DENSITY OF A MINIMAL SUBMANIFOLD AND TOTAL CURVATURE OF ITS BOUNDARY

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Abstract

We show that given an $(n-1)$-dimensional submanifold Γ of $\mathbb{R}^m$ the total absolute curvature of Γ equals the average over all n-planes $R^n \subset \mathbb{R}^m$ of the total Gauss-Kronecker curvature of the orthogonal projection of Γ onto the R^n. As a consequence, we prove that given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$ and two compact convex hypersurfaces Γ_i of $R_i^n, i = 1, 2$, a nonflat minimal submanifold spanned by $\Gamma := \Gamma_1 \cup \Gamma_2$ is embedded.

1 Introduction

Fenchel [F1] showed that the total curvature of a closed space curve $\gamma \subset \mathbb{R}^m$ is at least 2π, and it equals 2π if and only if γ is a plane convex curve. Fáry [Fa] and Milnor [M] independently proved that a simple knotted regular curve has total curvature larger than 4π. These two results indicate that a Jordan curve which is curved at most *double* the minimum is isotopically simple. But in fact minimal surfaces spanning such Jordan curves must be simple as well. Indeed, Nitsche [N] showed that an analytic Jordan curve in $\mathbb{R}^3$ with total curvature at most 4π bounds exactly one minimal disk. Moreover, Ekholm, White and Wienholtz [EWW] proved that a minimal surface spanning such a Jordan curve in $\mathbb{R}^m$ is embedded.

Given an n-dimensional submanifold M of $\mathbb{R}^m$, there are two well-studied ways of defining the total curvature of M: the higher-dimensional Gauss-Bonnet integral $\int_M \Omega$ as defined in [AW] and [C1]; and the total absolute curvature of M, $\int_M K^* dV_M$ as defined by Chern and Lashof in [CL] (see section 2 below). Chern and Lashof proved that $\int_M K^*\, dV_M \geq 2$, with equality if and only if M is a convex hypersurface in an $(n+1)$-dimensional plane. It is known that if $\int_M K^* dV_M < 3$ then M is homeomorphic to $\mathbb{S}^n$ and that if $\int_M K^* dV_M < 4$ then M is homeomorphic to $\mathbb{S}^n$, $\mathbb{R}P^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$ or to $CayP^2$ (for $n = 16$). [EK].

In the light of Ekholm-White-Wienholtz's theorem, it is quite natural to conjecture that *an n-dimensional minimal submanifold $\Sigma \subset \mathbb{R}^m$ spanning a compact connected submanifold Γ^{n-1} with total absolute curvature < 4 is embedded.* In this paper we prove a theorem in the spirit of this conjecture: given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$ and two compact convex hypersurfaces Γ_i^{n-1} of $R_i^n, i = 1, 2$, a nonflat minimal submanifold spanned by $\Gamma := \Gamma_1 \cup \Gamma_2$ is embedded.

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In [Fa] Fáry showed that the total curvature of a space curve γ in $\mathbb{R}^m$ is equal to the average over all 2-planes $R^2 \subset \mathbb{R}^m$ of the total curvature of the orthogonal projection of γ onto the R^2. In this paper we also extend Fáry's theorem to show that given an $(n-1)$-dimensional submanifold Γ of $\mathbb{R}^m$ the total absolute curvature of Γ equals the average over all n-planes $R^n \subset \mathbb{R}^m$ of the total absolute curvature of the projection of Γ onto the R^n.

2 Total absolute curvature

Consider a submanifold M^n of Euclidean space $\mathbb{R}^m$. As discussed above, in high dimension and codimension there are two types of total curvature: one intrinsic (Allendörfer-Weil-Chern-Gauss-Bonnet), and one extrinsic (Chern-Lashof). In this section we shall review Chern-Lashof's total absolute curvature. This total curvature may be understood in terms of Gauss-Kronecker curvature of hypersurfaces.

Let M^n be an oriented hypersurface immersed in $\mathbb{R}^{n+1}$. A unit normal vector ν to M at $p \in M$ defines the Gauss map $G_1 : M \to \mathbb{S}^n$. The determinant of the differential G_{1*} is called the *Gauss-Kronecker curvature* of M, which we shall denote GK_M. It follows that for M compact,

$$\int_M GK_M \, dV_M = c_n \deg(G_1), \quad c_n := \mathrm{Vol}(\mathbb{S}^n).$$

Furthermore, if n is even, H. Hopf [H] showed

$$\int_M GK_M \, dV_M = \frac{1}{2} c_n \, \chi(M). \tag{1}$$

Now let M be an n-dimensional submanifold of $\mathbb{R}^m$. The volume form of the unit normal bundle $N_1 M$ of M is $dV_M \wedge d\sigma_{m-n-1}$ where the restriction of $d\sigma_{m-n-1}$ to a fiber of $N_1 M$ at p is the volume form of the sphere of unit normal vectors at $p \in M$. Define the Gauss map $G_1 : N_1 M \to \mathbb{S}^{m-1}$ by $G_1(p, \nu) = \nu$ and let $d\sigma_{m-1}$ be the volume form of $\mathbb{S}^{m-1}$. Then the *Lipschitz-Killing curvature* $G(p, \nu)$ of M at (p, ν) is defined to be the scalar $G(p, \nu)$ such that

$$G_1^*(d\sigma_{m-1}) = G(p, \nu) \, dV_M \wedge d\sigma_{m-n-1}.$$

Then $G(p, \nu)$ is exactly the volume expansion ratio of G_1, that is,

$$G(p, \nu) = \lim_{D \to \{p\}} \frac{\mathrm{Vol}(G_1(D))}{\mathrm{Vol}(D)},$$

where $\mathrm{Vol}(G_1(D))$ denotes the signed volume of $G_1(D)$. In fact, $G(p, \nu)$ has the following geometric interpretation [CL]: $G(p, \nu)$ is equal to the Gauss-Kronecker curvature at p of the orthogonal projection of M onto the $(n+1)$-dimensional plane $L(\nu)$ spanned by $T_p M$ and ν.

Let π be the canonical projection of $N_1 M$ into M. The integrals

$$K(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} G(p, \nu) \, d\sigma_{m-n-1} \quad \text{and} \quad K^*(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} |G(p, \nu)| \, d\sigma_{m-n-1}$$

are called the *total curvature* and the *total absolute curvature* of M at p, respectively. The values

$$\tau(M) := \int_M K \, dV_M, \quad \text{and} \quad \tau^*(M) := \int_M K^* \, dV_M$$

are called the *total curvature* and the *total absolute curvature of* M, respectively. Lipschitz and Killing have shown that $K(p)$ is an intrinsic quantity of M at p for n even (see [SS] for a more general result). However, $K(p) = 0$ for n odd. Both $\tau(M)$ and $\tau^*(M)$ remain unchanged even if the ambient space $\mathbb{R}^m$ is embedded into $\mathbb{R}^k$, $k > m$.

For $M^n \subset \mathbb{R}^m$, Fenchel [F2] generalized Hopf's theorem (1):

$$\int_M K\, dV_M = \chi(M). \tag{2}$$

However, Chern and Lashof [CL] proved that

$$\int_M K^*\, dV_M \geq 2, \tag{3}$$

with equality if and only if M is a convex hypersurface in an $(n+1)$-dimensional plane, and that if $\int_M K^* dV_M < 3$ then M is homeomorphic to $\mathbb{S}^n$. Moreover, Morse theory tells us that

$$\int_M K^*\, dV_M \geq \sum_i \beta_i,$$

where β_i is the i-th Betti number of M.

3 Vision angle versus average density

A minimal submanifold Σ^n in $\mathbb{R}^m$ has the remarkable property that the density of Σ at $p \in \Sigma$ is bounded above by that of the cone $C = p ⋇ \partial\Sigma$ at its vertex p. (We assume that Σ with its boundary is compact.) Recall that the *density* of Σ is defined as

$$\Theta_\Sigma(p) := \lim_{r \to 0} \frac{\mathrm{Vol}\left(\Sigma \cap B_r^m(p)\right)}{\mathrm{Vol}\left(B_r^n(p)\right)}.$$

Further, the density of C has the interesting property that it equals the average of the densities of the orthogonal projections of C onto n-planes in $\mathbb{R}^m$. These properties will be verified in this section.

In what follows, we shall write $\overline{\nabla}$ for the Euclidean connection on $\mathbb{R}^m$, and $\nabla = \nabla_M$ for the induced connection on a submanifold M.

Lemma 1. *Let* Σ *be an* n*-dimensional minimal submanifold of* $\mathbb{R}^m$*,* p *a point of* $\mathbb{R}^m$*, and* C *an* n*-dimensional piecewise smooth cone with vertex* p*. Define the Euclidean distance function* $r(x) = \mathrm{dist}(p, x), x \in \mathbb{R}^m$*. Let* $Y_1 = r\overline{\nabla} r$ *and* $Y_2 = r^{1-n}\overline{\nabla} r$*, and define* $\mathrm{div}_\Sigma Y_i = \mathrm{tr}_\Sigma \overline{\nabla} Y_i = \sum_j \langle \overline{\nabla}_{e_j} Y_i, e_j \rangle$*,* $\{e_1, \ldots, e_n\}$ *being an orthonormal frame of* Σ*. Then*
(a) *On* Σ*,* $\mathrm{div}_\Sigma Y_1 = n$ *and* $\mathrm{div}_\Sigma Y_2 \geq 0$*;*
(b) *On* C*,* $\mathrm{div}_C Y_1 = n$ *and* $\mathrm{div}_C Y_2 = 0$.

Proof. Given an n-dimensional submanifold $M \subset \mathbb{R}^m$, it is well known that

$$\triangle_M\, x := (\triangle_M\, x_1, \ldots, \triangle_M\, x_m) = \vec{H},$$

where $\vec{H}$ is the mean curvature vector of M, the trace of its second fundamental form. Hence the rectangular coordinate functions $x_1, \dots, x_m$ are harmonic on a minimal submanifold Σ^n of $\mathbb{R}^m$. If we take p as the origin, then since $\vec{H} = 0$ on Σ,

$$\mathrm{div}_\Sigma(Y_1) = \mathrm{div}_\Sigma(r\overline{\nabla}r) = \frac{1}{2}\triangle_\Sigma r^2 + \langle r\overline{\nabla}r, \vec{H}\rangle = \frac{1}{2}\sum \triangle_\Sigma x_i^2 = \sum x_i \triangle_\Sigma x_i + \sum |\nabla x_i|^2 = n.$$

On the cone C, since $\vec{H}$ is perpendicular to $r\overline{\nabla}r = x \in C$, we have

$$\mathrm{div}_C(Y_1) = \mathrm{div}_C(r\overline{\nabla}r) = \frac{1}{2}\triangle_C r^2 + \langle r\overline{\nabla}r, \vec{H}\rangle = \frac{1}{2}\sum \triangle_\Sigma x_i^2 = \langle x, \vec{H}\rangle + \sum |\nabla x_i|^2 = n.$$

On the other hand, for $M = \Sigma$ or C,

$$\mathrm{div}_M Y_2 \;=\; \mathrm{div}_M(r^{-n}Y_1) = -nr^{-n-1}\langle \nabla r, Y_1\rangle + r^{-n}\mathrm{div}_M(Y_1) = nr^{-n}\Big(-|\nabla r|^2 + 1\Big).$$

Note that $|\nabla r| \le 1$ on $M = \Sigma$ and $|\nabla r| \equiv 1$ on $M = C$. This completes the proof. □

Theorem 1. *Let Σ be a stationary n-rectifiable set with boundary Γ in $\mathbb{R}^m$, an open dense subset of Σ being a smooth minimal submanifold. Let C be the cone $p ⨳ \Gamma$, $p \in \mathbb{R}^m$. Then*

$$\Theta_\Sigma(p) \le \Theta_C(p),$$

with equality if and only if $\Sigma = C$ and C is star-shaped with respect to p.

Proof. Compute the first variation of volume with respect to the (Lipschitz continuous) variation vector field

$$Y := r^{1-n}\overline{\nabla}r \quad \text{for } \; r \ge \varepsilon$$

and

$$Y := \varepsilon^{-n} r\overline{\nabla}r \quad \text{for } \; r \le \varepsilon.$$

Then the first variation of Σ with respect to the flow with velocity field Y [Si, p. 80] is

$$\int_\Sigma \mathrm{div}_\Sigma Y \, dV_\Sigma,$$

which must equal

$$\int_\Gamma \langle Y, \nu_\Sigma\rangle \, dV_\Gamma,$$

where ν_Σ is the outward unit normal vector to Γ tangent to Σ.

Computing the divergence on smooth subsets of the stationary set Σ, we find by Lemma 1 (a)

$$\mathrm{div}_\Sigma Y \ge 0 \quad \text{for } \; r \ge \varepsilon, \tag{4}$$

with equality at points where $\overline{\nabla}r$ lies in the tangent space, and

$$\mathrm{div}_\Sigma Y = n\varepsilon^{-n} \quad \text{for } \; r \le \varepsilon.$$

It follows that for each small ε,

$$\frac{\mathrm{Vol}(\Sigma \cap B_\varepsilon(p))}{|B_1^n|\varepsilon^n} \le \frac{1}{n|B_1^n|}\int_\Gamma r^{1-n}\langle \overline{\nabla}r, \nu_\Sigma\rangle \, dV_\Gamma, \quad |B_1^n| := \mathrm{Vol}(B_1^n(0)). \tag{5}$$

Now apply Stokes' theorem to the integral of $\mathrm{div}_C Y$ on C:

$$\int_C \mathrm{div}_C Y\, dV_C = \int_{\partial C} \langle Y, \nu_C\rangle = \int_\Gamma \langle Y, \nu_C\rangle,$$

where ν_C is the outward unit cornomal to Γ on C, and we used the fact that $Y \perp \nu_C$ on $\partial C \setminus \Gamma$. Therefore by Lemma 1 (b)

$$\frac{\mathrm{Vol}(C \cap B_\varepsilon(p))}{|B_1^n|\varepsilon^n} = \frac{1}{n|B_1^n|}\int_\Gamma r^{1-n}\langle \overline{\nabla} r, \nu_C\rangle\, dV_\Gamma. \tag{6}$$

Note here that

$$0 \le \langle \overline{\nabla} r, \nu_C\rangle$$

and

$$\langle \overline{\nabla} r, \nu_\Sigma\rangle \le \langle \overline{\nabla} r, \nu_C\rangle. \tag{7}$$

Thus, letting $\varepsilon \to 0$ in (5) and (6), we get the desired density estimate. If equality holds, then we must have equalities in (4) and (7), which implies $\Sigma = C$ and $\partial r/\partial\nu \ge 0$. □

Definition. Let π_p be the radial projection of $\mathbb{R}^m \setminus \{p\}$ onto $\partial B_1(p)$, the unit sphere centered at $p \in \mathbb{R}^m$. Define the *vision angle at* p of an $(n-1)$-rectifiable set $\Gamma \subset \mathbb{R}^m$ by

$$\Pi_p(\Gamma) = \mathrm{Vol}(\pi_p(\Gamma)),$$

and the *vision angle* of Γ by

$$\Pi(\Gamma) = \sup{}_{p\in\mathbb{R}^m} \Pi_p(\Gamma)).$$

Here the volume $\mathrm{Vol}(\pi_p(\Gamma))$ counts multiplicity.

Clearly we have for any $p \in \mathbb{R}^m$ and $C := p ⨳ \Gamma$

$$c_{n-1}\Theta_C(p) = \Pi_p(\Gamma^{n-1}) \le \Pi(\Gamma), \quad c_{n-1} := \mathrm{Vol}(\mathbb{S}^{n-1}),$$

and hence we get the following corollaries to Theorem 1.

Corollary 1. *If $\Gamma \subset \mathbb{R}^m$ is an $(n-1)$-dimensional compact manifold, then any stationary rectifiable set Σ spanning Γ satisfies*

$$c_{n-1}\Theta_\Sigma(p) \le \Pi_p(\Gamma).$$

for all $p \in \Sigma$.

Corollary 2. *If $\Gamma \subset \mathbb{R}^m$ is an $(n-1)$-dimensional compact manifold with $\Pi(\Gamma) < 2c_{n-1}$, then any immersed minimal submanifold Σ spanning Γ is embedded.*

Proof. An immersed submanifold Σ with $\Theta_\Sigma < 2$ everywhere has no self-intersection. □

Remark. It may appear inappropriate to view $\Pi(\Gamma)$ as a total curvature. But it has its own merit, as the following example demonstrates. Define an immersed closed C^1 curve $\gamma \subset \mathbb{R}^2$ (the unit square plus four small loops at the corners) by

$$\gamma = \partial([-1,1]^2) \cup \{(x,y) : |x| > 1, |y| > 1, [(|x|-1)^2 + (|y|-1)^2]^{3/2} = \varepsilon(|x|-1)(|y|-1)\}$$

and define a Jordan curve $\Gamma \subset \mathbb{R}^n$ to be an embedded C^2 curve C^1-close to γ. Then for small ε,

$$\int_\Gamma |\vec{k}| ds > 6\pi, \qquad \text{however,} \qquad \Pi(\Gamma) \approx 3\pi.$$

Hence by Corollary 2 any immersed minimal surface Σ spanning Γ is embedded since $2c_1 = 4\pi$, although the Ekholm-White-Wienholtz theorem [EWW] cannot give the same conclusion.

Let $G_n(\mathbb{R}^m)$ denote the *Grassmann manifold* of n-planes through the origin in $\mathbb{R}^m$, equipped with the unique $\mathbb{O}(m)$-invariant probability measure, and let $\text{Ave}_{P \in G_n(\mathbb{R}^m)}$ be the average over all $P \in G_n(\mathbb{R}^m)$. Denote by ψ_P the orthogonal projection of $\mathbb{R}^m$ onto $P \in G_n(\mathbb{R}^m)$.

Lemma 2. *Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n \subset \mathbb{R}^m$ centered at the origin O of $\mathbb{R}^m$ and let D be a domain in $\mathbb{S}^{n-1}$. Then*

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = \Theta_{O \divideontimes D}(O).$$

Proof. Assume that $a(D) > 0$ is a positive real number such that

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = a(D) \cdot \Theta_{O \divideontimes D}(O). \tag{8}$$

Letting D shrink to a point $x \in \mathbb{S}^{n-1}$, one can define a function $a : \mathbb{S}^{n-1} \to \mathbb{R}$ given by

$$a(x) := \lim_{D \to \{x\}} \frac{\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\}}{\Theta_{O \divideontimes D}(O)}.$$

Then one can easily see that

$$a(D) = \frac{\int_D a(x) dV_{\mathbb{S}^{n-1}}}{\text{Vol}(D)}.$$

Note here that $\mathbb{O}(n)$ is transitive on $\mathbb{S}^{n-1}$ and that the elements of $\mathbb{O}(n)$ preserve the volume form $dV_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$. Therefore one concludes that for all $x \in \mathbb{S}^{n-1}$,

$$a(x) \equiv c \quad \text{for a positive constant } c$$

and hence for any domain $D \subset \mathbb{S}^{n-1}$,

$$a(D) \equiv c.$$

Therefore it follows from (8) that

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = c \cdot \Theta_{O \divideontimes D}(O) \tag{9}$$

for any domain $D \subset \mathbb{S}^{n-1}$. However, for almost all $P \in G_n(\mathbb{R}^m)$,

$$\Theta_{\psi_P(O \divideontimes \mathbb{S}^{n-1})}(O) = \Theta_{O \divideontimes \mathbb{S}^{n-1}}(O) = 1.$$

Thus $c = 1$ in (9), which completes the proof. $\square$

Theorem 2. Let $\Gamma^{n-1} \subset \mathbb{R}^m$ be a compact submanifold. Then

$$\mathit{\Pi}_q(\mathit{\Gamma}^{n-1}) = \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)} \{\mathit{\Pi}_{\psi_P(q)}(\psi_P(\mathit{\Gamma}))\}.$$

Proof. The cone $q \divideontimes \Gamma$ can be thought of as a union of infinitesimal cones $q \divideontimes \Delta\Gamma_i$ and then one can apply Lemma 2 to each $q \divideontimes \Delta\Gamma_i$. Hence

$$\begin{aligned} \Pi_q(\Gamma) &= c_{n-1}\, \Theta_{q \divideontimes \Gamma}(q) \\ &= c_{n-1}\, \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)} \{\Theta_{\psi_P(q \divideontimes \Gamma)}(\psi_P(q))\} \\ &= \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)} \{\Pi_{\psi_P(q)}(\psi_P(\Gamma))\}. \qquad \square \end{aligned}$$

4 Gauss-Kronecker versus total absolute curvature

In this section we generalize Fáry's theorem to any dimension n and to any codimension $m-n$.

Theorem 3 Let Γ^{n-1} be a smooth submanifold of $\mathbb{R}^m, m \geq n$. Then

$$\frac{c_{n-1}}{2} \int_\Gamma K^* dV_\Gamma = \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)} \int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}| \, dV_{\psi_P(\Gamma)}$$

.

To prove Theorem 3 we first need to recall Chern-Lashof's theorem [CL]:

Theorem CL. *Let $L(\eta)$ be the linear space of dimension n spanned by the tangent space $T_q\Gamma$ to $\Gamma^{n-1} \subset \mathbb{R}^m$ at q and the normal vector $\eta(q)$. Then the Lipschitz-Killing curvature $G(q\,,\eta)$ of Γ at q is equal to the Gauss-Kronecker curvature at q of the orthogonal projection of Γ into $L(\eta)$.*

Recall that the total absolute curvature of Γ at q is defined to be

$$K^*(q) = \frac{1}{c_{m-1}} \int_{\eta \in N_1\Gamma(q)} |G(q\,,\eta)| d\sigma_{m-n}\,, \tag{10}$$

where $d\sigma_{m-n}|_{N_1\Gamma(q)}$ is the volume form of the sphere of unit normals at $q \in \Gamma$.

We also need a special case of Jacobi's theorem [Sh]:

Theorem J. *Let M be an $n \times n$ orthogonal matrix partitioned into submatrices $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A and D are square matrices. Then $|\det A| = |\det D|$.*

Proof of Theorem J. First suppose that $n = 2k$ is even, and that A and D have size $k \times k$. Write $MM^T = I$ in terms of the matrices A, B, C, D to get $AC^T = -BD^T$, hence $(-1)^k \frac{\det A}{\det D} = \frac{\det B}{\det C}$; and then $M^T M = I$ to get $A^T B = -C^T D$, hence $(-1)^k \frac{\det A}{\det D} = \frac{\det C}{\det B}$, its own reciprocal. Thus we have a proof in case the square submatrices A and D have the same size.

In general, suppose A has size $k \times k$ and D has size $(n-k) \times (n-k)$ with $2k > n$. Let V be the span of the first k columns of M and W the span of the remaining $n-k$ columns. Then A arises from the projection of V onto $\mathbb{R}^k \subset \mathbb{R}^n$ and D arises from the projection of W onto $\mathbb{R}^{n-k} \subset \mathbb{R}^n$. Consider $U := V \cap \mathbb{R}^k$. Then dim $U = 2k - n$ and the orthogonal projection of V onto $\mathbb{R}^k$ leaves U fixed. Let X be the orthogonal complement of U in V with dim $X = n-k$. Then we have the space $X \oplus W$ of dimension $2(n-k)$. On $X \oplus W$ the orthogonal matrix arising from the bases of X and W satisfies the equal determinant property as proved in the first step. Hence the orthogonal matrix arising from the bases of V and W has the same property because U is fixed under the orthogonal projection of V onto $\mathbb{R}^k$. □

Proof of Theorem 3. The key ingredient of Theorem 3 is to transform the integral over $G_n(\mathbb{R}^m)$ into that over the $(m-n)$-sphere $N_1\Gamma(q)$. Therefore the starting point of its proof is to evenly divide the volume form $d\mu_n^m$ of $G_n(\mathbb{R}^m)$ by the volume form $d\sigma_{m-n}$ of $N_1\Gamma(q)$ at each point $q \in \Gamma$. To do so, we need to evenly partition $G_n(\mathbb{R}^m)$ into infinitesimal subsets with representatives from $N_1\Gamma(q)$.

For integers $1 \le \ell \le k$, let μ_ℓ^k be the $\mathbb{O}(k)$-invariant measure on $G_\ell(\mathbb{R}^k)$ such that $d\mu_\ell^k$ is the volume form of $G_\ell(\mathbb{R}^k)$. Note that

$$g_\ell^k := \int_{G_\ell(\mathbb{R}^k)} d\mu_\ell^k = 2\, \frac{c_{k-1}c_{k-2}\cdots c_{k-\ell}}{c_{\ell-1}c_{\ell-2}\cdots c_1 c_0} = g_{k-\ell}^k\,, \tag{11}$$

[S]. It is easy to see that for μ_n^m-almost any $P \in G_n(\mathbb{R}^m)$ there exists a unique $\eta \in N_1\Gamma(q)$ *up to sign* such that $\eta \in P$. In fact, this is true for all P which meet $T_q\Gamma$ transversally. Such n-planes P form a dense open set U of full measure in the Grassmann manifold [T].

Let $\eta^\perp$ be the hyperplane in $\mathbb{R}^m$ that is perpendicular to η. Then $P_1 := P \cap \eta^\perp$ is an $(n-1)$-plane in $\eta^\perp$. Hence every $P \in U$ uniquely determines (up to sign) a pair $(\eta(P), P_1) \in N_1\Gamma(q) \times G_{n-1}(\mathbb{R}^{m-1})$. For $\eta \in N_1\Gamma(q)$, define $G_\eta = \{P \in G_n(\mathbb{R}^m) : P \ni \eta\}$. Then we see that G_η is $G_{n-1}(\eta^\perp) \cong G_{n-1}(\mathbb{R}^{m-1})$, and U is foliated by $G_\eta,\ \eta \in N_1\Gamma(q)$.

Using the *one-to-two* correspondence $P \leftrightarrow (\eta(P), P_1)$, let's define two immersions $\varphi_1 : N_1\Gamma(q) \to G_n(\mathbb{R}^m)$ and $\varphi_2 : G_{n-1}(\mathbb{R}^{m-1}) \to G_n(\mathbb{R}^m)$ by $\varphi_1(\eta) = (\eta, P_1)$, for fixed $P_1 \subset \eta^\perp$, $P_1 \in G_{n-1}(\mathbb{R}^m)$, and $\varphi_2(P_1) = (\eta, P_1)$ for fixed $\eta \in N_1\Gamma$ with $\eta \perp P_1$. Then φ_2 is an isometry whereas φ_1 deforms the volume form $d\sigma_{m-n}$. Also, let $\varphi : N_1\Gamma(q) \times G_{n-1}(\mathbb{R}^{m-1}) \to G_n(\mathbb{R}^m)$ be the immersion defined by $\varphi(\eta(P), P_1) = P$. Here we need to see how φ deforms $N_1\Gamma(q)$ in relation to $G_{n-1}(\mathbb{R}^{m-1})$. Let φ_* denote the differential of φ and φ^* its dual.

(i) Now what is the relationship between $d\mu_n^m$ and $d\sigma_{m-n} d\mu_{n-1}^{m-1}$? We claim the following:

$$\varphi^*(d\mu_n^m(P)) = f(P)\, d\sigma_{m-n}(\eta(P))\, d\mu_{n-1}^{m-1}(P_1), \tag{12}$$

where $f(P)$ is the determinant of the orthogonal projection onto P from the n-plane $Q := L(\eta)$ spanned by $\eta = \eta(P)$ and $T_q\Gamma$. Suppose that $v_1, ..., v_{m-n}$ and $w_1, ..., w_{(n-1)(m-n)}$

are orthonormal bases of the tangent spaces of $N_1\Gamma(q)$ and $G_{n-1}(\mathbb{R}^{m-1})$, respectively. Since φ_2 is an isometry,

$$\varphi_*(w_1 \wedge ... \wedge w_{(n-1)(m-n)}) = w_1 \wedge ... \wedge w_{(n-1)(m-n)}.$$

Moreover $\varphi_*(v_1 \wedge ... \wedge v_{m-n})$ is orthogonal to $\varphi_*(w_1 \wedge ... \wedge w_{(n-1)(m-n)})$. Hence

$$|\varphi_*(v_i)| = |\mathrm{proj}_{P^\perp} v_i|,$$

where $\mathrm{proj}_{P^\perp}$ denotes the orthogonal projection onto $P^\perp$, and

$$|\varphi_*(v_1 \wedge ... \wedge v_{m-n})| = |\mathrm{proj}_{P^\perp}(v_1 \wedge ... \wedge v_{m-n})| = |\det \psi|,$$

ψ being the orthogonal projection of $Q^\perp$ onto $P^\perp$. Therefore

$$|\varphi_*(v_1 \wedge ... \wedge v_{m-n} \wedge w_1 \wedge ... \wedge w_{(n-1)(m-n)})| = |\det \psi|,$$

and thus the claim follows from Theorem J.

(ii) We claim that under the map ψ_P, GK changes by the factor of $1/f(P)^2$. This is because first, one can easily find orthonormal vectors $v_1, ..., v_{n-1}$ on $T_q\Gamma$ such that $\psi_P(v_1), ..., \psi_P(v_{n-1})$ are orthogonal on P; second, introduce rectangular coordinates $x_1, ..., x_n$ on $L(\eta)$ whose coordinate axes are parallel to $v_1, ..., v_{n-1}, \eta$; third, if $S \subset L(\eta)$ is the graph of $x_n = h(x_1, x_2, ..., x_{n-1})$ and S_a the graph of $x_n = h(a_1x_1, a_2x_2, ..., a_{n-1}x_{n-1})$, then

$$GK_S(0, ..., 0) = \det(\mathrm{Hess}\, h)(0, ..., 0)$$

and

$$GK_{S_a}(0, ..., 0) = (a_1a_2 \cdots a_{n-1})^2 GK_S(0, ..., 0);$$

fourth, S_a can be thought of as the hypersurface $\psi_P(S)$ in P with $\det(\psi_P|_{L(\eta)}) = 1/a_1a_2\cdots a_{n-1}$.

With (i) and (ii) we are now ready to complete the proof.

$$\begin{aligned}
&\mathrm{Ave}_{P\in G_n(\mathbb{R}^m)} \int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}|\, dV_{\psi_P(\Gamma)} = \frac{1}{g_n^m} \int_{P\in G_n(\mathbb{R}^m)} \int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}|\, dV_{\psi_P(\Gamma)} d\mu_n^m \\
= &\frac{1}{g_n^m} \int_\Gamma \int_{P\in G_n(\mathbb{R}^m)} f(P) |GK_{\psi_P(\Gamma)}|\, d\mu_n^m dV_\Gamma \\
= &\frac{1}{2g_n^m} \int_\Gamma \int_{\eta\in N_1\Gamma(q)} \int_{P\ni\eta} f(P)^2 |GK_{\psi_P(\Gamma)}|\, d\mu_{n-1}^{m-1}\, d\sigma_{m-n}\, dV_\Gamma \\
&\qquad \text{(by (12) with factor of } \tfrac{1}{2} \text{ due to the } 1:2 \text{ correspondence)} \\
= &\frac{g_{n-1}^{m-1}}{2g_n^m} \int_\Gamma \int_{\eta\in N_1\Gamma(q)} |GK_{\psi_{L(\eta)}(\Gamma)}|\, d\sigma_{m-n}\, dV_\Gamma \qquad \text{(by (ii))} \\
= &\frac{c_{n-1}}{2c_{m-1}} \int_\Gamma \int_{\eta\in N_1\Gamma(q)} |G(q,\eta)|\, d\sigma_{m-n}\, dV_\Gamma \qquad \text{(by (11) and Theorem CL)} \\
= &\frac{c_{n-1}}{2} \int_\Gamma K^*\, dV_\Gamma. \qquad \text{(by (10))} \qquad \square
\end{aligned}$$

5 Embeddedness of minimal submanifolds

In Theorem 2 we have seen that the cone angle of a compact set is equal to the average (in the Grassmannian) of the cone angle of its projection. In Theorem 3 we have proved that the total absolute curvature of a compact set equals the average of the total absolute Gauss-Kronecker curvature of its projection. As a comparison between these theorems it is tempting to propose the following:

Conjecture. For any minimal submanifold $\Sigma^n \subset \mathbb{R}^m$ spanning Γ^{n-1}, any point $q \in \Sigma$ and $P \in G_n(\mathbb{R}^m)$,

$$\Pi_{\psi_P(q)}(\psi_P(\Gamma)) \leq \int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}| dV_{\psi_P(\Gamma)}.$$

If this conjecture is true, one could combine Theorems 1, 2 and 3 to prove a higher-dimensional extension of Ekholm-White-Wienholtz's theorem as follows:

If Σ is a minimal submanifold spanning an $(n-1)$-dimensional compact manifold Γ, then

$$\Theta_\Sigma \leq \frac{1}{2}\int_\Gamma K^* dV_\Gamma.$$

Consequently one could prove the following as well:

If an $(n-1)$-dimensional compact connected manifold Γ satisfies $\int_\Gamma K^* dV_\Gamma < 4$ then any immersed minimal submanifold Σ spanning Γ is embedded.

The conjecture seems to be hard to prove as yet. But if we let Γ_i be a compact convex hypersurface of an n-plane $R_i^n \subset \mathbb{R}^m$, $i = 1, 2$, and define $\Gamma = \Gamma_1 \cup \Gamma_2$ then the conjecture holds for this Γ because

$$\Pi_{\psi_P(q)}(\psi_P(\Gamma_i)) \leq c_{n-1} = \int_{\psi_P(\Gamma_i)} |GK_{\psi_P(\Gamma_i)}| dV_{\psi_P(\Gamma_i)}, \tag{13}$$

where equality holds if and only if q is in R_i^n and inside Γ_i. Thus we have the following:

Theorem 4. *Given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$, let Γ_i be a compact convex hypersurface in R_i^n, $i = 1, 2$. If $\Gamma = \Gamma_1 \cup \Gamma_2$ then any n-dimensional minimal submanifold spanning Γ is either a union of two flat domains of R_i^n or is nonflat and has no self intersection.*

Proof of Theorem 4. Obviously $\int_\Gamma K^* dV_\Gamma = 4$. Thus by Theorems 1, 2, 3 we have $\Theta_\Sigma \leq 2$. If $\Theta_\Sigma = 2$, (13) implies Σ is flat. If $\Theta_\Sigma < 2$, Σ is nonflat and has no self intersection. In the latter case Σ is either smooth or has isolated singularities.

Remark. It should be mentioned that R. Schoen [Sc] proved a theorem which is a special case of Theorem 4:

If $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are $(m-2)$-spheres in parallel $(m-1)$-planes with the line ℓ joining their centers being orthogonal to these hyperplanes, then any immersed minimal submanifold Σ spanning Γ is a hypersurface of revolution with axis ℓ. In particular, Σ is a catenoid or a pair of plane disks.

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