Generating Axially Symmetric Minimal Hyper-surfaces in $\mathbf{R}^{1,3}$

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Abstract

It is shown that, somewhat similar to the case of classical Bäcklund transformations for surfaces of constant negative curvature, infinitely many axially symmetric minimal hypersurfaces in 4-dimensional Minkowski-space can be obtained, in a non-trivial way, from any given one by combining the scaling symmetries of the equations in light cone coordinates with a non-obvious symmetry (the analogue of Bianchi's original transformation) - which can be shown to be involutive/correspond to a space-reflection.

Recently [1, 2] various signs of integrability for axially symmetric membranes in 4-dimensional space time,

$$x^\nu(t,\varphi,\theta) = \begin{pmatrix} t \\ r(t,\varphi)\begin{matrix}\cos\theta \\ \sin\theta\end{matrix} \\ z(t,\varphi)\end{pmatrix} = \begin{pmatrix} \tau+\zeta/2 \\ R(\tau,\mu)\begin{matrix}\cos\Psi \\ \sin\Psi\end{matrix} \\ \tau-\zeta/2\end{pmatrix} = \tilde{x}^\nu(\tau,\mu,\Psi), \tag{1}$$

were revealed, including the use of

$$\frac{v'}{E} = \dot{z}, \qquad \dot{v} = r^2\frac{z'}{E}, \tag{2}$$

($\cdot$ and $'$ indicating derivatives with respect to t and φ; the constant E sometimes put equal to 1), where $r(t,\varphi)$ and $z(t,\varphi)$, describing the shape of the hypersurface, satisfy

$$\begin{aligned} &\dot{r}r' + \dot{z}z' = 0, \quad \text{and} \quad \dot{r}^2 + \dot{z}^2 + r^2\Big(\frac{r'^2+z'^2}{E^2}\Big) = 1; \\ &E^2\ddot{z} = (r^2z')', \qquad\quad E^2\ddot{r} = (r^2r')' - r(r'^2+z'^2) \end{aligned} \tag{3}$$

(the second line being implied by the first, provided $\dot{r}z' \neq r'\dot{z}$), while in light-cone coordinates, the first order equations of (3) take the form

$$\dot{\zeta} = \frac{1}{2}\Big(\dot{R}^2 + \frac{R^2 R'^2}{\eta^2}\Big), \qquad \zeta' = \dot{R}R', \tag{4}$$

implying

$$\eta^2 \ddot{R} = R(RR')' \tag{5}$$
$$\eta^2 \ddot{\zeta} = (R^2\zeta')', \tag{6}$$

with $R(\tau,\mu) = r(t,\varphi)$ and $\zeta = t - z$, $\cdot$ and $'$ denoting derivatives with respect to

$$\tau = \frac{1}{2}(t + z(t,\varphi)) \quad \text{and} \quad \mu = \frac{1}{2\eta}(E\varphi + v(t,\varphi)), \tag{7}$$

(2) implying

$$\eta\dot{\mu} = \frac{r^2}{E}\tau', \qquad \eta\frac{\mu'}{E} = \dot{\tau}, \tag{8}$$

where η is a constant (related to boosts in the z-direction, i. e. Minkowski rotations in the (t,z)-plane), often put equal to 1 just like E. Due to (5), respectively (6), (4) also implies the existence of a function $\kappa(\tau,\mu)$ satisfying

$$\frac{\kappa'}{\eta} = \dot{\zeta}, \quad \dot{\kappa} = R^2\frac{\zeta'}{\eta}. \tag{9}$$

(1), (8), and (9) are all generalizations of the Minkowski version of Cauchy-Riemann equations, i. e. for $r^2 = 1$ and $R^2 = 1$ all 6 functions would satisfy the linear wave equation with respect to their independent variables (t,φ for (1) and (8), τ,μ for (9)).

Just as $g(f(w))$ is a holomorphic function of w, if f and g are, the following 'non-linear' (respectively r-dependent) generalization holds:
Theorem 1:
Let $r(t,\varphi)$ be given, as well as $(x(t,\varphi), y(t,\varphi))$ and $(\phi(t,\varphi), T(t,\varphi))$ satisfy

$$x' = \dot{y}, \quad \dot{x} = r^2 y' \tag{10}$$
$$\phi' = \dot{T}, \quad \dot{\phi} = r^2 T' \tag{11}$$

$X(T,\phi) = x(t,\varphi)$ and $Y(T,\phi) = y(t,\varphi)$ (with $R(T,\phi) = r(t,\varphi)$) will then satisfy

$$X' = \dot{Y}, \quad \dot{X} = R^2 Y' \tag{12}$$

provided the change of variables $(t,\varphi) \mapsto (T,\phi)$ is invertible i. e. $\dot{T}^2 - r^2 T'^2 \neq 0$.
Proof: We note that,

$$\partial_t = \dot{T}\partial_T + \dot{\phi}\partial_\phi, \quad \partial_\varphi = T'\partial_T + \phi'\partial_\phi\,. \tag{13}$$

So, using (11), (10) implies

$$\begin{pmatrix} \dot{T} & T' \\ r^2 T' & \dot{T} \end{pmatrix} \begin{pmatrix} X' - \dot{Y} \\ \dot{X} - R^2 Y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

hence (12) follows. For the construction of infinitely many non-trivial solutions of (5), the following observation will be crucial:

Theorem 2:
Given (5), respectively (4) and (9), $\rho(\zeta,\kappa) = R(\tau,\mu)$ will satisfy $\ddot{\rho} = \rho(\rho\rho')'$ i.e. a new solution of (5) can be generated by rewriting a starting solution R in terms of ζ,κ (obtained by solving (4) and (9)).
Proof: Note that

$$\begin{pmatrix} \tau_\zeta & \tau_\kappa \\ \mu_\zeta & \mu_\kappa \end{pmatrix} = \begin{pmatrix} \dot{\zeta} & \zeta' \\ \dot{\kappa} & \kappa' \end{pmatrix}^{-1} = \frac{1}{\delta}\begin{pmatrix} \kappa' & -\zeta' \\ -\dot{\kappa} & \dot{\zeta} \end{pmatrix}, \tag{14}$$

where $\delta = \eta(\dot{\zeta}^2 - R^2\frac{\zeta'^2}{\eta^2}) = \frac{\eta}{4}(\dot{R}^2 - R^2\frac{R'^2}{\eta^2})^2 =: \eta\mathcal{L}^2$, gives

$$\begin{aligned} \dot{\rho} &= \rho_\zeta = (\tau_\zeta\partial_\tau + \mu_\zeta\partial_\mu)R = ... = \frac{\dot{R}}{\mathcal{L}} \\ \rho' &= \rho_\kappa = (\tau_\kappa\partial_\tau + \mu_\kappa\partial_\mu)R = ... = -\frac{R'}{\eta\mathcal{L}}. \end{aligned} \tag{15}$$

A lengthy, but straigthforward, calculation of $\delta\ddot{\rho} = (\kappa'\partial_\tau - \dot{\kappa}\partial_\mu)\frac{\dot{R}}{\mathcal{L}}$ and $\delta\rho'' = -(-\zeta'\partial_\tau + \dot{\zeta}\partial_\mu)\frac{R'}{\eta\mathcal{L}}$ then gives the desired result. Note also that (15) implies

$$\frac{1}{2}(\dot{\rho}^2 - \rho^2\rho'^2)\cdot\frac{1}{2}(\dot{R}^2 - \frac{R^2R'^2}{\eta^2}) = 1. \tag{16}$$

It would at first sight be tempting to apply Theorem 2 directly multiple times, to obtain infinitely many solutions from a given one. However, (16), as well as (ζ,τ), as functions of t,φ, related to (τ,μ) by the simple reflection $z \to -z$ (the factors of 2 being irrelevant, respectively cancelling) indicate that the transformation is involutive. Indeed, considering

$$X' = \dot{Y} = \frac{1}{2}(\dot{\rho}^2 + \rho^2\rho'^2) \quad \text{and} \quad \dot{X} = \rho^2 Y' = \rho^2\dot{\rho}\rho' \tag{17}$$

in τ,μ coordinates, using $(\eta = 1)$,

$$\partial_\zeta = \frac{1}{\mathcal{L}^2}(\kappa'\partial_\tau - \dot{\kappa}\partial_\mu), \quad \partial_\kappa = \frac{1}{\mathcal{L}^2}(-\zeta'\partial_\tau + \dot{\zeta}\partial_\mu), \tag{18}$$

gives

$$\begin{aligned} \kappa'\dot{x} - \dot{\kappa}x' = R^2(-\zeta'\dot{y} + \dot{\zeta}y') = \mathcal{L}^2\rho^2\dot{\rho}\rho' = -R^2\dot{R}R' \\ -\zeta'\dot{x} + \dot{\zeta}x' = \kappa'\dot{y} - \dot{\kappa}y' = \frac{1}{2}\mathcal{L}^2(\dot{\rho}^2 + \rho^2\rho'^2) = \frac{1}{2}(\dot{R}^2 + R^2R'^2) \end{aligned}$$

which implies

$$\begin{pmatrix} \dot{x} \\ x' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \dot{y} \\ y' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i. e. up to some trivial additive constants

$$x(\tau,\mu) = \mu, \quad y(\tau,\mu) = \tau. \tag{19}$$

Just as with the original Bäcklund transformation [4] which (as observed by Lie) were a combination of Bianchi's original transformation [5] and trivial scaling symmetries (see [6] for a nice discusion of $K = -1$ surfaces) one may combine Theorem 2 with the ('trivial') observation that

$$R_{\alpha,\gamma}(\tau,\mu) = \alpha R(\alpha\gamma\tau, \gamma\mu) \tag{20}$$

satisfies (5), if R does, to obtain infinitely many solutions of (5) which are <u>not</u> simply results of (20) (namely applying theorem 2 <u>and</u> (20) infinitely many times, always in pairs).

As nice as this seems, explicit non-trivial elementary examples of the construction are not easy to come by:

$$R(\tau,\mu)=\sqrt{2}\frac{\mu}{\tau},\qquad \zeta=-\frac{\mu^2}{\tau^3}=-\frac{R^2}{2\tau},\ \text{and}\ \ \kappa=\frac{\mu^3}{\tau^4} \tag{21}$$

is invariant under both (20) and the involutary transformation of theorem 2,

$$R\mapsto R^*(\tau,\mu):=\rho(\tau,\mu) \tag{22}$$

as $\frac{\kappa}{\zeta}=-\frac{\mu}{\tau}$, hence $\rho(\zeta,\tau)=-\sqrt{2}\frac{\kappa}{\zeta}$, i.e. $R^*=-R$ (which of course is always a solution, to be identified with R as the radius of the rotated curve should, by definition, be positive.) (21) corresponds to $\mathcal{M}_3=\{x^\nu\in\mathbf{R}^{1,3}|t^2+x^2+y^2=z^2\}$, moving hyperboloids, which in orthonormal parametrization were shown in [7] to read

$$z(t,\varphi)=\pm t\sqrt{\frac{\sqrt{1+8\frac{\varphi^2}{t^4}}+1}{2}},\qquad r(t,\varphi)=|t|\sqrt{\frac{\sqrt{1+8\frac{\varphi^2}{t^4}}+1}{2}}, \tag{23}$$

solving (2), hence also the second-order equations ($E=1$)

$$E^2\ddot{r}=(r^2r')'-r(r'^2+z'^2),\quad E^2\ddot{z}=(r^2z')'. \tag{24}$$

On the other hand, applying Theorem 2 to the minimal hypersurface

$$R=\sqrt{2}\frac{\sqrt{\mu^2+\epsilon}}{\tau},\qquad \zeta=-\frac{\mu^2+\epsilon/3}{\tau^3},\qquad \kappa=\frac{\mu^3+\epsilon\mu}{\tau^4}, \tag{25}$$

(where ϵ is a constant), whose level set form,

$$\mathcal{M}_3=\{x^\nu\in\mathbf{R}^{1,3}|(t^2+x^2+y^2-z^2)(t+z)^2=\frac{16\epsilon}{3}\} \tag{26}$$

was, 30 years after Dirac's spherically symmetric solution [8] the first non-trivial, polynomial solution [9], is already very difficult to fully work out in detail, as one has to solve polynomial equations of high degree(s) to obtain $\mu(\zeta,\kappa)$ and $\tau(\zeta,\kappa)$ and hence $\rho(\zeta,\kappa)$, from (25) –or to find an explicit orthonormal parametrization of (26) by inverting

$$t=\tau-\frac{\mu^2+\epsilon/3}{2\tau^3}\qquad E\varphi=\mu+\frac{\mu^3+\epsilon\mu}{2\tau^4}\,. \tag{27}$$

Before going to the next example, note that (20) will give

$$\zeta_{\alpha,\gamma}(\tau,\mu)=\alpha^3\gamma\,\zeta(\alpha\gamma\tau,\gamma\mu),\quad \kappa_{\alpha,\gamma}(\tau,\mu)=\alpha^4\gamma\,\kappa(\alpha\gamma\tau,\gamma\mu) \tag{28}$$

and the effect of scale transformations in (t,φ) parametrization can be calculated according to

$$\begin{aligned} t &= \tau+\frac{\alpha^3\gamma}{2}\,\zeta(\alpha\gamma\tau,\gamma\mu)\\ E\varphi &= \mu+\frac{\alpha^4\gamma}{2}\,\kappa(\alpha\gamma\tau,\gamma\mu) \end{aligned} \tag{29}$$

the inversion of which gives $\tau = \tau_{\alpha,\gamma}(t,\varphi), \mu = \mu_{\alpha,\gamma}(t,\varphi)$ from which

$$z_{\alpha,\gamma}(t,\varphi) = \tau - \frac{1}{2}\zeta_{\alpha,\gamma}, \quad \text{and} \quad r_{\alpha,\gamma}(t,\varphi) = R_{\alpha,\gamma}(\tau,\mu) \tag{30}$$

can in principle be calculated.

Consider now, as a third (class of) example(s)

$$R = \tau\sqrt{\mu}, \quad \zeta = \frac{\mu\tau}{2} + \frac{\tau^5}{40}, \quad \kappa = \frac{\tau^4}{8}\mu + \frac{\mu^2}{4} \tag{31}$$

respectively their scaled versions,

$$\begin{aligned} R_{\alpha,\gamma} &= R_{\beta=\alpha^2\gamma^{3/2}}(\tau,\mu) = \beta\tau\sqrt{\mu} \\ \zeta_\beta &= \beta^2(\frac{\mu\tau}{2} + \beta^2\frac{\tau^5}{40}) = \frac{R_\beta^2}{2\tau} + \frac{\beta^4\tau^5}{40} \\ \kappa_\beta &= \beta^4\frac{\tau^4\mu}{8} + \beta^2\frac{\mu^2}{4} \end{aligned} \tag{32}$$

corresponding to

$$\mathcal{M} = \{x^\nu \in \mathbf{R}^{1,3} |\ t^2 - x^2 - y^2 - z^2 = C(t+z)^6\}, \quad \text{with} \quad C = \frac{\beta^4}{1280} > 0 \tag{33}$$

(discussed in [2, 1] and 'quantized' in [10]). In order to obtain R^*, respectively ρ, by writing R in (31)/(32) as a function of ζ and κ one would have to solve polynomial equations like ($\beta = 1$)

$$(2\zeta - \frac{\tau^5}{20})(\frac{9}{20}\tau^5 + 2\zeta) = 4\tau^2\kappa. \tag{34}$$

As a final example, consider

$$R = \sqrt{2}F(\tau)\mu, \quad \zeta = F\dot{F}\mu^2, \quad \kappa = F^4\mu^3 + \frac{1}{3}\mu^3 \tag{35}$$

where F is an elliptic function satisfying

$$\dot{F}^2 = F^4 + 1\,. \tag{36}$$

Here one can compute that

$$\rho(\zeta,\kappa) = \zeta^{1/2}F(\kappa\zeta^{-3/2}) \tag{37}$$

showing that the involutary transformation $R \mapsto \rho$ of Theorem 2 maps two self-similar solutions (with different exponents) onto each other.

In order to appreciate the non-triviality of implementing the $z \mapsto -z$ symmetry on solutions $R(\tau,\mu), \zeta(\tau,\mu)$ of (4),(5) and (6), consider for example (25)/(26): although Theorem-2 is difficult to apply explicitly (as for that μ and τ would be needed as explicit functions of ζ and κ), one may observe that ζ^4/κ^3 depends only on μ hence $\mu = \mu(\zeta\kappa^{-3/4})$, implying $\tau = \zeta^{-1/3}h(\zeta\kappa^{-3/4})$ and $\rho = \zeta^{1/3}f(\zeta\kappa^{-3/4})$. Theorem 2 thus predicts a new solution $\tilde{R}(\tau,\mu)$, that is of the form

$$\tilde{R}(\tau,\mu) = \tau^{1/3}f(\mu\tau^{-3/4} := \xi). \tag{38}$$

Both (4), and the $z \mapsto -z$ variant of (26),

$$(2\tau\tilde{\zeta} + \tilde{R}^2)\tilde{\zeta}^2 = \frac{16\epsilon}{3} = C\ (< 0); \tag{39}$$

then imply/suggest that

$$\tilde{\zeta}(\tau, \mu) = \tau^{-1/3} g(\xi). \tag{40}$$

Indeed, (38)/(40) consistently reduces (39) and (4) to

$$2g^3 + f^2 g^2 = C\,(< 0) \tag{41}$$

and the 2 ODE's

$$g' = f'(\frac{1}{3}f + \xi f') \tag{42}$$

$$2(\xi g' - \frac{g}{3}) = (\frac{1}{3}f + \xi f')^2 + \frac{9}{16}\xi^{14/3} f^2 (f')^2. \tag{43}$$

Using (from (41))

$$f = -\frac{1}{g}\sqrt{C - 2g^3}, \quad f' = \frac{g'}{g^2\sqrt{C - 2g^3}}(C + g^3), \tag{44}$$

(42) can be written as

$$\begin{aligned} -\frac{d\xi}{\xi} &= -3\frac{dg}{g}\frac{(C+g^3)^2}{(C+4g^3)(C-2g^2)} \overset{(z=-g^3)}{=} \frac{dz}{z}\frac{(z-C)^2}{(2z+C)(4z-C)} \\ &= dz\Big(-\frac{1}{z} + \frac{3/4}{(z+C/2)} + \frac{3/8}{(z-C/4)}\Big) \end{aligned} \tag{45}$$

implying

$$E^2 \xi = \frac{z}{(z+C/2)^{3/4}(z-C/4)^{3/8}}. \tag{46}$$

(43), on the other hand gives

$$\begin{aligned} 6(C+4g^3)(C-2g^3)g^3 - 6g^3(C+g^3)^2 - (C-2g^3)(C+g^3)^2 - (C+4g^3)(C-2g) \\ +2(C+g^3)(C+4g^3)(C-2g^3) = \frac{9}{16}\frac{\xi^{8/3}}{g^2}(C-2g^3)^2(C+4g^3) \end{aligned} \tag{47}$$

with a (at first sight 'weird') factor $\frac{\xi^{8/3}}{g^2}$, which however, using (46) (with $E^2 = 1$) is exactly what is needed to make (43) consistent with (42)/(41) (both sides of (47) are equal to $-9z^2(C - 4z)$, provided the integration constant E^2 is chosen to be equal to 1).

So, while (25) can not be explicitly inverted, the solution that is obtained from it via Theorem 2, respectively $z \mapsto -z$ in (26), can be obtained 'almost explicitly', the function $g(\xi)$ in (40) solving a polynomial equation of degree 27,

$$\xi^8 (g^3 + C/4)^3 (g^3 - C/2)^6 = g^{24}. \tag{48}$$

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