Received: 11 August 2017 | Revised: 4 June 2018 | Accepted: 8 June 2018

DOI: 10.1002/mana.201700303

ORIGINAL PAPER

MATHEMATISCHE NACHRICHTEN

Some minimal submanifolds generalizing the Clifford torus

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Funding information
NRF, Grant/Award Numbers: SRC-GAIA, 2011-0030044

Abstract

The Clifford torus is a product surface in $\mathbb{S}^3$ and it is helicoidal. It will be shown that more minimal submanifolds of $\mathbb{S}^n$ have these properties.

KEYWORDS
Clifford torus, helicoidal, minimal surface

MSC (2010)
53A10, 49Q05

1 | INTRODUCTION

The Clifford torus is the simplest minimal surface in $\mathbb{S}^3$ besides the great sphere. Similarly in higher dimension we have a generalized Clifford torus $\mathbb{S}^p\left(\sqrt{\frac{p}{p+q}}\right) \times \mathbb{S}^q\left(\sqrt{\frac{q}{p+q}}\right)$ which is minimal in $\mathbb{S}^{p+q+1}$.

In Euclidean space there is an easy theorem that $\Sigma_1 \times \Sigma_2$ is minimal in $\mathbb{R}^{n_1+n_2}$ if Σ_1 and Σ_2 are minimal in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. While one cannot expect the same theorem to hold literally in $\mathbb{S}^n$, we will prove an analogous theorem as follows:

If Σ_1^m is minimal in $\mathbb{S}^p$ and Σ_2^n is minimal in $\mathbb{S}^q$, then $\sqrt{\frac{m}{m+n}}\,\Sigma_1^m \times \sqrt{\frac{n}{m+n}}\,\Sigma_2^n$ is minimal in $\mathbb{S}^{p+q+1}$.

There is another way of proving the minimality of the Clifford torus Σ in $\mathbb{S}^3$. It is well known that Σ is (doubly) foliated by great circles and Σ divides $\mathbb{S}^3$ into two congruent domains D_1, D_2. For every great circle ℓ in Σ consider the rotation ρ_ℓ of $\mathbb{S}^3$ about ℓ by 180°. One can show that

$$\rho_\ell(\Sigma) = \Sigma, \quad \rho_\ell(D_1) = D_2, \quad \rho_\ell(D_2) = D_1, \quad \rho_\ell(p) = p, \quad \text{for all} \quad p \in \ell. \tag{1.1}$$

More generally, if a hypersurface Σ^{n-1} of a Riemannian manifold M^n has an isometry ρ (in place of ρ_ℓ) satisfying (1.1) at every point p of Σ^{n-1}, Σ^{n-1} is said to be *helicoidal* in M. In Proposition 3.2 we show that the generalized Clifford torus $\mathbb{S}^p(1/\sqrt{2}) \times \mathbb{S}^p(1/\sqrt{2})$ is helicoidal in $\mathbb{S}^{2p+1}$. In Theorem 3.3 we will prove that every helicoidal hypersurface of M is minimal.

Recently Tkachev [3] and Hoppe–Linardopoulos–Turgut [2] found algebraic minimal hypersurfaces N_1 in $\mathbb{S}^{n^2-1}$ and N_2 in $\mathbb{S}^{2n^2-n-1}$, respectively:

$$N_1 = \left\{(x_{11}, x_{12}, \dots, x_{nn}) \in \mathbb{R}^{n^2} : (x_{ij}) \text{ is an } n \times n \text{ matrix with zero determinant}\right\} \cap \ \mathbb{S}^{n^2-1};$$

$$N_2 = \left\{(x_{11}, x_{12}, \dots, x_{2n2n}) \in \mathbb{R}^{4n^2} : (x_{ij}) \text{ is a } 2n \times 2n \text{ skew-symmetric matrix with zero determinant}\right\} \ \cap \ \mathbb{S}^{4n^2-1}$$

is similar to a minimal hypersurface in $\mathbb{S}^{2n^2-n-1}$. In this paper we give a new proof of their minimality, showing that they are helicoidal.

2 | PRODUCT MANIFOLDS

Let Σ^m be an m-dimensional submanifold of $\mathbb{S}^p$ and let Σ^n be an n-dimensional submanifold of $\mathbb{S}^q$. Denote by $a\Sigma^m$ and $b\Sigma^n$ the homothetic expansions of $\Sigma^m \subset \mathbb{R}^{p+1}$ and $\Sigma^n \subset \mathbb{R}^{q+1}$ with factors of a and b, respectively.

Theorem 2.1. *If $\Sigma_1^{n_1}$ is minimal in $\mathbb{S}^p$ and $\Sigma_2^{n_2}$ is minimal in $\mathbb{S}^q$, then $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1} \times \sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$ is minimal in $\mathbb{S}^{p+q+1}$.*

Proof. Let $\varphi_1, \dots, \varphi_{n_1}$ be the local coordinates of $\Sigma_1^{n_1}$ such that $\mathbf{m_1} : (\varphi_1, \dots, \varphi_{n_1}) \in D_1 \subset \mathbb{R}^{n_1} \to \mathbf{m_1}(\varphi_1, \dots, \varphi_{n_1}) \in \Sigma_1^{n_1} \subset \mathbb{S}^p \subset \mathbb{R}^{p+1}$ is a local immersion. Similarly, $\varphi_{n_1+1}, \dots, \varphi_{n_1+n_2}$ are local coordinates of $\Sigma_2^{n_2}$ with a local immersion $\mathbf{m_2} : D_2 \subset \mathbb{R}^{n_2} \to \Sigma_2^{n_2} \subset \mathbb{S}^q \subset \mathbb{R}^{q+1}$. Clearly $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1} \times \sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2} \subset \mathbb{R}^{p+q+2}$ is a submanifold of $\mathbb{S}^{p+q+1}$. Let's define a local immersion $\hat{\mathbf{m}} : D_1 \times D_2 \subset \mathbb{R}^{n_1+n_2} \to \mathbb{S}^{p+q+1} \subset \mathbb{R}^{p+q+2}$ by

$$\hat{\mathbf{m}}(\varphi_1, \dots, \varphi_{n_1+n_2}) = \begin{pmatrix} \sqrt{\frac{n_1}{n_1+n_2}}\mathbf{m_1}(\varphi_1, \dots, \varphi_{n_1}) \\ \sqrt{\frac{n_2}{n_1+n_2}}\mathbf{m_2}(\varphi_{n_1+1}, \dots, \varphi_{n_1+n_2}) \end{pmatrix}.$$

Then $\hat{\mathbf{m}}$ is an immersion into $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1} \times \sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$. The metric of $\hat{\mathbf{m}}$ is

$$ds^2 = \sum_{A,B=1}^{n_1+n_2} \hat{g}_{AB} d\varphi_A d\varphi_B,$$

where $(\hat{g}_{AB})$ is the block matrix

$$(\hat{g}_{AB}) = \begin{pmatrix} \frac{n_1}{n_1+n_2} g_{ab} & O \\ O & \frac{n_2}{n_1+n_2} g_{a'b'} \end{pmatrix}$$

with

$$a, b = 1, \dots, n_1, \quad a', b' = n_1 + 1, \dots, n_1 + n_2,$$

and

$$g_{ab} = \frac{\partial \mathbf{m_1}}{\partial \varphi_a} \cdot \frac{\partial \mathbf{m_1}}{\partial \varphi_b}, \qquad g_{a'b'} = \frac{\partial \mathbf{m_2}}{\partial \varphi_{a'}} \cdot \frac{\partial \mathbf{m_2}}{\partial \varphi_{b'}}.$$

Moreover,

$$(\hat{g}^{AB}) = \begin{pmatrix} \frac{n_1+n_2}{n_1} g^{ab} & O \\ O & \frac{n_1+n_2}{n_2} g^{a'b'} \end{pmatrix}$$

and

$$\hat{g} = \det(\hat{g}_{AB}) = \frac{n_1^{n_1} n_2^{n_2}}{(n_1 + n_2)^{n_1+n_2}} g g', \quad g = \det(g_{ab}), \quad g' = \det(g_{a'b'}).$$

Let $\Delta_{n_1}, \Delta_{n_2}, \Delta_{n_1+n_2}$ denote the Laplacians on $\Sigma_1^{n_1}, \Sigma_2^{n_2}$ and $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1} \times \sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$, respectively. Since $\Sigma_1^{n_1}, \Sigma_2^{n_2}$ are minimal, we have

$$\Delta_{n_1} \mathbf{m_1} = \frac{1}{\sqrt{g}} \sum_{a,b} \frac{\partial}{\partial \varphi_a}\left(\sqrt{g} g^{ab} \frac{\partial}{\partial \varphi_b} \mathbf{m_1}\right) = -n_1 \mathbf{m_1},$$

$$\Delta_{n_2} \mathbf{m_2} = \frac{1}{\sqrt{g'}} \sum_{a',b'} \frac{\partial}{\partial \varphi_{a'}}\left(\sqrt{g'} g^{a'b'} \frac{\partial}{\partial \varphi_{b'}} \mathbf{m_2}\right) = -n_2 \mathbf{m_2}.$$

Hence

$$
\begin{aligned}
\Delta_{n_1+n_2}\hat{\mathbf{m}} &= \frac{1}{\sqrt{\hat{g}}}\sum_{A,B}\frac{\partial}{\partial\varphi_A}\left(\sqrt{\hat{g}}\hat{g}^{AB}\frac{\partial}{\partial\varphi_B}\hat{\mathbf{m}}\right)\\
&= \frac{n_1+n_2}{n_1}\frac{1}{\sqrt{g}}\sum_{a,b}\frac{\partial}{\partial\varphi_a}\begin{pmatrix}\sqrt{g}g^{ab}\frac{\partial}{\partial\varphi_b}\sqrt{\frac{n_1}{n_1+n_2}}\mathbf{m_1}\\ O\end{pmatrix} + \frac{n_1+n_2}{n_2}\frac{1}{\sqrt{g'}}\sum_{a',b'}\frac{\partial}{\partial\varphi_{a'}}\begin{pmatrix} O\\ \sqrt{g'}g^{a'b'}\frac{\partial}{\partial\varphi_{b'}}\sqrt{\frac{n_2}{n_1+n_2}}\mathbf{m_2}\end{pmatrix}\\
&= -(n_1+n_2)\begin{pmatrix}\sqrt{\frac{n_1}{n_1+n_2}}\mathbf{m_1}\\ O\end{pmatrix} - (n_1+n_2)\begin{pmatrix} O\\ \sqrt{\frac{n_2}{n_1+n_2}}\mathbf{m_2}\end{pmatrix}\\
&= -(n_1+n_2)\hat{\mathbf{m}}.
\end{aligned}
$$

Thus $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1}\times\sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$ is minimal. □

Remark 2.2. Even if $\Sigma_1^{n_1}\subset\mathbb{S}^{n_1+1}$ and $\Sigma_2^{n_2}\subset\mathbb{S}^{n_2+1}$ are hypersurfaces, $\sqrt{\frac{n_1}{n_1+n_2}}\Sigma_1^{n_1}\times\sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$ has codimension 3 in $\mathbb{S}^{n_1+n_2+3}$. But if $\Sigma_1^{n_1}=\mathbb{S}^{n_1}$ one can say that $\Sigma_1^{n_1}$ is trivially minimal in $\mathbb{S}^{n_1}$ and then $\sqrt{\frac{n_1}{n_1+n_2}}\mathbb{S}^{n_1}\times\sqrt{\frac{n_2}{n_1+n_2}}\Sigma_2^{n_2}$ is minimal with codimension 2 in $\mathbb{S}^{n_1+n_2+2}$. Furthermore, $\sqrt{\frac{n_1}{n_1+n_2}}\mathbb{S}^{n_1}\times\sqrt{\frac{n_2}{n_1+n_2}}\mathbb{S}^{n_2}$ is minimal with codimension 1 in $\mathbb{S}^{n_1+n_2+1}$.

3 | HELICOIDAL

Just as the Clifford torus is helicoidal in $\mathbb{S}^3$, so is the helicoid in $\mathbb{R}^3$. For a more general setting we introduce the following definition.

Definition 3.1. Let M be a complete Riemannian manifold and let Σ be an embedded hypersurface of M. Assume that Σ divides M into two domains D_1 and D_2. Suppose that at any point p of Σ there is an isometry φ of M such that

$$\varphi(p)=p,\quad \varphi(\Sigma)=\Sigma,\quad \varphi(D_1)=D_2,\quad \varphi(D_2)=D_1.$$

Then we say that Σ is *helicoidal* in M.

Proposition 3.2. *The generalized Clifford torus* $\Sigma^{2p}=\mathbb{S}^p(1/\sqrt{2})\times\mathbb{S}^p(1/\sqrt{2})$ *is helicoidal in* $\mathbb{S}^{2p+1}$.

Proof. Let ξ be the reflection of $\mathbb{R}^{2p+2}$ defined by

$$\xi(x_1,\dots,x_{2p+2})=(x_{p+2},x_{p+3},\dots,x_{2p+2},x_1,x_2,\dots,x_{p+1}).$$

If D_1, D_2 are the domains of $\mathbb{S}^{2p+1}$ divided by Σ^{2p}, then

$$\xi(\Sigma^{2p})=\Sigma^{2p},\ \ \xi(D_1)=D_2,\ \ \xi(D_2)=D_1$$

and $\xi(p)=p$ if and only if

$$p=(x_1,\dots,x_{p+1},x_1,\dots,x_{p+1}).$$

For any $q\in\Sigma^{2p}$, there exists an isometry η of $\mathbb{S}^{2p+1}$ mapping q to p such that

$$\eta(\Sigma^{2p})=\Sigma^{2p},\quad \eta(D_1)=D_1,\quad \eta(D_2)=D_2.$$

Hence

$$\eta^{-1}\circ\xi\circ\eta(q)=q,\quad \eta^{-1}\circ\xi\circ\eta(\Sigma^{2p})=\Sigma^{2p},$$

and

$$\eta^{-1} \circ \xi \circ \eta(D_1) = D_2, \quad \eta^{-1} \circ \xi \circ \eta(D_2) = D_1.$$

So $\eta^{-1} \circ \xi \circ \eta$ is the desired isometry. □

Theorem 3.3. *Every helicoidal hypersurface Σ of a Riemannian manifold M^n is minimal in M wherever Σ is twice differentiable.*

Proof. Let $\vec{H}$ be the mean curvature vector of Σ at a point $p \in \Sigma$, that is,

$$\vec{H} = \sum_{i=1}^{n-1} \left(\bar{\nabla}_{e_i} e_i\right)^{\perp},$$

where $\bar{\nabla}$ is the Riemannian connection on M and $e_1, \dots, e_{n-1}$ are orthonormal vectors of Σ at p. Since $\varphi(\Sigma) = \Sigma$ and p is a fixed point of φ, one sees that $\varphi_*(e_1), \dots, \varphi_*(e_{n-1})$ are also orthonormal on Σ at p. Hence

$$\varphi_*\left(\vec{H}\right) = \sum_{i=1}^{n-1} \left(\bar{\nabla}_{\varphi_*(e_i)} \varphi_*(e_i)\right)^{\perp} = \sum_{i=1}^{n-1} \left(\bar{\nabla}_{e_i} e_i\right)^{\perp} = \vec{H}. \tag{3.1}$$

On the other hand, the condition $\varphi(D_1) = D_2$ implies that if $\vec{H}$ points into D_1 then $\varphi_*\left(\vec{H}\right)$ points into D_2. Likewise, if $\vec{H}$ points into D_2, then $\varphi_*\left(\vec{H}\right)$ should point into D_1. Therefore $\varphi_*\left(\vec{H}\right) = -\vec{H}$, which together with (3.1) implies $\vec{H} = 0$ at p. As p is arbitrarily chosen, one concludes that Σ is minimal. □

Incidentally, $\mathbb{S}^1\left(1/\sqrt{2}\right) \times \mathbb{S}^1\left(1/\sqrt{2}\right)$ is congruent in $\mathbb{S}^3$ to

$$\mathbb{S}^3 \cap \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \det \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} = 0 \right\}.$$

Also $\mathbb{S}^2\left(1/\sqrt{2}\right) \times \mathbb{S}^2\left(1/\sqrt{2}\right)$ is congruent in $\mathbb{S}^5$ to

$$\mathbb{S}^5 \cap \left\{ (x_1, \dots, x_6) \in \mathbb{R}^6 : \det \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix} = 0 \right\}.$$

When is the zero determinant set minimal? With regard to this question, the following two theorems have been recently proved.

Theorem 3.4 (Tkachev, [3]). $\Sigma = \left\{(x_{11}, x_{12}, \dots, x_{nn}) \in \mathbb{R}^{n^2} : \left(x_{ij}\right)\right.$ *is an $n \times n$ real matrix with zero determinant*$\left.\right\}$ *is a minimal hypercone in $\mathbb{R}^{n^2}$.*

Theorem 3.5 (Hoppe–Linardopoulos–Turgut, [2]).

$$\Sigma = \left\{(x_{11}, x_{12}, \dots, x_{2n\,2n}) \in \mathbb{R}^{4n^2} : \left(x_{ij}\right) \textit{ is a } 2n \times 2n \textit{ skew-symmetric matrix with zero determinant}\right\}$$

is congruent to a minimal hypercone in $\mathbb{R}^{2n^2-n}$.

They obtained these theorems from the harmonicity of x_{ij} on Σ. Here we will give a new proof by showing that Σ is helicoidal.

Proof of Theorem 3.4. Let M_n be the set of all real $n \times n$ matrices. One can identify M_n with $\mathbb{R}^{n^2}$. Define

$$\Sigma = \{X \in M_n : \det X = 0\}.$$

Then Σ is an $\left(n^2 - 1\right)$-dimensional algebraic variety in $\mathbb{R}^{n^2}$. Σ divides $\mathbb{R}^{n^2}$ into two domains D_+ and D_- with

$$D_+ = \{X \in M_n : \det X > 0\}, \quad D_- = \{X \in M_n : \det X < 0\}.$$

Let's introduce an inner product $\langle\ ,\ \rangle$ in M_n by

$$\langle X, Y\rangle = \operatorname{tr}\left(X^T Y\right), \quad X, Y \in M_n.$$

Given $A \in O(n)$, define $\varphi_A : M_n \to M_n$ by $\varphi_A(X) = AX$. Then φ_A is an isometry on M_n because

$$\langle \varphi_A(X), \varphi_A(Y)\rangle = \langle AX, AY\rangle = \operatorname{tr}\left(X^T A^T A Y\right) = \operatorname{tr}\left(X^T Y\right) = \langle X, Y\rangle.$$

Clearly

$$\varphi_A(\Sigma) = \Sigma.$$

Moreover, if $A \in SO(n)$, then

$$\varphi_A(D_+) = D_+ \quad \text{and} \quad \varphi_A(D_-) = D_-,$$

and if $A \in O(n) \setminus SO(n)$, then

$$\varphi_A(D_+) = D_- \quad \text{and} \quad \varphi_A(D_-) = D_+.$$

Choose any $X \in \Sigma$. Then the column vectors of X are linearly dependent. Let P be an $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ containing all the column vectors of X and let $v \in \mathbb{R}^n$ be a nonzero normal vector of P. Then there exists $A \in O(n) \setminus SO(n)$ such that P is an eigenspace of A with eigenvalue 1 and v an eigenvector of A with eigenvalue -1. Hence

$$\varphi_A(X) = X \quad \text{and} \quad \varphi_A(D_+) = D_-, \quad \varphi_A(D_-) = D_+.$$

Therefore Σ is helicoidal in $\mathbb{R}^{n^2}$ and so by Theorem 3.3 it is minimal in $\mathbb{R}^{n^2}$ away from its singular set. Σ is a cone since $\det X$ is a homogeneous polynomial. □

It is known that the determinant of a $2n \times 2n$ skew-symmetric matrix A can be written as the square of the Pfaffian of A. The Pfaffian $pf(A)$ of $A = \left(a_{ij}\right)$ is defined as follows. Let ω be a 2-vector

$$\omega = \sum_{i<j} a_{ij} e_i \wedge e_j,$$

where $\{e_1, \ldots, e_{2n}\}$ is the standard basis of $\mathbb{R}^{2n}$. Then $pf(A)$ is defined by

$$\frac{1}{n!}\omega^n = pf(A)\, e_1 \wedge \cdots \wedge e_{2n}.$$

One computes

$$pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1)\sigma(2i)}.$$

Moreover,

$$pf\left(B^T A B\right) = \det(B)\, pf(A) \tag{3.2}$$

for any skew-symmetric matrix A and any $2n \times 2n$ matrix B.

Proof of Theorem 3.5. Define

$$N = \left\{ X \in M_{2n} : X = \begin{pmatrix} 0 & x_1 & x_2 & \cdot & \cdot & x_{2n-1} \\ -x_1 & 0 & x_{2n} & \cdot & \cdot & x_{4n-3} \\ -x_2 & -x_{2n} & 0 & \cdot & \cdot & x_{6n-6} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & x_{2n^2-n} \\ -x_{2n-1} & -x_{4n-3} & -x_{6n-6} & \cdot & -x_{2n^2-n} & 0 \end{pmatrix} \right\}$$

and

$$\Sigma = \{X \text{ is a } 2n \times 2n \text{ skew-symmetric marix with } \det X = 0\}.$$

Then Σ is a hypersurface in the $(2n^2 - n)$-dimensional subspace N of $\mathbb{R}^{4n^2}$. Let

$$D_+ = \{X \text{ is a } 2n \times 2n \text{ skew-symmetric matrix with } pf(X) > 0\},$$

$$D_- = \{X \text{ is a } 2n \times 2n \text{ skew-symmetric matrix with } pf(X) < 0\}.$$

For any $A \in O(2n)$ define $\psi_A : M_{2n} \to M_{2n}$ by

$$\psi_A(X) = A^T X A.$$

One sees that $\psi_A(X)$ is skew-symmetric if X is. Hence

$$\psi_A : N \to N \quad \text{and} \quad \psi_A(\Sigma) = \Sigma.$$

ψ_A is an isomety since

$$\langle \psi_A(X), \psi_A(Y)\rangle = \left\langle A^T X A, A^T Y A\right\rangle = \mathrm{tr}\left(A^T X^T A A^T Y A\right) = \mathrm{tr}\left(A^T X^T Y A\right) = \mathrm{tr}\left(A A^T X^T Y\right) = \langle X, Y\rangle.$$

Every skew-symmetric matrix can be reduced to a block diagonal form by a special orthogonal matrix. In particular, every $2n \times 2n$ skew symmetric matrix X with zero determinant can be transformed by an orthogonal matrix Q to the form

$$Q^T X Q = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & \lambda_k & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & -\lambda_k & 0 & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix} := \Lambda, \tag{3.3}$$

where $\lambda_1, \ldots, \lambda_k$ are real.

Define a $2n \times 2n$ block matrix

$$J = \begin{pmatrix} I_{2n-2} & O^T \\ O & K \end{pmatrix},$$

where I_{2n-2} is the $(2n-2) \times (2n-2)$ identity matrix, O is the $2 \times (2n-2)$ zero matrix and $K = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then for any $X \in \Sigma$ we have an orthogonal matrix Q such that

$$Q^T X Q = \Lambda \quad \text{and} \quad J \Lambda J = \Lambda.$$

Hence

$$J Q^T X Q J = Q^T X Q.$$

Therefore

$$\left(Q J Q^T\right) X \left(Q J Q^T\right) = X, \quad Q J Q^T \neq I, \quad \det\left(Q J Q^T\right) = -1.$$

QJQ^T is orthogonal because

$$\left(Q J Q^T\right)\left(Q J Q^T\right)^T = Q J Q^T Q J Q^T = Q J J Q^T = Q Q^T = I.$$

Let $B = QJQ^T$. Then by (3.2)

$$pf(\psi_B(Y)) = -pf(Y)$$

for any skew-symmetric matrix Y and hence

$$\psi_B(X) = X, \quad \psi_B(\Sigma) = \Sigma, \quad \psi_B(D_+) = D_-, \quad \psi_B(D_-) = D_+.$$

Therefore Σ is helicoidal and thus minimal in N everywhere it is twice differentiable.

Let $\mu : N \to \mathbb{R}^{2n^2-n}$ be the map defined by

$$\mu(X) = \frac{1}{\sqrt{2}}(x_1, x_2, \dots, x_{2n^2-n}),$$

where

$$X = \begin{pmatrix} 0 & x_1 & x_2 & \cdot & \cdot & x_{2n-1} \\ -x_1 & 0 & x_{2n} & \cdot & \cdot & x_{4n-3} \\ -x_2 & -x_{2n} & 0 & \cdot & \cdot & x_{6n-6} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & x_{2n^2-n} \\ -x_{2n-1} & -x_{4n-3} & -x_{6n-6} & \cdot & -x_{2n^2-n} & 0 \end{pmatrix}.$$

Then μ is an isometry. Therefore $\mu(\Sigma)$ is a minimal hypercone in $\mathbb{R}^{2n^2-n}$. □

Questions 3.5.

1. A generalized helicoid is defined in [1] to be the locus of the minimal cone $O \times (\mathbb{S}^n(1/\sqrt{2}) \times \mathbb{S}^n(1/\sqrt{2}))$ when the multi-screw motion in $\mathbb{R}^{2n+3}$ is applied to the cone. That generalized helicoid is minimal. Instead of $\mathbb{S}^n$, let's consider its minimal submanifold M. Then the cone $O \times \left(\frac{1}{\sqrt{2}}M \times \frac{1}{\sqrt{2}}M\right)$ is minimal in $\mathbb{R}^{2n+2}$. If we apply the multi-screw motion in $\mathbb{R}^{2n+3}$ to the cone, is its locus minimal?
2. In the proof of Theorem 3.4 the hyperplane P is assumed to contain all the column vectors of the matrix X. The minimal hypercone Σ of the theorem may have a singularity other than the origin. Is it true that the rank of X is related with the Hausdorff dimension of the singular set of Σ?

ACKNOWLEDGEMENTS

Jaigyoung Choe was supported in part by NRF 2011-0030044, SRC-GAIA.

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How to cite this article: Choe J, Hoppe J. Some minimal submanifolds generalizing the Clifford torus. *Mathematische Nachrichten*. 2018;291:2536–2542. https://doi.org/10.1002/mana.201700303