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On the Area of Minimal Surfaces in a Slab

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Consider a non-planar orientable minimal surface Σ in a slab which is possibly with genus or with more than two boundary components. We show that there exists a catenoidal waist $\mathcal{W}$ in the slab whose flux has the same vertical component as Σ such that $\text{Area}(\Sigma) \geq \text{Area}(\mathcal{W})$, provided the intersections of Σ with horizontal planes have the same orientation.

1 Introduction

The catenoid is the first nontrivial minimal surface discovered. It was Euler [6] who found it in 1744 in the process of proving that when the catenary is rotated about an axis it generates a surface of smallest area. In 1860, Bonnet [3] showed that the catenoid is the only non-planar minimal surface of revolution. More recently, Schoen [11] also characterized the catenoid as the unique complete minimal surface with finite total curvature and with two embedded ends. Similarly, Lopez and Ros [8] showed that the catenoid is the only complete embedded non-planar minimal surface of finite total curvature and genus zero. Moreover, they proved that the catenoid is the unique complete embedded minimal surface in $\mathbb{R}^3$ of Morse index one [7]. Finally, Collin [5] characterized the catenoid as the unique properly embedded minimal annulus.

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More recently, rotationally symmetric compact pieces of the catenoid have been characterized as minimal annuli in a slab in two ways. Pyo [10] showed that the catenoid in a slab is the only minimal annulus meeting the boundary of the slab in a constant angle. Also it was proved by Bernstein and Breiner [1] that all embedded minimal annuli in a slab have area bigger than or equal to the minimum area of the catenoids in the same slab. That minimum is attained by the catenoidal waist along the boundary of which the rays from the center of the slab are tangent to the waist. This waist is said to be maximally stable because its proper subset is stable and any subset of the catenoid properly containing the waist is unstable (see Proposition 1, [4]).

Bernstein and Breiner conjectured that the theorem should also hold for an immersed minimal surface possibly with genus and/or with more than two boundary components in a slab. In this article, we prove this conjecture provided the minimal surface is orientable and the intersections of the minimal surface with horizontal planes have the same orientation. The orientation of a horizontal section is induced by the surface so that its conormal is upward at regular points. If the minimal surface in a slab has more than two boundary components or non-zero genus, it may happen that a horizontal section of the surface consists of two or more closed curves which have both clockwise and counterclockwise orientations. It may also happen that a horizontal section contains a curve of rotation number zero, since the surface is not assumed to be embedded. If none of this happens (see Definition 3.1 for a more precise statement), then we can prove the conjecture. Figure 1 shows the horizontal sections $\Sigma \cap P$ with the same orientation and with the mixed orientations. An explicit example of a surface with mixed orientations can be obtained by intersecting a Costa–Hoffman–Meeks surface with three catenoidal ends with a sufficiently thick slab.

For their proof, Bernstein and Breiner used Osserman–Schiffer's theorem [9] that the length $L(u)$ of the curve $\{u = \text{const}\}$ for a harmonic function u on a minimal surface

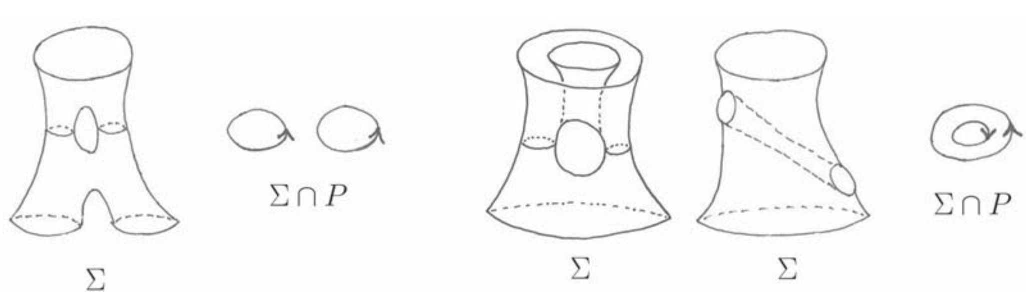

Fig. 1. Surfaces whose horizontal sections have the same orientation (left picture), mixed orientations (middle and right pictures).

Σ satisfies

$$L''(u) \geq L(u),$$

where equality holds if and only if Σ is the catenoid or an annulus in $\mathbb{R}^2$. By comparison, we directly compute the area of Σ using the conformal metric $\mathrm{d}s^2 = \cosh^2 \kappa(u,v)(\mathrm{d}u^2 + \mathrm{d}v^2)$ on Σ, take the average of $\kappa(u,v)$ along $\{u = \text{const}\}$ and use the convexity of the function $\cosh^2$. It should be noted that the topology of Σ is irrelevant in this averaging process.

At the end of this article, we propose some open problems in relation to our theorem.

2 Preliminaries

2.1 Weierstrass representation

We recall that a conformal minimal immersion from a Riemannian surface Σ into $\mathbb{R}^3$ can be written as

$$x = \mathrm{Re} \int \left(\frac{1}{2}(1 - g^2)\omega, \frac{i}{2}(1 + g^2)\omega, g\omega \right)$$

where g is a meromorphic function on Σ (the *Gauss map*) and ω a holomorphic 1-form on Σ such that

- The zeros of ω have even orders,
- A point is a pole of g of order $m \geq 1$ if and only if it is a zero of ω of order $2m$.

This is called the *Weierstrass representation*. In particular, critical points of the height function (i.e., horizontal points of $x(\Sigma)$) are zeros and poles of g, and the order of such a point as a critical point of the height function coincides with its order as a zero or a pole of g. Moreover, the induced metric is given by

$$\mathrm{d}s^2 = \frac{1}{4}(1 + |g|^2)^2 |\omega|^2.$$

Away from critical points of the height function, we can choose a local complex coordinate w such that the height function is given by $\mathrm{Re}(w)$, that is, $g\omega = \mathrm{d}w$. In this case, the metric reads as

$$\mathrm{d}s^2 = \cosh^2 \kappa |\mathrm{d}w|^2, \quad \kappa = \ln|g| = \mathrm{Re}(\log g), \quad \cosh \kappa = \frac{1}{2}\left(|g| + \frac{1}{|g|}\right).$$

2.2 Flux

Let Σ be an oriented minimal surface in $\mathbb{R}^3$ and γ an oriented smooth closed curve on Σ. Let τ be the unit tangent vector field on γ compatible with the orientation of γ. We define the *conormal* ν as the image of τ by the 90°-rotation on Σ. Note that ν is determined by the orientations of Σ and γ and changing one of these orientations changes ν into $-\nu$. The *flux vector* (or *force*) of Σ along γ is defined by

$$\int_\gamma \nu \mathrm{d}s.$$

It is well known that two homologous closed oriented curves on Σ have the same flux vector.

3 Theorem

Given the catenoid $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 : \cosh z = \sqrt{x^2 + y^2}\}$, $\mathcal{C}_a^b := \mathcal{C} \cap \{a \le z \le b\}$ is called a catenoidal waist. If there exists a point $p = (0, 0, c)$ such that the rays emanating from p are tangent to $\mathcal{C}$ along the boundary circles $\mathcal{C} \cap \{z = a, b\}$, then $\mathcal{C}_a^b$ is called a *maximally stable* waist. This is because the homotheties centered at p generate a Jacobi field J on $\mathcal{C}_a^b$ with $|J| > 0$ and vanishing only on $\partial\mathcal{C}_a^b$. Let $\beta > 0$ be the unique solution to the equation

$$\tanh z = 1/z. \tag{3.1}$$

Then it is easy to see that the tangent to the graph of $r = \cosh z$ at $z = \beta$ passes through the origin. It follows that $\mathcal{C}_{-\beta}^{\beta}$ is maximally stable. The image of a catenoidal waist (respectively, a maximally stable waist) by a translation or a homothety will still be called a catenoidal waist (respectively, a maximally stable waist).

Definition 3.1. Consider the horizontal slab $\mathrm{H}_{-a}^{a} := \{-a \le z \le a\}$ in $\mathbb{R}^3$. Let Σ be a non-planar orientable compact immersed minimal surface in H_{-a}^{a} of $\mathbb{R}^3$ with $\partial\Sigma \subset \partial\mathrm{H}_{-a}^{a}$. We choose an orientation on Σ and we then give to any horizontal section of Σ (i.e., the intersection of Σ with a horizontal plane) the orientation so that its conormal (see Section 2.2) is upward at regular points (in other words, the orientation of $\Sigma \cap \{z = t\}$ is that of $\partial(\Sigma \cap \{-a \le z \le t\})$). We say that *the horizontal sections of* Σ *have the same orientation* if the rotation numbers of all components of all horizontal sections are either all positive or all negative (here, the rotation number is that of a curve in a horizontal plane oriented so that the normal to the plane is upward). □

Theorem 3.2. Let Σ be a non-planar orientable compact immersed minimal surface in the horizontal slab H^a_{-a} with $\partial\Sigma \subset \partial\mathrm{H}^a_{-a}$ such that the horizontal sections of Σ have the same orientation in the sense of Definition 3.1. Then there exists a catenoidal waist $\mathcal{W} \subset \mathrm{H}^a_{-a}$ whose flux has the same vertical component as Σ such that

$$\mathrm{Area}(\Sigma) \geq \mathrm{Area}(\mathcal{W}). \tag{3.2}$$

Also

$$\mathrm{Area}(\mathcal{W}) \geq \mathrm{Area}\left(\frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}\right), \tag{3.3}$$

where β satisfies $\tanh\beta = \frac{1}{\beta}$ and $\frac{a}{\beta}\mathcal{C}^{\beta}_{-\beta}$ is the homothetic expansion of the catenoid $\mathcal{C}^{\beta}_{-\beta}$ by the factor of $\frac{a}{\beta}$. The boundary circles of $\frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}$ lie on the boundary of H^a_{-a} and $\frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}$ is maximally stable. Moreover,

$$\mathrm{Area}(\Sigma) = \mathrm{Area}\left(\frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}\right)$$

if and only if $\Sigma = \frac{a}{\beta}\,\mathcal{C}^{\beta}_{-\beta}$ up to translation. □

Proof. We use the orientation conventions explained in Definition 3.1. Without loss of generality, we can assume that all horizontal sections are oriented clockwise, that is, with negative rotation numbers.

The minimality of Σ in $\mathbb{R}^3$ implies that the Euclidean coordinates x, y, z of $\mathbb{R}^3$ are harmonic on Σ. The critical points of x, y, z are isolated on Σ. Let $u = z|_\Sigma$ and define $v = u^*$, the harmonic conjugate of u on Σ. Note that v is multivalued on Σ but $\mathrm{d}v$ is well defined there. Let $\tilde{\Sigma}$ be the set of regular points of z; then $w = u + iv$ is a local complex parameter on $\tilde{\Sigma}$. The Gauss map $g : \Sigma \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function which is used to express the metric of Σ (see Section 2.1):

$$\mathrm{d}s^2 = \cosh^2\kappa|\mathrm{d}w|^2, \;\; \kappa = \ln|g| = \mathrm{Re}(\log g), \;\; \cosh\kappa = \frac{1}{2}\left(|g| + \frac{1}{|g|}\right).$$

The key idea of the proof is to take the average of the harmonic function $\kappa(u, v)$ along the level curves of u, $\gamma_c : \{u = \mathrm{c}\}$. To do so, we need to find the total variation of v along γ_c:

$$\int_{\gamma_c} \mathrm{d}v = \int_{\gamma_c} \mathrm{d}z^* := -f(c). \tag{3.4}$$

The unit conormal ν to γ_c on Σ defines the *flux* (see Section 2.2) of Σ along γ_c as follows:

$$\int_{\gamma_c} \nu \mathrm{d}s = \int_{\gamma_c} \begin{pmatrix} \mathrm{d}x(\nu) \\ \mathrm{d}y(\nu) \\ \mathrm{d}z(\nu) \end{pmatrix} \mathrm{d}s = -\int_{\gamma_c} \begin{pmatrix} \mathrm{d}x^*(\tau) \\ \mathrm{d}y^*(\tau) \\ \mathrm{d}z^*(\tau) \end{pmatrix} \mathrm{d}s,$$

where τ is the unit tangent to γ_c, i.e., -90°-rotation of ν on Σ. Hence $f(c)$ equals the vertical component of the flux of Σ along γ_c, which is in fact constant for $-a \le c \le a$. From now on, we simply denote this constant by f. This constant is positive since the conormal ν is upward, by our choice of orientation.

The average $h(u)$ of $\kappa(u, v)$ along γ_u is defined by

$$h(u) = -\frac{1}{f} \int_{\gamma_u} \kappa(u, v) \mathrm{d}v.$$

It should be remarked that the average is taken over all the components of $\gamma(u)$ and so that the topology of Σ is irrelevant to $h(u)$.

Let us compute $h'(u)$ away from the critical values of the height. By the Cauchy–Riemann equations,

$$h'(u) = -\frac{1}{f} \int_{\gamma_u} \kappa_u(u, v) \mathrm{d}v = -\frac{1}{f} \int_{\gamma_u} \kappa_v^*(u, v) \mathrm{d}v.$$

Since

$$\log g = \ln |g| + i \arg g = \kappa(u, v) + i \kappa^*(u, v),$$

we have $\kappa^* = \arg g$ and so $\int_{\gamma_u} \kappa_v^*(u, v) \mathrm{d}v$ equals the total variation of $\arg g$ on γ_u, i.e., 2π times the total rotation number $r(\gamma_u)$ of the set γ_u which is the union of a finite number of closed curves. Hence

$$h'(u) = -\frac{2\pi \cdot r(\gamma_u)}{f}.$$

By our hypothesis, γ_u consists of n closed curves with the same orientation assumed clockwise. Hence

$$r(\gamma_u) \le -n \le -1. \tag{3.5}$$

Moreover, this rotation number is constant on any interval without critical value of the height function.

The function h is piecewise linear and h' is a step function. We have $\log g(w) = \pm\infty$ at the points where the tangent plane to Σ is horizontal, that is, $g(w) = 0, \infty$. We recall that these points are isolated. So $h'(u)$ is not defined at the height u of the horizontal points. Even so, $h(u)$ is continuous at this height. This can be proved as follows.

We claim that the 1-form $\kappa(u, v)\mathrm{d}v$ extends continuously at a horizontal point p. Let Z be a complex coordinate around p such that $Z(p) = 0$,

$$\frac{\mathrm{d}w}{\mathrm{d}Z} = \mathrm{O}(Z^m) \quad \text{as} \quad Z \to 0$$

and

$$g(Z) = Z^m \quad \text{or} \quad g(Z) = Z^{-m}$$

for some positive integer m (such a complex coordinate exists by the characterization of horizontal points, see Section 2.1). Hence

$$\kappa(u, v)\mathrm{d}v = \mathrm{Im}(\log|g(w)|\mathrm{d}w) = \mathrm{O}(Z^m \log|Z|\mathrm{d}Z).$$

This proves the claim. And since γ_u depends continuously on u, this proves that h is continuous at the height of a horizontal point.

We now compute the area of Σ:

$$\mathrm{Area}(\Sigma) = \int_{-a}^{a}\int_{\gamma_u} \cosh^2 \kappa(u, v)\mathrm{d}v\,\mathrm{d}u \geq \int_{-a}^{a} f \cosh^2 h(u)\mathrm{d}u,$$

where we have the inequality due to the convexity of the function $\cosh^2$. Recall that all the components of γ_u are assumed to have the same orientation. Hence (3.5) implies that $h(u)$ is an increasing function with slope at least $2\pi/f$. If $h(u)$ vanishes at some height d, define

$$k(u) = \frac{2\pi}{f}(u - d).$$

(See Figure 2.) If $h(u)$ has no zero and $\min h(u) = h(-a) > 0$, define

$$k(u) = \frac{2\pi}{f}(u + a) + h(-a),$$

and if $\max h(u) = h(a) < 0$, define

$$k(u) = \frac{2\pi}{f}(u - a) + h(a).$$

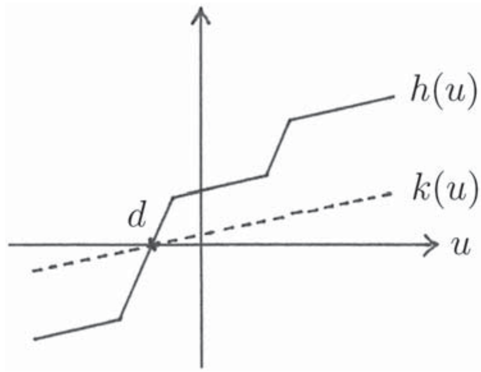

Fig. 2. Graphs of the functions h and k in the case where h vanishes.

Then we have for every u

$$h(u) \le k(u) \le 0 \quad \text{or} \quad 0 \le k(u) \le h(u).$$

It follows that

$$\cosh h(u) \ge \cosh k(u).$$

Therefore

$$\text{Area}(\Sigma) \ge \int_{-a}^{a} f \cosh^2 h(u) \mathrm{d}u \ge \int_{-a}^{a} f \cosh^2 k(u) \mathrm{d}u. \tag{3.6}$$

By the way, for some $d_0 \in \mathbb{R}$

$$k(u) = \frac{2\pi}{f} u + d_0 = \text{Re}\left(\frac{2\pi}{f} w + d_0\right).$$

Consider the Weierstrass data on $(\mathbb{C}/if\mathbb{Z}, w)$:

$$G = \exp\left(\frac{2\pi}{f} w + d_0\right) \text{ and the 1-form } \frac{1}{G(w)} \mathrm{d}w.$$

These data give rise to the catenoid which we denote as $\mathcal{C}(2\pi/f; d_0)$. So

$$\begin{aligned} \text{Area}(\Sigma) &\ge \int_{-a}^{a} f \cosh^2 k(u) \mathrm{d}u \\ &= \int_{-a}^{a} \int_{0}^{f} \cosh^2 \ln \left| \exp\left(\frac{2\pi}{f} w + d_0\right) \right| \mathrm{d}v \mathrm{d}u \\ &= \text{Area}(\mathcal{C}(2\pi/f; d_0) \cap \mathrm{H}_{-a}^{a}), \end{aligned}$$

which proves (3.2) with $\mathcal{W} = \mathcal{C}(2\pi/f; d_0) \cap \mathrm{H}^a_{-a}$. Note here that

$$\begin{aligned}\mathrm{Area}(\mathcal{C}(2\pi/f; d_0) \cap \mathrm{H}^a_{-a}) &= \int_{-a}^{a} f \cosh^2\left(\frac{2\pi}{f}u + d_0\right) \mathrm{d}u \\ &= \frac{f^2}{4\pi} \sinh\left(\frac{4\pi}{f}a\right) \cosh(2d_0) + af \\ &\geq \frac{f^2}{4\pi} \sinh\left(\frac{4\pi}{f}a\right) + af,\end{aligned}$$

that is,

$$\mathrm{Area}(\mathcal{C}(2\pi/f; d_0) \cap \mathrm{H}^a_{-a}) \geq \mathrm{Area}(\mathcal{C}(2\pi/f; 0) \cap \mathrm{H}^a_{-a}). \tag{3.7}$$

Now we need to find the catenoidal waist in H^a_{-a} with smallest area. This was already proved by Bernstein–Breiner [1] but is also proved here for completeness. This is a straightforward computation.

We observe that, by (3.7), a least area catenoidal waist is of the form $\mathcal{C}(\lambda; 0) \cap \mathrm{H}^a_{-a}$ for some $\lambda > 0$, i.e., symmetric with respect to the plane $\{z = 0\}$. So we have only to consider catenoidal waists of this form. The catenoid $\mathcal{C}(\lambda; 0)$ has $2\pi/\lambda$ as the vertical component of the flux along a horizontal circle. Hence

$$A(\lambda) := \mathrm{Area}(\mathcal{C}(\lambda; 0) \cap \mathrm{H}^a_{-a}) = \frac{2\pi}{\lambda} \int_{-a}^{a} \cosh^2(\lambda u) \mathrm{d}u = \frac{2\pi a}{\lambda} + \frac{\pi}{\lambda^2} \sinh(2\lambda a),$$

and

$$A'(\lambda) = -\frac{2\pi a}{\lambda^2} - \frac{2\pi}{\lambda^3} \sinh(2\lambda a) + \frac{2a\pi}{\lambda^2} \cosh(2\lambda a).$$

Since $A'(\lambda) = 0$ has a unique solution, $\mathrm{Area}(\mathcal{C}(\lambda; 0) \cap \mathrm{H}^a_{-a})$ must have a unique minimum. At that minimum we can easily show that λ satisfies

$$\tanh(\lambda a) = \frac{1}{\lambda a}.$$

Then (3.1) implies that $\beta = \lambda a$ and hence

$$\mathrm{Area}(\mathcal{C}(2\pi/f; 0) \cap \mathrm{H}^a_{-a}) \geq \mathrm{Area}(\mathcal{C}(\beta/a; 0) \cap \mathrm{H}^a_{-a}). \tag{3.8}$$

Since the central waist circle of $\mathcal{C}(\beta/a; 0)$ has radius a/β, we have

$$\mathcal{C}(\beta/a; 0) = \frac{a}{\beta} \mathcal{C}.$$

Therefore

$$\mathcal{C}(\beta/a; 0) \cap \mathrm{H}^a_{-a} = \frac{a}{\beta}\mathcal{C}^\beta_{-\beta},$$

which, together with (3.7) and (3.8), gives (3.3). Clearly, $\frac{a}{\beta}\mathcal{C}^\beta_{-\beta}$ is maximally stable. If

$$\mathrm{Area}(\Sigma) = \mathrm{Area}\left(\frac{a}{\beta}\mathcal{C}^\beta_{-\beta}\right), \tag{3.9}$$

then all the inequalities in (3.6), (3.7), (3.8) should become equalities. Thus

$$\Sigma = \frac{a}{\beta}\mathcal{C}^\beta_{-\beta}.$$

Similarly

$$\mathrm{Area}(\Sigma) = \mathrm{Area}(\mathcal{W})$$

if and only if $\Sigma = \mathcal{W} = \mathcal{C}(2\pi/f; d_0) \cap \mathrm{H}^a_{-a}$. This completes the proof. ■

In conclusion, we would like to propose the following:

Problems

1. Does there exist a minimal surface with two boundary components in a slab which has a horizontal section with mixed orientations? (See Figure 1.)
2. Let $\Sigma \subset \mathbb{R}^3$ be an immersed minimal annulus in a slab H such that $\Sigma \cap P$ is a figure-eight curve for any horizontal plane $P \subset \mathrm{H}$. And let $\mathcal{W}$ be the least area maximally stable catenoidal waist in H. Is it true that $\mathrm{Area}(\Sigma) \geq \mathrm{Area}(\mathcal{W})$? More generally, is it possible to remove in Theorem 3.2 the hypotheses on the orientability and on the rotation numbers of the level sets of the surface?
3. Given a minimal hypersurface Σ in a slab H of $\mathbb{R}^n$, $n \geq 4$, show that its volume is bigger than that of an $(n-1)$-dimensional catenoid in H (see [2] for the higher dimensional catenoid), or their volumes are equal if and only if Σ is the maximally stable catenoidal waist symmetric with respect to the mid-hyperplane of H.
4. Given a minimal surface Σ in a slab $\mathbb{H}^2 \times [-a, a]$ of the homogeneous manifold $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane, prove a theorem similar to Theorem 3.2.

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