Math. Ann. 323, 143–156 (2002) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080100300

# Sufficient conditions for constant mean curvature surfaces to be round

**Jaigyoung Choe**

Received: 3 October 2000 / Published online: 1 February 2002 – © Springer-Verlag 2002

**Abstract.** We study under what condition a constant mean curvature surface can be round: i) If the boundary of a compact immersed disk type constant mean curvature surface in $\mathbf{R}^3$ consists of lines of curvature and has less than 4 vertices with angle $< \pi$, then the surface is spherical; ii) A compact immersed disk type capillary surface with less than 4 vertices in a domain of $\mathbf{R}^3$ bounded by spheres or planes is spherical; iii) The mean curvature vector of a compact embedded capillary hypersurface of $\mathbf{R}^n$ with smooth boundary in an unbounded polyhedral domain with unbalanced boundary should point inward; iv) If the $k$th order ($2 \leq k \leq n-1$) mean curvature of a compact immersed constant mean curvature hypersurface of $\mathbf{R}^n$ without boundary is constant, then the hypersurface is a sphere.

A round sphere has constant mean curvature. But not all constant mean curvature surfaces in $\mathbf{R}^3$ are round as there is a famous counterexample: Wente's torus [W1]. With some condition on a constant mean curvature surface, however, one can derive the roundness of the surface. Hopf [Ho] assumed a constant mean curvature surface to be an immersed sphere and showed it is a round sphere. Alexandrov [A] added the hypotheses of compactness and embeddedness and proved that such a constant mean curvature surface is round. The classical isoperimetric inequality for immersed surfaces states that the area minimizing surface among all constant mean curvature surfaces enclosing a fixed volume counting multiplicity is round. Nitsche [N] found that an immersed disk type constant mean curvature surface $S$ in a ball which makes a constant contact angle with the boundary sphere of the ball along $\partial S$ is a spherical cap. Also Barbosa and do Carmo [BC, W2] imposed stability on a compact constant mean curvature surface $S$ and obtained the roundness of $S$.

In this paper we give more sufficient conditions which guarantee that the given constant mean curvature surfaces are (part of) a round sphere. First, Hopf's theorem mentioned above will be extended from an immersed sphere to an immersed disk. Hopf showed that the constancy of the mean curvature gives rise to a holomorphic function $\Phi$. The zeros of $\Phi$ are then shown to be related to the topology

J. Choe

Department of Mathematics, Seoul National University, Seoul, 151-742, Korea

(e-mail: choe@math.snu.ac.kr)

Supported by KRF-2000-015-DP0031 through RIBS-SNU

of the surface. While the zeros of $\Phi$ in Hopf's case are always interior points, our zeros of $\Phi$ may occur on the boundary of the disk. Therefore we should estimate the order of the zeros of $\Phi$ and will prove that if a compact immersed disk type constant mean curvature surface $S$ is $C^{2,\alpha}$ up to $\partial S$ and if $\partial S$ is $C^{2,\alpha}$ up to its singular points (called vertices), consists of lines of curvature, and has less than 4 vertices with angle $< \pi$, then $S$ is part of a round sphere. This theorem has a nice application for a capillary surface $S$ in a polyhedral domain as $\partial S$ then becomes lines of curvature. A capillary surface $S$ in a domain $U$ is a constant mean curvature surface which meets $\partial U$ in a constant contact angle along $\partial S$. So our second theorem is that if a compact immersed disk type capillary surface in a domain bounded by planes or spheres has less than four vertices on its boundary, then the surface is spherical. Third, we are interested in the conjecture that the only compact embedded capillary hypersurface of $\mathbf{R}^n$ with smooth boundary in a domain bounded by a family of hyperplanes and spheres which are unbalanced (see the definition right before Theorem 3) is part of a round sphere. A partial answer is given to this conjecture by Theorem 3 which says that a compact embedded capillary hypersurface with smooth boundary other than the spherical ones cannot exist if its mean curvature vector points outward. Finally we turn to constant mean curvature hypersurfaces without boundary. The $k$th order mean curvature of a hypersurface is an elementary symmetric polynomial of degree $k$ in the principal curvatures of the hypersurface. It is proved in Theorem 4 that if the $k$th order mean curvature of a compact immersed constant mean curvature hypersurface of $\mathbf{R}^n$ is constant for some $k$, $2 \le k \le n-1$, then the hypersurface is a sphere.

We wish to thank Finn and McCuan for their helpful comments on this work.

## 1. Extension of Hopf's theorem

In this section let us first review Hopf's method and then extend it to a disk type surface. Let $S$ be a surface in $\mathbf{R}^3$ which is the image of a conformal immersion $X$ of a unit disk $D = \{(u, v) \in \mathbf{R}^2 : u^2 + v^2 < 1\}$ into $\mathbf{R}^3$. Suppose $u$ and $v$ are the isothermal coordinates on $S$ determined by $X$. The metric of $S$ is written

$$ds^2 = E(du^2 + dv^2).$$

Define a unit normal vector $\overline{X}$ to $S$ by $\overline{X} = X_u \times X_v / |X_u \times X_v|$. With the frame $\{X_u,\ X_v,\ \overline{X}\}$ along $S$ one can write

$$\begin{aligned} X_{uu} &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L\overline{X}, \\ X_{uv} &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M\overline{X}, \\ X_{vv} &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N\overline{X}. \end{aligned} \tag{1}$$

Note that

$$\frac{1}{2}E_u = \langle X_{uu},\ X_u\rangle = \langle X_{uv},\ X_v\rangle,\ \frac{1}{2}E_v = \langle X_{uv},\ X_u\rangle = \langle X_{vv},\ X_v\rangle.$$

Hence

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \frac{E_u}{2E},\ \Gamma_{12}^1 = \Gamma_{22}^2 = \frac{E_v}{2E}.$$

Also

$$\frac{1}{2}E_u = \langle X_v,\ X_{uv}\rangle = -\langle X_{vv},\ X_u\rangle,\ \frac{1}{2}E_v = \langle X_u,\ X_{vu}\rangle = -\langle X_{uu},\ X_v\rangle.$$

Hence

$$\Gamma_{22}^1 = -\frac{E_u}{2E},\ \Gamma_{11}^2 = -\frac{E_v}{2E}.$$

Therefore (1) becomes

$$\begin{aligned} X_{uu} &= \frac{E_u}{2E}X_u - \frac{E_v}{2E}X_v + L\overline{X}, \\ X_{uv} &= \frac{E_v}{2E}X_u + \frac{E_u}{2E}X_v + M\overline{X}, \\ X_{vv} &= -\frac{E_u}{2E}X_u + \frac{E_v}{2E}X_v + N\overline{X}. \end{aligned} \tag{2}$$

Let us think of $\overline{X}$ as the Gauss map $\overline{X} : S \to S^2$. Define the *mean curvature* $H$ of $S$ to be the trace of $-d\overline{X}$. Since

$$\langle \overline{X}_u,\ X_u\rangle = -\langle \overline{X},\ X_{uu}\rangle = -L,\ \langle \overline{X}_u,\ X_v\rangle = -\langle \overline{X},\ X_{uv}\rangle = -M,$$

$$\langle \overline{X}_v,\ X_u\rangle = -\langle \overline{X},\ X_{uv}\rangle = -M,\ \langle \overline{X}_v,\ X_v\rangle = -\langle \overline{X},\ X_{vv}\rangle = -N,$$

one has

$$\overline{X}_u = -\frac{L}{E}X_u - \frac{M}{E}X_v,\ \overline{X}_v = -\frac{M}{E}X_u - \frac{N}{E}X_v. \tag{3}$$

Thus

$$H = \frac{L+N}{E}. \tag{4}$$

Now let us derive the Codazzi equations. From (1) we get

$$L = \langle X_{uu},\ \overline{X}\rangle,\ M = \langle X_{uv},\ \overline{X}\rangle,\ N = \langle X_{vv},\ \overline{X}\rangle.$$

Hence by (2), (3), and (4)

$$\begin{aligned} L_v - M_u &= \langle X_{uu},\ \overline{X}_v\rangle - \langle X_{uv},\ \overline{X}_u\rangle = \frac{E_v}{2E}(L+N) = \frac{1}{2}E_v H, \\ M_v - N_u &= \langle X_{uv},\ \overline{X}_v\rangle - \langle X_{vv},\ \overline{X}_u\rangle = -\frac{E_u}{2E}(L+N) = -\frac{1}{2}E_u H. \end{aligned} \tag{5}$$

But by differentiating $EH = L + N$, we get

$$E_v H = -EH_v + L_v + N_v,\ E_u H = -EH_u + L_u + N_u.$$

Therefore the Codazzi equations (5) can be written as

$$(L-N)_u + 2M_v = EH_u,\ (L-N)_v - 2M_u = -EH_v \tag{6}$$

Here let us introduce the complex coordinates

$$w = u + iv,\ \bar{w} = u - iv.$$

Then

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right),\ \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Put

$$\Phi(w, \bar{w}) = L - N - 2iM. \tag{7}$$

Then adding the first equation of (6) to $i$ times the second gives

$$\Phi_{\bar{w}} = EH_w.$$

From this Hopf concluded the following.

**Lemma 1** [H]. *$\Phi$ is holomorphic on a constant mean curvature surface.*

Let us give a sketchy proof of Hopf's theorem about a constant mean curvature immersion of a sphere. It follows from (3) that

$$-4\langle X_w, \overline{X}_w\rangle = -\langle X_u - iX_v, \overline{X}_u - i\overline{X}_v\rangle = \Phi.$$

Similarly, if $\Psi(z, \bar{z})$ denotes the function analogous to $\Phi(w, \bar{w})$ for another complex coordinate $z = x + iy$, then

$$\Psi = -4\langle X_z, \overline{X}_z\rangle.$$

Hence

$$\Phi dw^2 = \Psi dz^2. \tag{8}$$

This formula implies that $\Phi dw^2$ is a holomorphic quadratic differential. But a standard theorem about Riemann surfaces states that on a compact Riemann surface of genus 0, there exists no holomorphic quadratic differential $\Phi dw^2$ except the trivial one, $\Phi \equiv 0$. Therefore $L \equiv N$ and $M \equiv 0$, which implies by (3) that all the points of the constant mean curvature surface $S$ are umbilic and thus $S$ is a round sphere.

Now, how can we extend Hopf's arguments to a surface with nonempty boundary? Fortunately, Hopf gave a second proof for his theorem. His second proof is based on the lines of curvature. The lines of curvature of a smooth surface flow smoothly except at umbilic points. They rotate (bend) sharply around umbilic points. So one can naturally define the rotation index of the lines of curvature at an umbilic point. Being the zeros of $\Phi$, the umbilic points are isolated. Moreover the holomorphicity of $\Phi$, as it turns out, requires the umbilic points to have a negative rotation index. But the total sum of the rotation indices over a finite number of umbilic points of a compact surface equals its Euler characteristic, which is positive on $S^2$. Thus an immersed sphere of constant mean curvature is round.

Let $X : D_h \to S$ be a conformal immersion of a half disk $D_h = \{(u, v) \in D : v \geq 0\}$ into a regular surface $S$ mapping the diameter $l$ of $D_h$ into $\partial S$. The lines of curvature of $S$ can be pulled back by $X$ to a line field on $D_h$. If $X(l)$ is a line of curvature of $S$, then this line field can be extended smoothly to a line field $F$ on $D$ by reflection through the diameter. Clearly $F$ has the well defined rotation index at $X^{-1}(p)$ for an umbilic point $p \in \partial S$ and furthermore, the rotation index is independent of the choice of the immersion $X$. So one can naturally define the rotation index of the lines of curvature at an umbilic point $p \in \partial S$ to be *half* the rotation index of $F$ at $X^{-1}(p)$. But one needs to show that the umbilic points on $\partial S$ are isolated and then one ought to estimate the rotation index of $F$ at those umbilic points. With these in mind we do some preliminary work.

From (3) it follows that the principal curvature $\kappa$ and the principal vector $\begin{pmatrix} a \\ b \end{pmatrix}$ satisfy

$$-\frac{1}{E}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \kappa \begin{pmatrix} a \\ b \end{pmatrix}.$$

Taking an infinitesimal tangent vector $\begin{pmatrix} du \\ dv \end{pmatrix}$ to a line of curvature and eliminating $\kappa$, one obtains the differential equation for the lines of curvature

$$-Mdu^2 + (L - N)dudv + Mdv^2 = 0.$$

This equation can be rephrased in complex coordinates as

$$\mathrm{Im}(\Phi dw^2) = 0,$$

which is equivalent to

$$\arg dw = \frac{m\pi}{2} - \frac{1}{2}\arg\Phi. \ (m \text{ an integer})$$

Therefore the rotation index of the lines of curvature is

$$I = \frac{1}{2\pi}\delta(\arg dw) = -\frac{1}{4\pi}\delta(\arg\Phi),$$

where $\delta$ denotes the variation as one winds once around an isolated umbilic point $p$. If $p$ is in the interior of $S$ and is a zero of order $n$ of $\Phi$, then $\delta(\arg\Phi) = 2\pi n$ and consequently

$$I = -\frac{n}{2} \leq -\frac{1}{2}. \tag{9}$$

At a boundary umbilic point $p$, suppose that $\Phi$ has a zero (a pole, respectively) of order $n(-n > 0$, respectively). Then

$$I = \frac{1}{2}\left[-\frac{1}{4\pi}\delta(\arg\Phi)\right] = -\frac{n}{4}. \tag{10}$$

In comparison with the interior umbilic points, the boundary umbilic points have some similarities and differences as follows.

**Lemma 2.** *Let $S$ be an immersed constant mean curvature surface which is of class $C^{2,\alpha}$ up to and including the boundary $\partial S$ and let $\partial S$ consist of curves which are $C^{2,\alpha}$ up to and including the singular points of $\partial S$ called the vertices of $S$. If the regular components of $\partial S$ are lines of curvature, then we have the following.*
(a) *The boundary umbilic points of $S$ are isolated;*
(b) *At a boundary umbilic point which is not a vertex of $S$ the rotation index of lines of curvature is not bigger than $-1/4$;*
(c) *At a vertex of $S$ with angle $< \pi$ the rotation index is $\leq 1/4$, and at a vertex with angle $> \pi$ the rotation index is $\leq -1/4$.*

*Proof.* One may assume without loss of generality that $S$ is of disk type. First we show that $S$ has a conformal parametrization which is $C^{2,\alpha}$ up to and including $\partial S$ except the vertices of $S$. Fix an interior point $q \in S$ and find a harmonic function $\rho(p)$ on $S$ which vanishes on $\partial S$ and grows like $\log \operatorname{dist}(p, q)$ near $q$. Clearly such $\rho$ exists in $C^{2,\alpha}(\bar{S} \sim \{q, \text{vertices of } S\})$. Let $\theta$ be a conjugate harmonic function of $\rho$. Then $\rho$ and $\theta$ give rise to the desired conformal immersion $X : \bar{D} \to \bar{S}$ such that $X^{-1}(p) = e^{\rho + i\theta}$ and $X(0) = q$. Now let $X_1 : D_h \to S$ be a conformal immersion of a half disk $D_h \subset D$ into $S$ mapping the diameter $l$ of $D_h$ into $\partial S$ in a $C^{2,\alpha}$ manner. From (3) and (7) one sees that the imaginary part of the holomorphic function $\Phi$ vanishes along $l$. Hence $\Phi$ can be extended to a holomorphic function (still denoted $\Phi$) defined in the whole disk $D$ except at some vertices of $S$. Therefore the nonvertex boundary umbilic points are isolated. Let us now show that a vertex of $S$ cannot be an accumulation point of the boundary umbilic points. One might call a vertex $p$ an umbilic point provided the two lines of curvature emanating from $p$ are not perpendicular, but $\Phi$ does not necessarily vanish at $p$. This is because the regularity of $\bar{S}$ implies that the second fundamental form is bounded on $\bar{S}$ and hence, in view of (3), $|\Phi(p)| \leq E(p) = 0$ or $\infty$ depending on whether the angle of $S$ at $p$ is $> \pi$ or $< \pi$. In case $\Phi(p) = 0$, $p$ cannot be an accumulation point of the boundary umbilic points since the zeros of the extended $\Phi$ are isolated. In case $\Phi(p) = \infty$, $p$ cannot be an accumulation point of zeros of $\Phi$. Thus in either case (a) follows. For (b) one just uses the holomorphicity of the extended $\Phi$ and (10). For (c) one needs to estimate the order of $E$ in (3) at a vertex as in the following lemma.

**Lemma 3.** *Let $S$ and $\partial S$ be the same as in* Lemma 2 *and suppose $p$ is a vertex of $S$ with angle $\xi$. If $\xi < \pi$ and $p$ is a singularity of $\Phi$, then $p$ is a simple pole, and if $\xi > \pi$, then $p$ is a zero of $\Phi$.*

*Proof.* In view of (3) and the boundedness of the second fundamental form on $\bar{S}$, one can see that $\Phi$ cannot grow faster than $E$. First we claim that the regularity of $\partial S$ implies $p$, if it is a singularity of $\Phi$, is not essential. Let $R \subset S$ be a neighborhood of $p$ which is $C^{2,\alpha}$ diffeomorphic to a sector $D_s \subset D$ with angle $\xi$ at the origin. Find the harmonic function $\theta$ which satisfies the mixed boundary condition that $\theta = 0$ and $\theta = \xi$ on two components of $(\partial S \sim \{p\}) \cap \partial R$, respectively, and $\frac{\partial \theta}{\partial \nu} = 0$ on $\partial R \sim \partial S$, $\nu$ being the unit normal to $\partial R \sim \partial S$. If $\rho$ is a harmonic function conjugate to $\theta$, then the conformal immersion $X_2 : D_s \to R$ defined by $X_2^{-1} = e^{\rho + i\theta}$ is in $C^{2,\alpha}(\bar{D}_s)$, and the conformal factor $E_2$ of $X_2$ is bounded. Let $X_3(w) = w^{\xi/\pi}$ and define $X = X_2 \circ X_3 : D_h \to R$. Then $X$ is a conformal immersion and its conformal factor $E$ grows at most like a pole at $p$. Hence $p$ cannot be an essential singularity of $\Phi$.

Suppose $\xi < \pi$. Then $E(p) = \infty$ and $0 \le |\Phi(p)| \le \infty$. Since $M = 0$ along $\partial S$, we have only to compute the blow-up rate of $L - N$ on $\partial S$. Let $X_1 : D_h \to S$ be the immersion as in the proof of the preceding lemma. Assume that $X_1(0) = p$ and $\partial S$ is parametrized by arclength $s$ such that $X_1(u) = c(s)$, $-1 < u < 1$, and $p = c(0)$. Then $s$ can be thought of as a function of $u$, $s = s(u)$ with $s(0) = 0$. Note that along $l$, $E(w) = (ds/du)^2$. Suppose $0 < s(u) \le -a/\log u$ for all $u$, $0 < u < \epsilon$ $(< 1)$, and for some positive constant $a$. Then there exists a sequence $\{u_i\}$ of positive numbers approaching 0 such that

$$0 \le s'(u_i) \le \frac{d}{du}\left(\frac{-a}{\log u}\right)(u_i) = \frac{a}{u_i(\log u_i)^2}$$

and hence

$$E(u_i) \le \frac{a^2}{u_i^2(\log u_i)^4} < \frac{a^2}{u_i^2}.$$

Therefore $p$ is a simple pole of $\Phi$ if $|\Phi(p)| = \infty$. Suppose now that $s(u)$ grows faster than $-a/\log u$ for any $a > 0$. Then $0 < 1/s(u) < -(\log u)/a$ for $0 < u < \epsilon$. So one can find a sequence of positive numbers $u_i \to 0$ such that for any $a > 0$, $0 > -s'(u_i)/[s(u_i)]^2 > -1/au_i$, or $0 < s'(u_i) < [s(u_i)]^2/au_i < 1/au_i$ since $s(u_i) \to 0$. In this case again, $p$ is a simple pole. Now suppose $\xi > \pi$. Then $E(p) = 0$ and hence (3) implies that $p$ is a zero of $\Phi$.

Going back to the proof of Lemma 2, one notes that at a simple pole of $\Phi$, $\delta(\arg\Phi) = -2\pi$. Thus (c) follows from (10).

**Theorem 1.** *Let $S \subset \mathbf{R}^3$ be a compact immersed disk type constant mean curvature surface which is $C^{2,\alpha}$ up to and including $\partial S$ and whose boundary is $C^{2,\alpha}$ up to and including its vertices. Suppose the regular components of $\partial S$ are lines of curvature. If the number of vertices of $S$ with angle $< \pi$ is less than or equal to* 3, *then $S$ is part of a sphere.*

*Proof.* To begin, let us remark from the Poincarè-Hopf theorem [Ho] that if $V$ is a line field on the domain $D$ with a finite number of singularities which is the pull-back under $X : D \to S$ of the lines of curvature on $S$, then the sum of the rotation indices of $V$ at the singularities in $\bar{D}$ is equal to 1. Therefore $S$ has a nonempty set $Y$ of singularities. However, note here that the singularities of the lines of curvature on $S$ occur not only at the umbilic points (the zeros of $\Phi$) but also at the vertices of $S$ (the poles or zeros of $\Phi$). Suppose $Y$ is finite. Let $p_i, q_j, r_k$, and $s_l$ be the interior umbilic points, nonvertex boundary umbilic points, vertices with angle $> \pi$, and vertices with angle $< \pi$, respectively. Then (9), Lemma 2(b), and (c) imply

$$\Sigma_{p=p_i,q_j,r_k,s_l} I(p) \leq \Sigma_i(-1/2) + \Sigma_j(-1/4) + \Sigma_k(-1/4) + \Sigma_l(1/4) \leq \Sigma_l(1/4).$$

Hence if $0 \leq l \leq 3$, then $\Sigma_p I(p) \leq 3/4$, which contradicts the Poincaré-Hopf theorem. Therefore $Y$ is infinite and has an accumulation point $q$. If $\kappa_1$ and $\kappa_2$ are the principal curvatures of $S$, then $Y$ is the zero set of the continuous function $\kappa_1 - \kappa_2$, and hence $q \in Y$ and $Y$ is closed. But the points of $Y$ except some vertices are also the zeros of $\Phi$, and the zero set of $\Phi$ is either open or finite. Thus $Y = S$, and so $S$ is spherical.

*Remark 1.* The number of vertices being less than or equal to 3 in Theorem 1 is a critical condition. There is a counterexample with 4 vertices: Let $M$ be a cylinder $S^1 \times \mathbf{R}^1 \subset \mathbf{R}^3$ and $S \subset M$ a rectangular region which is bounded by two straight lines and two circles. Then $S$ is a compact embedded disk type constant mean curvature surface with 4 vertices at each of which the rotation index equals 1/4.

## 2. Capillary surfaces

A *capillary surface* $S$ in a domain $U$ is an immersed constant mean curvature surface which meets $\partial U$ at a constant contact angle along $\partial S$. When $\partial U$ is a piecewise smooth surface one may assume the contact angles to be distinct on each smooth component of $\partial U$. In this section we will study when a capillary surface is spherical or rotational and when it does not exist.

It was H. A. Schwarz [Sc] who showed that a minimal surface with piecewise linear boundary is the conjugate surface of a capillary surface with $H \equiv 0$ in a polyhedron. By the Weierstrass representation formula two conjugate minimal surfaces are isometric. Therefore a compact minimal surface with constant contact angle in a polyhedral domain which has less than 4 vertices must be flat since a minimal surface bounded by three line segments is flat. We extend this fact to capillary surfaces with $H \neq 0$ in the following theorem which in fact comes as a corollary of Theorem 1.

**Theorem 2.** *Let $U \subset \mathbf{R}^3$ be a domain bounded by spheres or planes and let $S$ be a compact immersed disk type capillary surface in $U$ which is $C^{2,\alpha}$ up to and including $\partial S$ and whose boundary is $C^{2,\alpha}$ up to and including its vertices. If $S$ has less than 4 vertices with angle $< \pi$, then $S$ is spherical.*

*Proof.* The Terquem-Joachimsthal theorem [S1] says that if $C = S_1 \cap S_2$ is a line of curvature in $S_1$, then $C$ is also a line of curvature in $S_2$ if and only if $S_1$ and $S_2$ intersect at a constant angle along $C$. Hence each smooth component of $\partial S$ is a line of curvature of $S$. Thus the theorem follows from Theorem 1.

*Remark 2.* If $S_0 = \partial U$ is neither spherical nor planar, one cannot expect Theorem 2 to hold. Even if $S_0$ is of constant mean curvature, there is no hope. However, if $S_0$ is the union of two constant mean curvature surfaces $S_1$ and $S_2$, and the triad of surfaces $S$, $S_1$, and $S_2$ meet each other along a curve $\gamma \subset \partial S \cap \partial S_1 \cap \partial S_2$ at the angle of 120°, then $\gamma$ can be thought of as a removable singularity. Indeed $\gamma$ makes no contribution in the evaluation of some geometric integrals along $\partial S \cup \partial S_1 \cup \partial S_2$ (see [Ch]). Motivated by this property of $\gamma$, we would like to propose the following.

**Conjecture.** *The singular curves of a stationary compound soap bubble in $\mathbf{R}^3$ are lines of curvature.*

An affirmative answer to this conjecture will give a remarkable result about the multiple bubbles as follows. Define the symmetric double bubble, triple bubble, and quadruple bubble to be simply connected compound soap bubbles which respectively enclose 2, 3, and 4 congruent regions homeomorphic to a ball. Then every regular surface of a compound soap bubble which is an immersion of a symmetric double bubble, triple bubble, or quadruple bubble has less than four vertices and therefore by Theorem 1 and the conjecture above the regular surfaces are spherical. The compound soap bubble of our conjecture is just assumed to be stationary. But there are some recent results on the area minimizing compound soap bubbles. In 1995 Hass, Hutchings, and Schlafly [HHS] succeeded in proving that an area minimizing compound soap bubble enclosing two regions of equal volume is a symmetric double bubble. Furthermore in 2000 Hutchings, Morgan, Ritoré, and Ros [HMRR] showed that an area minimizing compound soap bubble enclosing two regions of any prescribed volume is the standard double bubble consisting of three spherical caps.

*Remark 3.* It should be mentioned that Finn and McCuan [FM] also obtained a result similar to Theorem 2 for embedded $S$ with $H \geq 0$, $U$ bounded by planes only, and $C^{1,\alpha}$ behavior at the vertices. Also their definition of vertices is significantly different from ours.

From now on let us deal with the capillary surfaces without a vertex. A disk type capillary surface in a ball was initially studied by Nitsche [N]; He showed that it is necessarily a spherical cap. The Delaunay surfaces and the catenoid, being annular rotational surfaces, meet a sphere or a pair of parallel planes at constant angles. Indeed, adopting the Alexandrov reflection method, Wente [W3] showed that every embedded annular capillary surface in a slab is a surface of revolution. But he also constructed many examples of immersed annular capillary surfaces lying in a ball or slab which are not surfaces of revolution [W4]. However, an immersed annular capillary surface with zero mean curvature in a slab must be part of the catenoid. This is due to the harmonicity of both the height function and $\log |g|$, $g$ being the Gauss map.

As for the capillary surfaces with more than two boundary components, some existence results are known. Lawson [L] showed there exist capillary surfaces in a triangular prism and in a square prism; Schwarz [Sc] and Smyth [Sm] constructed capillary surfaces in a cube and in a tetrahedron, respectively. Of course in these domains there also exist much simpler ones: the spherical capillary surfaces. But the existence of a nonspherical capillary surface in the domain above is partly because the piecewise linear boundary of the domain is balanced. Therefore in a domain $U$ like a wedge or an octant whose boundary is not well balanced, it is conjectured that *the only capillary surface is the spherical one*. In regard to this conjecture for the case of a wedge, McCuan [M] gave a partial affirmative answer under an assumption on the contact angles that there are no embedded constant mean curvature spanners in a wedge if $\theta_1 + \theta_2 \leq \pi + \alpha$ where $\theta_i$ are the contact angles and $\alpha$ is the dihedral angle for the wedge. In the following theorem we give another partial answer by showing that a nonspherical capillary surface $S$ does not exist if the mean curvature vector of $S$ points outward or vanishes. We consider a domain $U$, more general than a wedge or an octant, whose boundary $\partial U$ is piecewise smooth and *unbalanced*. $D \subset \partial U$ is said to be unbalanced if there exists a parallel vector field $v$ such that for any outward unit normal $\eta$ to $D$, $\langle \eta, v \rangle$ is positive.

**Theorem 3.** *Let $S$ be a compact embedded capillary hypersurface with smooth boundary in a domain $U \subset \mathbf{R}^n$ with piecewise smooth boundary $\partial U$ such that $\partial S$ is disjoint from the singular set of $\partial U$. Let $\Omega$ be the compact set bounded by $\partial U$ and $S$, and $\eta$ the outward unit normal to $\partial \Omega$. Suppose that $\partial \Omega \sim S$ is unbalanced and is a disjoint union of open disks $D_i \subset \partial U$. Let $\theta_i$, $0 < \theta_i < \pi$, be the contact angle of $S$ with $D_i$, $\mathbf{H} = H\eta$ the mean curvature vector of $S$, and*

$\mathbf{H_0} = H_0\eta$ *the mean curvature vector of* $\partial\Omega \sim S$. *If* $H + H_0 \cos\theta_i \geq 0$ *for all* $i$, *then no such capillary hypersurface* $S$ *exists.*

*Proof.* Let $\nu$ be the outward unit conormal to $\partial S$ on $S$, $\nu_t$ the projection of $\nu$ on $\partial U$, and $\nu_n$ the component of $\nu$ normal to $\partial U$. Then

$$|\nu_t| = |\cos\theta_i| \text{ and } \nu_n = (\sin\theta_i)\eta.$$

First, one has for the constant $H$

$$0 = \int_{\partial\Omega} H\eta = \int_S \mathbf{H} + \sum_i \int_{D_i} H\eta. \tag{11}$$

If $X$ denotes the position vector from a fixed point, then

$$\Delta X = \mathbf{H} \text{ on } S \text{ and } \Delta X = H_0\eta \text{ on } \partial\Omega \sim S. \tag{12}$$

Integrating (12) gives

$$\int_S \mathbf{H} = \sum_i \int_{\partial D_i} \nu = \sum_i \int_{\partial D_i} \nu_t + \sum_i \int_{\partial D_i} (\sin\theta_i)\eta, \tag{13}$$

and

$$\cos\theta_i \int_{D_i} H_0\eta = \int_{\partial D_i} \nu_t. \tag{14}$$

In (13) $\eta$ along $\partial D_i$ is the unit normal to $D_i$, not to $S$. By combining (11), (13), and (14) one gets

$$0 = \sum_i \int_{D_i} (H + H_0 \cos\theta_i)\eta + \sum_i \int_{\partial D_i} (\sin\theta_i)\eta.$$

But this contradicts the hypothesis that $\cup_i D_i$ is unbalanced.

*Remark 4.* Concus-Finn-McCuan [CFM] derived the same result when $U$ is a wedge in $\mathbf{R}^3$.

## 3. Higher order mean curvature

In this section let us give a sufficient condition for a compact immersed hypersurface $S \subset \mathbf{R}^n$ without boundary to be a round sphere $S^{n-1}$. First of all, the $k$th *order mean curvature* $H_k$ of $S$ is defined to be the elementary symmetric polynomial of degree $k$ in the principal curvatures $\kappa_1, ..., \kappa_{n-1}$ of $S$. For notational simplicity let us normalize $H_k$ by the following identity

$$(1 + \kappa_1 t) \cdots (1 + \kappa_{n-1} t) = 1 + \binom{n-1}{1} H_1 t + \binom{n-1}{2} H_2 t^2 + ... + \binom{n-1}{n-1} H_{n-1} t^{n-1}.$$

As the mean curvature $H_1$ is related to the first variation of the volume of $S$, so is $H_k$ related with the $k$th variation of the volume of $S$. Let $S_r$ be the parallel hypersurface of $S$ which is the set of all points in $\mathbf{R}^n$ at a distance $r$ from $S$ on one side of $S$ for sufficiently small $r > 0$. According to Weyl's tube formula [Wy, Re] the volume $V(r)$ of $S_r$ is a polynomial of degree $n-1$ the coefficient of $r^k$ of which is a constant multiple of $\int_S H_k$.

Minkowski had found another integral formula involving $H_k$ as follows. Let $X$ be the position vector as before. Then

$$\Delta|X|^2 = 2(n-1) + 2\langle \mathbf{H}, X\rangle \text{ on } S. \tag{15}$$

Integrating (15) on $S$ and denoting $A = \text{Volume}(S)$, $\mathbf{H} = (n-1)H_1\eta$, one gets

$$0 = A + \int_S H_1\langle \eta, X\rangle. \tag{16}$$

(16) also holds for $S_r$, which, interestingly, becomes a polynomial equation of $r$. Equating the like terms gives the Minkowski formula [Hs]:

$$0 = \int_S H_{k-1} + \int_S H_k\langle \eta, X\rangle, \tag{17}$$

(16) can be recovered from (17) by setting $k = 1$. If we set $k = 0$, (17) reduces just nominally to

$$0 = nV + \int_S \langle \eta, X\rangle, \tag{18}$$

where $V$ is the volume of the immersion $S$ counting multiplicity.

Being elementary symmetric polynomials, $H_k$ satisfies the following inequalities:

$$H_k^{k-1} \le H_{k-1}^k, \tag{19}$$

provided $H_l > 0$ on $S$ for some $l \ge k$. Here equality holds only at umbilic points of $S$. Indeed (19) holds if $H_k \equiv$ const since $S$ has a point where all principal curvatures are positive, which is obviously true for $S$ compact. Applying (19) inductively, one sees that if $H_k \equiv$ const, then

$$H_k \le H_1^k, \tag{20}$$

here again equality holds only at umbilic points. We are now ready to prove the following.

**Theorem 4.** *Let $S \subset \mathbf{R}^n$ be a compact immersed constant mean curvature hypersurface without boundary. If the* kth *order mean curvature of $S$ is constant for some $k$, $2 \le k \le n-1$, then $S$ is a hypersphere.*

*Proof.* Integrate $H_k^{(k-1)/k} \leq H_{k-1}$ over $S$ to get

$$H_k^{(k-1)/k} A \leq \int_S H_{k-1} \leq -H_k \int_S \langle \eta, X \rangle = n H_k V \text{ (by (17), (18))}.$$

Hence

$$(A/V)^k \leq n^k H_k \leq n^k H_1^k \text{ (by (20))}.$$

Since $H_1$ is constant, (16) and (18) imply

$$H_1 = A/(nV).$$

Therefore equality holds in (20) and $S$ is umbilic everywhere. Thus $S$ is a hypersphere [S2].

*Remark 5.* The constancy of $H_1$ can be replaced by the embeddedness of $S$. A. Ros [Ro] proved that the hypersphere is the only embedded compact hypersurface in $\mathbf{R}^n$ with $H_k \equiv$ const for some $k$, $1 \leq k \leq n-1$.

## References


[A] A.D. Alexandrov: Uniqueness theorems for surfaces in the large. I, II, Amer. Math. Soc. Transl. (2) **21**, 341–388 (1962)

[BC] J. L. Barbosa, M. do Carmo: Stability of hypersurfaces with constant mean curvature. Math. Z. **185**, 339–353 (1984)

[Ch] J. Choe: Sharp isoperimetric inequalities for stationary varifolds and area minimizing flat chains mod k. Kodai Math. J. **19**, 177–190 (1996)

[CFM] P. Concus, R. Finn, J. McCuan: Liquid bridges, edge blobs, and Scherk-type capillary surfaces. To appear in Indiana Univ. Math. J.

[FM] R. Finn, J. McCuan: Vertex theorems for capillary drops on support planes. Math. Nachr. **209**, 115–135 (2000)

[HHS] J. Hass, M. Hutchings, R. Schlafly: The double bubble conjecture. Elec. Res. Ann. AMS, **1**, 98–102 (1995)

[Ho] H. Hopf: Differential Geometry in the Large. Lect. Notes Math. 1000, Springer Verlag, Berlin, 1989

[Hs] C. C. Hsiung: Some integral formulas for closed hypersurfaces. Math. Scand. **2**, 286–294 (1954)

[HMRR] M. Hutchings, F. Morgan, M. Ritoré, A. Ros: Proof of the double bubble conjecture. preprint

[L] H. B. Lawson, Jr.: Lectures on Minimal Submanifolds. Publish or Perish, Berkeley, 1980

[M] J. McCuan: Symmetry via spherical reflection and spanning drops in a wedge. Pac. J. Math. **180**, 291–324 (1997)

[N] J. C. C. Nitsche: Stationary partitioning of convex bodies. Arch. Rat. Mech. Anal. **89**, 1–19 (1985); corrigendum in Arch. Rat. Mech. Anal. **95**, 389 (1986)

[Re] R. C. Reilly: Variational properties of functions of the mean curvatures for hypersurfaces in space forms. J. Diff. Geom. **8**, 465–477 (1974)

[Ro] A. Ros: Compact hypersurfaces with constant higher order mean curvatures. Revista Mat. Ibero. **3**, 447–453 (1987)

[Sc] H. A. Schwarz: Gesammelte Mathematische Abhandlungen. Band I und II, Springer, Berlin, 1890

[Sm] B. Smyth: Stationary minimal surfaces with boundary on a simplex. Invent. Math. **76**, 411–420 (1984)

[S1] M. Spivak: A Comprehensive Introduction to Differential Geometry. Vol. III, Publish or Perish, Berkeley, 1979

[S2] M. Spivak: A Comprehensive Introduction to Differential Geometry. Vol. IV, Publish or Perish, Berkeley, 1979

[W1] H. C. Wente: Counterexample to a question of H. Hopf. Pac. J. Math. **121**, 193–243 (1986)

[W2] H. C. Wente: A note on the stability theorem of J. L. Barbosa and M. do Carmo for closed surfaces of constant mean curvature. Pac. J. Math. **147**, 375–379 (1991)

[W3] H. C. Wente: The symmetry of sessile and pendant drops. Pac. J. Math. **88**, 387–397 (1980)

[W4] H. C. Wente: Tubular capillary surfaces in a convex body. Advances in geometric analysis and continuum mechanics, Edited by P. Concus and K. Lancaster, International Press 1995, 288–298

[Wy] H. Weyl: On the volume of tubes. Amer. J. Math. **61**, 461–472 (1939)