $\vec{H}(p)=0$.
Since l and p are arbitrary $\vec{H}\equiv 0$ on the helicoid.

* $X(u,v)$: minimal, u,v: isothermal $\Leftrightarrow x,y,z$: harmonic $\Rightarrow$ complex analytic
$f:\mathbb{C}\to\mathbb{C}$ analytic $f(z)=f_1(u,v)+if_2(u,v)$, $z=u+iv$
$\Leftrightarrow \frac{\partial f_1}{\partial u}=\frac{\partial f_2}{\partial v}, \quad \frac{\partial f_1}{\partial v}=-\frac{\partial f_2}{\partial u}$: Cauchy-Riemann equation
$\varphi_1(z)=\frac{\partial x}{\partial u}-i\frac{\partial x}{\partial v}, \quad \varphi_2(z)=\frac{\partial y}{\partial u}-i\frac{\partial y}{\partial v}, \quad \varphi_3(z)=\frac{\partial z}{\partial u}-i\frac{\partial z}{\partial v}$

<u>Lemma</u> u,v are isothermal if and only if $\varphi_1^2+\varphi_2^2+\varphi_3^2\equiv 0$.
In case u,v are isothermal, $X(u,v)$ is minimal if and only if $\varphi_1,\varphi_2,\varphi_3$ are complex analytic.
pf) $\varphi_1^2+\varphi_2^2+\varphi_3^2=|X_u|^2-|X_v|^2-2i\langle X_u,X_v\rangle=E-G-2iF$.
$X_{uu}+X_{vv}=0$, $X_{uv}=X_{vu} \Leftrightarrow$ Cauchy-Riemann equations for $\varphi_1,\varphi_2,\varphi_3$.
3미지수, 1방정식 $\longrightarrow$ 2 미지수
$\varphi_1^2+\varphi_2^2+\varphi_3^2=(\varphi_1+i\varphi_2)(\varphi_1-i\varphi_2)+\varphi_3^2$: $f=\varphi_1-i\varphi_2$, $g=\frac{\varphi_3}{\varphi_1-i\varphi_2}$
$\therefore \varphi_1=\frac{1}{2}f(1-g^2)$, $\varphi_2=\frac{i}{2}f(1+g^2)$, $\varphi_3=fg$. $\left(\because \varphi_1+i\varphi_2=-\frac{\varphi_3^2}{\varphi_1-i\varphi_2}\right)$

$\therefore x=\mathrm{Re}\int\frac{1}{2}f(1-g^2)dz$, $y=\mathrm{Re}\int\frac{i}{2}f(1+g^2)dz$, $z=\mathrm{Re}\int fg\,dz$
f: holomorphic, g: meromorphic : Weierstrass representation formula

ex) $M=\mathbb{C}$, $g(z)=-e^z$, $f=-e^{-z}$: Catenoid
$\varphi_1=\sinh z$, $\varphi_2=-i\cosh z$, $\varphi_3=1$
$x=\mathrm{Re}\int\sinh z\,dz=\cos v\cdot\cosh u$, $y=\mathrm{Re}\int -i\cosh z\,dz=\mathrm{Re}(-i\sinh z)$
$=\sin v\cdot\cosh u$, $z=\mathrm{Re}\int dz=u$.
$\cosh z=\cosh u\cos v+i\sinh u\sin v$, $\sinh z=\sinh u\cos v+i\cosh u\sin v$

ex) Helicoid : $M=\mathbb{C}$, $g(z)=-ie^z$, $f=e^{-z}$.
$\varphi_1=\cosh z$, $\varphi_2=-i\sinh z$, $\varphi_3=-i$
$x=\cos v\cdot\sinh u$, $y=\sin v\cdot\sinh u$, $z=v$

ex) Enneper's surface : $M=\mathbb{C}$, $g(z)=z$, $f(z)=1$.
$x=\mathrm{Re}\int\frac{1}{2}(1-z^2)dz=\frac{1}{2}(u-\frac{u^3}{3}+uv^2)$, $y=\mathrm{Re}\int\frac{i}{2}(1+z^2)dz=\frac{1}{2}(-v+\frac{v^3}{3}-u^2v)$,
$z=\mathrm{Re}\int z\,dz=\frac{1}{2}(u^2-v^2)$.

11/4

ex) Scherk's surface : $M=\{|z|<1\}$, $g(z)=z$, $f(z)=\frac{4}{1-z^4}$
$X(u,v)=\left(-\arg\frac{z+i}{z-i}, -\arg\frac{z+1}{z-1}, \log\left|\frac{z^2+1}{z^2-1}\right|\right)$, $z=\log\frac{\cos y}{\cos x}$

* $g=\pi\circ N$, π: stereographic projection, N: Gauss map.

ex) Costa's surface : $M=\mathbb{C}/L$, L: square lattice generated by 1 and i.
$T^2=\mathbb{C}/L$ with punctures at $0, \frac{1}{2}, \frac{i}{2}$. $P(z)$: Weierstrass $\wp$-function for L. $P(z)=\frac{1}{z^2}+\sum_{\omega\in L-\{0\}}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)$, $f=P(z)$, $g=\frac{a}{P'(z)}$.
$P'^2=4(P^2-a^2)P$, $a=P(\frac{1}{2})=-P(\frac{i}{2})$, $P(\frac{1+i}{2})=0$.

* $r^2=x^2+y^2+z^2$, $\Delta(fh)=h\Delta f+2\langle\nabla f,\nabla h\rangle+f\Delta h$, $\therefore \Delta r^2=2r\Delta r+2\langle\nabla r,\nabla r\rangle$
$\Delta r^2=2x\Delta x+2y\Delta y+2z\Delta z+2\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]=2(|X_u|^2+|X_v|^2)$
$=4\lambda^2$: only locally defined. But $\iint_S\lambda^2\,du\,dv$ is globally defined.
$\therefore \iint_S\Delta r^2\,du\,dv=\iint_S 4\lambda^2\,du\,dv=\iint_S 4\,d\sigma=4\,\mathrm{Area}(S)$.

* Hoffman-Meeks conjecture:

g: genus, e: #(ends). There is no complete embedded minimal surface of finite total curvature with $g+2<e$.

* $S: (\varphi_1, \varphi_2, \varphi_3) \to (e^{i\theta}\varphi_1, e^{i\theta}\varphi_2, e^{i\theta}\varphi_3) \quad 0 \le \theta \le \frac{\pi}{2}$

$(e^{i\theta})^2(\varphi_1^2+\varphi_2^2+\varphi_3^2)=0$, $e^{i\theta}\varphi_k$ are holomorphic $\Rightarrow$ minimal surface:

associate min. surf. $\mathbf{X}_\theta(u,v) = (\mathrm{Re}\int e^{i\theta}\varphi_1 dz, \mathrm{Re}\int e^{i\theta}\varphi_2 dz, \mathrm{Re}\int e^{i\theta}\varphi_3 dz)$

$\mathbf{X}_{\frac{\pi}{2}}$: conjugate min. surf.

$E=G=\lambda^2 = |\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2 = |e^{i\theta}\varphi_1|^2+|e^{i\theta}\varphi_2|^2+|e^{i\theta}\varphi_3|^2$

$\therefore$ All the associate surfaces are locally isometric minimal surfaces to S.

$\mathbf{X}_\theta(u,v)$: $\mathbf{X}_0 \to \mathbf{X}_{\frac{\pi}{2}}$: isometric deformation of S to its conjugate surface. ex). Catenoid → Helicoid. Scherk's first surface → 2nd surf.

* u,v: defined on $D \subset S$. Divergence theorem for $\iint_D \Delta r^2 du dv$?

$\Delta r^2 = \mathrm{div}\,\mathrm{grad}\, r^2$ $\therefore \iint_D \Delta r^2 du dv = \int_{\partial D} \langle \mathrm{grad}\, r^2, n\rangle dl$, n: unit normal to ∂D, outward. $= \int_{\partial D} \frac{\partial r^2}{\partial n} dl$, $\frac{\partial}{\partial n}$: n 방향으로 단위길이 당 변화율.

$\mathbf{X}(u,v): D \to S$, conformal map, $d\mathbf{X}(n) = \lambda N$, $(d\mathbf{X}(dl) = ds)$, $d\mathbf{X}^*(ds) = \lambda dl$

$\lambda = |\mathbf{X}_u| = |\mathbf{X}_v|$, N: unit normal (tangent to $\mathbf{X}(D)$) to $\mathbf{X}(\partial D)$. $\mathbf{X}$ preserves r^2.

$\therefore \frac{\partial r^2}{\partial n} = \frac{\partial r^2}{\partial N}\cdot\lambda$, $\int_{\partial D} \frac{\partial r^2}{\partial n} dl = \int_{\mathbf{X}(\partial D)} \frac{\partial r^2}{\partial N} ds$: this vanishes along $\partial D \cap \partial D'$ and $\therefore 4\,\mathrm{Area}(S) = \iint_S \Delta r^2 du dv = \int_{\partial S} 2r \frac{\partial r}{\partial N} ds$. is independent of (u,v) on D and (u',v') on D'.

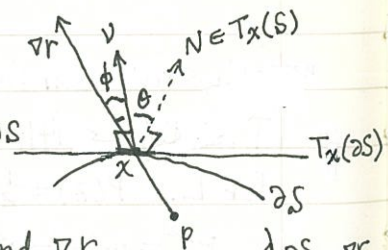
$\frac{\partial r}{\partial N} = \langle \nabla r, N\rangle$, ∇r: gradient in $\mathbb{R}^3$ $\therefore |\nabla r| = 1$.

$\therefore \frac{\partial r}{\partial N} = \cos\theta$, θ: angle between ∇r and N.

r = distance from p. N: outward unit conormal to ∂S

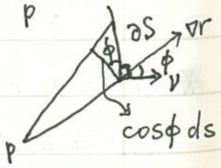
ν: unit normal to $T_x(\partial S)$ that is closest to ∇r

$\therefore \nu \in \mathrm{span}\{\nabla r, T_x(\partial S)\}$, ϕ = angle between ν and ∇r.



$\phi \le \theta \Rightarrow \frac{\partial r}{\partial N} = \cos\theta \le \cos\phi = \frac{\partial r}{\partial \nu}$

$\therefore 4\,\mathrm{Area}(S) \le \int_{\partial S} 2r\frac{\partial r}{\partial \nu} ds = 4\,\mathrm{Area}(p \divideontimes \partial S)$



$p \divideontimes \partial S$: the cone from p over $\partial S = \bigcup_{q \in \partial S} \overline{pq}$ $\quad \therefore \mathrm{Area}(S) \le \mathrm{Area}(p \divideontimes \partial S)$, $\forall p$.

$S \subset \mathbb{R}^n$, minimal $\Rightarrow x_1, x_2, \cdots, x_n$: harmonic wrt isothermal coordinates u,v.

$r^2 = x_1^2+x_2^2+\cdots+x_n^2$, $\Delta r^2 = 4\lambda^2$ $\cdots$ $\mathrm{Area}(S) \le \mathrm{Area}(p \divideontimes \partial S)$, $\forall p \in \mathbb{R}^n$.

This area comparison is obvious for area minimizing S.

이 부등식은 극대인 minimal surface 에 대해서도 성립한다.

11/9

* Application. minimal surface: generalized plane

Conjecture: $4\pi A \le L^2$ should also hold for minimal surfaces Σ in $\mathbb{R}^n$ and "=" $\Leftrightarrow$ Σ: disk in $\mathbb{R}^2$.

Partial results:

(i) Yes, if Σ is simply connected. (Carleman, 1921)

(ii) $\partial\Sigma$: connected (Reid, Hsiung, 1959)

(iii) Σ: doubly connected $\subset \mathbb{R}^3$ (Osserman-Schiffer, 1975)

(iv) Σ: doubly connected $\subset \mathbb{R}^n$ (Feinberg, 1977)

(v) $\#(\partial\Sigma) = 2$, $\Sigma \subset \mathbb{R}^3$ (Li-Schoen-Yau, 1983)

(vi) $\#(\partial\Sigma) = 2$, $\Sigma \subset \mathbb{R}^n$ (Choe, 1990)

(vii) Σ: triply connected, $\subset \mathbb{R}^3$ (Choe-Schoen. 2015)

$\mathrm{Area}(S) \le \mathrm{Area}(p \divideontimes \partial S)$ gives an easy proof of (i) & (ii).

S: area minimizing, $\#(\partial S)$: arbitrary : easy

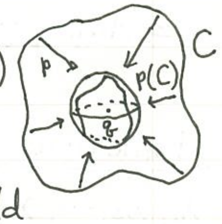
11/11

Theorem $\Sigma \subset \mathbb{R}^n$, minimal. $\mathrm{Angle}_p(\partial\Sigma) \ge 2\pi$, $\forall p \in \Sigma$.

$\mathrm{Angle}_p(\partial\Sigma) := \mathrm{Length}((p \divideontimes \partial\Sigma)^\infty \cap S_p^{n-1})$ = Length(radial projection of $\partial\Sigma$ on S_p^{n-1}).

Angle ?  $\ge 2\pi$ $\quad \therefore$ true in $\Sigma \subset \mathbb{R}^2$.

* Minimal surface in $\mathbb{R}^n$? $\quad \mathrm{Angle}_q(C) = \mathrm{Length}(p(C))$



$A'(0) = -\iint_S 2hH d\sigma$, $\mathbf{X}^t(u,v) = \mathbf{X}(u,v) + t h(u,v) N(u,v)$

$N(u,v)$: not unique on $S \subset \mathbb{R}^n$. hN: variation vector field

$hH = \langle hN, \frac{1}{2}(\mathbf{X}_{uu}+\mathbf{X}_{vv})\rangle$ $\quad \therefore A'(0) = -\iint_S 2\langle hN, \frac{1}{2}(\mathbf{X}_{uu}+\mathbf{X}_{vv})\rangle d\sigma$.

$\vec{H} := \frac{1}{2\lambda^2}(\mathbf{X}_{uu}+\mathbf{X}_{vv})$: mean curvature vector : normal to S

$\therefore \mathbf{X} \subset \mathbb{R}^n$ is minimal if $\vec{H} \equiv 0$.

$\Leftrightarrow x_1, x_2, \cdots, x_n$ are harmonic on $\mathbf{X}(D)$. $(x_1, \cdots, x_n)$: Euclidean coordinates of $\mathbb{R}^n$.

$\Leftrightarrow \varphi_i(z) = \frac{\partial x_i}{\partial u} - i\frac{\partial x_i}{\partial v}$ are holomorphic on $\mathbf{X}(D)$.