Contact surgery and Heegaard Floer theory

András I. Stipsicz

Contents

The fundamental theorem of Giroux — relating contact structures and open book decompositions — provides a way to study contact structures on closed 3–manifolds from a topological point of view. Contact surgery diagrams allow us to use some form of Kirby calculus in the study of contact 3–manifolds, and Heegaard Floer theory — through the Ozsváth–Szabó knot invariant of the binding of a compatible open book decomposition — gives a very sensitive contact invariant, which seems to be crucial in attacking the classification problem of tight contact structures on certain types of closed 3–manifolds. In these notes we collected the basic ideas of contact surgery and computation of contact Ozsváth–Szabó invariants. We paid special attention to explicit computations, hoping to convince the reader that the usual Heegaard Floer package, together with some simple homotopy–theoretic arguments might be used to derive exciting new results in contact topology.

The lectures these notes are based on were delivered at the 2004 Clay Mathematics Institute Summer School on Floer Homology, Gauge

theory and Low Dimensional Topology at the Alfréd Rényi Institute of Mathematics, Budapest. The author would like to thank Burak Ozbagci for his help in preparing the figures.

1. Contact 3–manifolds

General definitions. We start our discussion by recalling basic notions of contact topology — for a more complete treatment of the topics just mentioned here, see [**7, 11**].

Let Y be a given closed, oriented, smooth 3–manifold. A 1–form α is a (positive) *contact form* if $\alpha \wedge d\alpha > 0$ (with respect to the given orientation). A 2–plane field ξ is a positive, coorientable contact structure if there is a contact 1–form $\alpha \in \Omega^1(Y)$ such that $\xi = \ker \alpha$. By fixing α (up to multiplication by smooth functions $f \colon Y \to \mathbb{R}^+$) we also fix an orientation for the 2–plane field ξ: the basis $\{v_1, v_2\} \subset \xi_p$ is positive if $\{v_1, v_2, n\}$ with normal vector n satisfying $\alpha(n) > 0$ provides an oriented basis for T_pY. In this case the contact structure is *cooriented*.

Let (X, ω) be a given compact, symplectic 4–manifold, that is, X is a smooth, compact, oriented 4–manifold with possibly non–empty boundary and ω is a closed 2–form with $\omega \wedge \omega > 0$ (with respect to the given orientation). The contact 3–manifold (Y, ξ) is *compatible* with (X, ω), or (X, ω) is a *filling* of (Y, ξ) if $\partial X = Y$ as oriented manifolds and $\omega|_\xi \neq 0$. In this case (X, ω) is also called a *weak symplectic filling* of (Y, ξ).

A symplectic filling (X, ω) is a *strong filling* of (Y, ξ) if ω is exact near $\partial X = Y$ and there is a 1-form α near ∂X with $\omega = d\alpha$, $d\alpha|_\xi \neq 0$ and $\xi = \{\alpha|_Y = 0\}$. It can be shown that the existence of such α is equivalent to the existence of a vector field v defined near ∂X which is transverse to the boundary and is a symplectic dilation, that is, $\mathfrak{L}_v \omega = \omega$. In particular, for a strong filling (X, ω) the symplectic structure on a collar of the boundary can be shown to have a model in a symplectic manifold (in the *symplectization* of (Y, ξ)) which depends only on the contact structure ξ; therefore strong fillings are suitable for performing symplectic surgeries. Notice that if ω is nonexact near ∂X then (X, ω) is not a strong filling. It turns out that this is the only obstruction, more precisely

Lemma 1.1 (Eliashberg [**6**], Ohta–Ono [**24**]). *If (X, ω) is a weak filling of (Y, ξ) and ω is exact on a collar neighbourhood of ∂X then ω can be perturbed near ∂X to a symplectic form $\tilde{\omega}$ such that $(X, \tilde{\omega})$ is a strong filling of (Y, ξ).* $\square$

Since on a rational homology 3–sphere any 2–form is exact, this implies

Corollary 1.2. *Suppose that Y is a rational homology 3–sphere, i.e., $b_1(Y) = 0$. If (X, ω) is a weak filling of (Y, ξ) for some contact structure ξ then ω can be perturbed to provide a strong filling $(X, \tilde{\omega})$ of (Y, ξ).* □

The compact complex manifold (X, J) with complex structure J is a *Stein filling* of (Y, ξ) if $\partial X = Y$, ξ is given as the oriented 2–plane field of complex tangencies on Y and (X, J) is a Stein domain, that is, it admits a proper function $\varphi\colon X \to [0, \infty)$ with $\partial X = \varphi^{-1}(a)$ for some regular value $a \in \mathbb{R}$ which is *plurisubharmonic*, i.e., the 2–form $\omega_\varphi = -d^{\mathbb{C}} d\varphi$ is a Kähler form with associated Kähler metric g_φ. It is not hard to see that a Stein filling is always a strong filling and a strong filling is automatically a weak filling. For more about fillings see [**6**].

Example 1.3. It is easy to see that the 1–form $\alpha = dz + xdy$ induces a contact structure on the 3–dimensional Euclidean space $\mathbb{R}^3$. It turns out that this contact structure extends to the 3–sphere S^3. In addition, the resulting 2–plane field is isotopic to the 2–plane field of complex tangencies on S^3 when viewed as the boundary of the unit 4–ball in the complex vector space $\mathbb{C}^2$. The above structures are the *standard* contact structures on $\mathbb{R}^3$ and S^3, and we will denote them by ξ_{st}.

A knot $K \subset (Y, \xi)$ is called *Legendrian* if it is tangent to ξ, i.e., if ξ is defined by the 1–form α then $\alpha(TK) = 0$. Every knot can be smoothly isotoped to a Legendrian knot, in fact, for every knot there is a C^0–close Legendrian knot smoothly isotopic to it.

Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ (and so in (S^3, ξ_{st})) can be depicted by their *front projections* to the yz–plane, since according to the equation $x = -\frac{dz}{dy}$ the slope of the tangent of the front projection determines the x–coordinate. After possibly isotoping, every Legendrian knot admits a front projection with no triple points, transverse double points and $(2, 3)$-cusps instead of vertical tangencies. Conversely, any front projection having cusps instead of vertical tangencies and not admitting crossings with higher slope in front uniquely specifies a Legendrian knot. For this reason we will symbolize Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ (and so in (S^3, ξ_{st})) by their front projections.

Notice that if $L \subset (Y, \xi)$ is Legendrian, it admits a canonical framing: consider the unit orthogonal of the tangent vector of L in ξ. (When choosing the particular orthogonal, we take the orientation of the 2–plane field into account.) The resulting framing is called the *contact framing* of the Legendrian knot L. If L is null-homologous in Y then it admits another framing, induced by pushing off L along its existing Seifert surface. This latter framing is called the *Seifert framing*. When measuring the contact framing with respect to this Seifert framing we get an integer invariant of the Legendrian knot L called the *Thurston–Bennequin invariant* $\mathrm{tb}(L)$. Notice that since the Seifert framing is well–defined and independent of the chosen Seifert surface, the Thurston–Bennequin invariant depends only on the Legendrian knot L. Since knots in $\mathbb{R}^3$ and S^3 are all null–homologous, they all admit Thurston–Bennequin invariants. The computation of $\mathrm{tb}(L)$ from a front projection of L is an easy task: it is equal to

$$w(L) - \tfrac{1}{2}c(L),$$

where $w(L)$ is the *writhe* of the projection, i.e. the signed number of the double points of the projection, and $c(L)$ is the number of cusps in the projection. Since left and right cusps alternate among each other, it is easy to see that $\frac{1}{2}c(L) = c_r(L) = c_l(L)$ where $c_r(L)$ (and $c_l(L)$) stands for the right (resp. left) cusps of the projection.

Example 1.4. Figure 1 shows the front projection of a Legendrian knot smoothly isotopic to the right–handed trefoil. The writhe of this projection is 3, and has 4 cusps, hence the Thurston–Bennequin invariant of the Legendrian knot determined by the front projection is equal to 1.

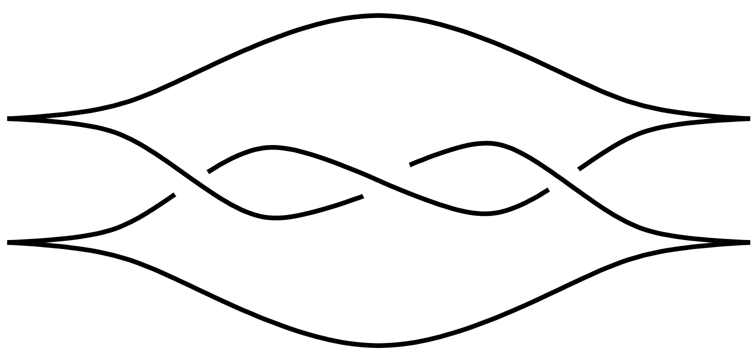

Figure 1. Front projection of a Legendrian trefoil knot

If $L \subset (Y, \xi)$ is null-homologous then there is another numerical invariant we can associate to it: consider a Seifert surface $\Sigma \subset (Y, \xi)$ and take the relative Euler class of ξ (as a 2–plane bundle) over Σ. For this to make sense we need to trivialize ξ over $\partial\Sigma = L$: choose the trivialization provided by the tangents of L together with their oriented normals in ξ. Note that in order to specify the tangents we need to fix an orientation on L. It is not hard to see that the resulting quantity, called the *rotation number* $\mathrm{rot}_\Sigma(L)$, will depend on the chosen Seifert surface and the orientation fixed on the knot. If $L \subset \mathbb{R}^3$ or S^3, however, the vanishing of the second homology group implies that the rotation number is independent of the chosen Seifert surface. If L is in $\mathbb{R}^3$ or in S^3, the rotation number can be computed for L given by a front projection by the formula

$$\mathrm{rot}(L) = \tfrac{1}{2}(c_d(L) - c_u(L)),$$

where $c_u(L)$ (resp. $c_d(L)$) denotes the number of up (resp. down) cusps of the projection.

Exercise 1.5. Compute the rotation number of the Legendrian trefoil given by Figure 1. Compute tb and rot for the knots given by Figures 4 and 6.

The crucial step for being able to do surgery in the contact category is to find canonical neighbourhoods. The following theorem provides such neighbourhoods for Legendrian curves. To state the theorem, consider the contact structure $\zeta_1 = \ker(\cos(2\pi\phi)dx - \sin(2\pi\phi)dy)$ on $S^1 \times \mathbb{R}^2$. (Here ϕ is the coordinate in the S^1-direction, while (x, y) are Cartesian coordinates on $\mathbb{R}^2$.)

Theorem 1.6 (Legendrian neighbourhood theorem). *If $K \subset (Y, \xi)$ is a Legendrian knot then there are neighbourhoods $U_1 \subset Y$ of K and $U_2 \subset S^1 \times D^2$ of $S^1 \times \{0\}$ such that $(U_1, \xi|_{U_1})$ and $(U_2, \zeta_1|_{U_2})$ are contactomorphic via a contactomorphism mapping K to $S^1 \times \{0\}$.* □

Overtwisted versus tight dichotomy.

Definition 1.7. A contact 3–manifold (Y, ξ) is *overtwisted* if there is an embedded 2–disk $D \subset Y$ which is tangent to ξ along its boundary. Such a disk D is called and *overtwisted disk*. If (Y, ξ) contains no overtwisted disk, we say that it is *tight*.

Theorem 1.8 (Eliashberg–Gromov). *If the contact 3–manifold (Y, ξ) is fillable then it is tight.* □

The above theorem is a major tool in proving tightness of contact structures. For a while, actually, it was unclear whether the reverse implication of the theorem is true or false, though it is now known to be false. As examples of this note show, there are many contact structures which are tight but not fillable. Regarding overtwisted contact structures we have Eliashberg's classification:

Theorem 1.9 (Eliashberg, [**4**]). *Two overtwisted contact structures on a closed 3–manifold Y are isotopic if and only if they are homotopic as oriented 2–plane fields. Moreover, for any oriented 2–plane field there is an overtwisted contact structure homotopic to it.* □

In short, the classification of overtwisted contact structures on a closed 3–manifold Y up to isotopy coincides with the classification of oriented 2–plane fields up to homotopy. This latter problem is homotopy–theoretic in nature: Fix a trivialization of TY and associate to an oriented 2–plane field ξ its unit orthogonal, providing a map $Y \to S^2$. In this way a trivialization of TY provides a bijection between homotopy types of oriented 2–plane fields and the set of homotopy classes of continuous maps from Y to S^2, $[Y, S^2]$. According to the Pontrjagin–Thom construction $[Y, S^2]$ can be identified with the framed cobordism classes of framed 1–manifolds in Y. The 1–manifold (through Poincaré duality) gives the spin^c structure induced by the oriented 2–plane field, while the framing gives an invariant in $\mathbb{Z}_d$ where d is the divisibility of the first Chern class of the 2–plane field. If $d = 0$, that is, the 2–plane field has torsion first Chern class, then this invariant (also called the *3–dimensional invariant* $d_3(\xi)$ of the 2–plane field ξ) is an element of an affine copy of $\mathbb{Z}$. An absolute $\mathbb{Q}$–lift of this invariant can be determined as follows: consider a compact almost–complex 4–manifold (X, J) such that $\partial X = Y$ and the 2–plane field of complex tangencies along ∂X is homotopic (as an oriented 2–plane field) to ξ. A homotopy–theoretic argument shows that such (X, J) always exists. Then

$$d_3(\xi) = \frac{1}{4}(c_1^2(X, J) - 3\sigma(X) - 2\chi(X)).$$

Notice that since $c_1(X, J)|_{\partial X} = c_1(\xi)$ is a torsion class, the square $c_1^2(X, J)$ is defined as a rational number rather than an integer. We will return to computations of 3–dimensional invariants in later sections. Tightness of a given contact structure turns out to be equivalent to a form of the adjunction inequality as given below.

Theorem 1.10 (Eliashberg). *The inequality*

$$tb_\Sigma(L) + |\mathrm{rot}_\Sigma(L)| \leq -\chi(\Sigma)$$

is satisfied for any Legendrian knot L and surface Σ with $\partial\Sigma = L$ in (Y, ξ) if and only if the contact 3-manifold (Y, ξ) is tight. □

Example 1.11. According to a result of Bennequin, the contact structure ξ_{st} induced by $\alpha = dz + xdy$ on $\mathbb{R}^3$ (and its extension to S^3, also denoted by ξ_{st}) is tight. On the other hand, the contact structure $\xi_1 = \ker \alpha_1$ for $\alpha_1 = \cos r dz + r \sin r d\theta$ in coordinates $(z, (r, \theta))$ on $\mathbb{R}^3$ can be easily shown to be overtwisted.

EXERCISE 1.12. Find an overtwisted disk $D \subset \mathbb{R}^3$ for the contact structure ξ_1 defined above.

The central problem of contact topology is to classify contact structures on 3–manifolds. Since overtwisted structures are classified by their homotopy type, the question reduces to understanding tight contact structures. Tight structures are much harder to find, and seem to carry important information about the geometry of the underlying 3–manifold, as is demonstrated by the successful application of contact topological arguments in the solution of several low–dimensional problems, see for example [**17, 18**], cf. also [**30**]. Great advances have been made in the recent past in classifying tight contact structures on some simple 3–manifolds, and this question is still in the focus of active research. In this note we would like to show an application of contact Ozsváth–Szabó invariants to solve the classification problem on certain classes of 3–manifolds.

2. Surgeries

Dehn surgeries. Suppose that $K \subset Y$ is a given knot in the closed 3–manifold Y. The operation of deleting a tubular neighbourhood of K and then regluing the solid torus $S^1 \times D^2$ is called *surgery* along the knot K. In order to specify the surgery uniquely, we have to determine the image of the simple closed curve $\{pt.\} \times \partial D^2$; the rest of the gluing is unique. For that matter, we need to fix a simple closed curve in the 2–torus $\partial(Y - \nu K) \cong T^2$. Since a simple closed curve in T^2 can be specified through its homology class, we only need to describe a homology class in $H_1(\partial(Y - \nu(K)); \mathbb{Z}) \cong \mathbb{Z}^2$. Such a class can be represented by a pair (p, q) of relatively prime integers provided there is a basis fixed in $H_1(\partial(Y - \nu(K)); \mathbb{Z})$. One basis element can be given by the boundary μ of a normal disk to K. By fixing an orientation on K, the orientation of Y equips this *meridian* μ with a canonical orientation. The other basis element, however, needs a choice. By fixing a framing of K, we get a longitude λ by pushing K off along the first basis vector of the framing. (Notice that longitudes and framings determine each other by this recipe.) Such a longitude can be chosen to be the second element of a basis in $H_1(\partial(Y - \nu(K)); \mathbb{Z})$, hence the surgery can be described by the pair (p, q) satisfying that the simple closed curve $\{pt.\} \times \partial D^2$ maps to a curve homologous to $p\mu + q\lambda$. By reversing the orientation of K, both μ and λ switch sign, and therefore the ratio $\frac{p}{q}$ remains unchanged. Notice that the ratio $\frac{p}{q}$ can take its values in $\mathbb{Q} \cup \{\infty\}$.

EXAMPLE 2.1. The surgery diagram of Figure 2 provides a description of the small Seifert fibered 3–manifold $M(r_1, r_2, r_3)$.

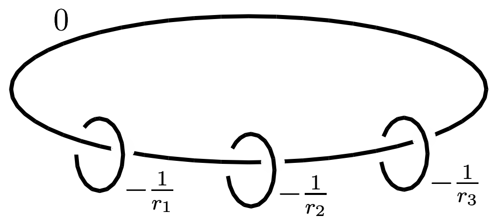

FIGURE 2. Surgery diagram for the Seifert fibered 3–manifold $M(r_1, r_2, r_3)$

EXERCISE 2.2. Verify the *slam-dunk* operation, i.e., that the two surgeries given by Figure 3 give diffeomorphic 3-manifolds. We assume that $n \in \mathbb{Z}$ and $r \in \mathbb{Q} \cup \{\infty\}$. (*Hint:* Perform surgery on K_2 first and isotop K_1 into the glued-up solid torus T. Since first we performed an integer surgery, K_1 will be isotopic to the core of T, hence when performing the second surgery we cut T out again and reglue it. Therefore it can be done by one surgery; the coefficient can be computed by first

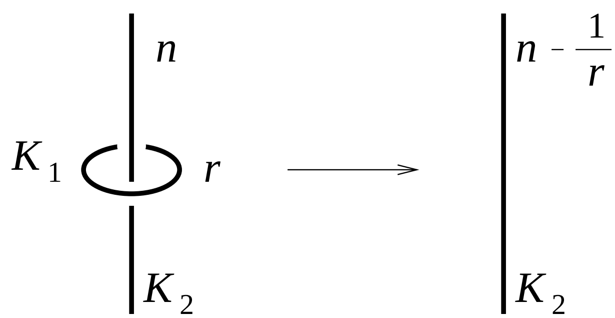

FIGURE 3. The slam-dunk operation

assuming $n = 0$ and then adding n extra twists. For more details see [**15**, pp 163-164].)

Contact surgery. The above surgery scheme can be extended to the contact category as follows. Suppose that $L \subset (Y, \xi)$ is a Legendrian knot in the given contact 3–manifold. Consider the contact framing on L and perform r–surgery with respect to this framing. The resulting 3–manifold is denoted by $Y_r(L)$. According to the classification of tight contact structures on solid tori [**16**], the contact structure

ξ admits an extension from $Y - \nu(L)$ to $Y_r(L)$ as a tight structure on the new glued–up torus provided $r \neq 0$. (The extension might not be tight on the whole closed 3–manifold $Y_r(L)$ but it is required to be tight on the solid torus of the surgery.) Such a tight extension is not unique in general; the different extensions can be determined by the continued fraction coefficients of r. Nevertheless, the extension is unique if $r \in \mathbb{Q}$ is of the form $\frac{1}{k}$ for some integer $k \in \mathbb{Z}$. In particular, according to the above, we have

PROPOSITION 2.3. *Let $\mathbb{L} = \mathbb{L}^+ \cup \mathbb{L}^- \subset (S^3, \xi_{st})$ be a given Legendrian link. The result of contact $(+1)$–surgery along components of $\mathbb{L}^+$ and contact (-1)–surgery along components of $\mathbb{L}^-$ uniquely specifies a contact 3–manifold $(Y_{\mathbb{L}}, \xi_{\mathbb{L}})$.* □

REMARK 2.4. The reason why this surgery construction works for contact 3–manifolds is that in the neighborhood of a Legendrian knot L the contact structure can be proved to be canonical, see Theorem 1.6.

In fact, the converse of this statement also holds, namely

THEOREM 2.5 (Ding–Geiges, [**2**]). *For a given contact 3–manifold (Y, ξ) there exists a Legendrian link $\mathbb{L} = \mathbb{L}^+ \cup \mathbb{L}^- \subset (S^3, \xi_{st})$ such that $(Y_{\mathbb{L}}, \xi_{\mathbb{L}}) = (Y, \xi)$.*

Before turning to the proof of this theorem we recall a useful observation regarding contact surgeries.

LEMMA 2.6 ([**2**]). *Suppose that $L \subset (Y, \xi)$ is Legendrian and L' is its Legendrian push–off. If (Y', ξ') is given by contact $(+1)$–surgery on L and contact (-1)–surgery on L' then Y' is diffeomorphic to Y and ξ' is isotopic to ξ.* □

EXERCISE 2.7. Verify that Y' is diffeomorphic to Y.

Proof of Theorem 2.5 (sketch). Perform contact $(+1)$–surgery on $L \subset (Y, \xi)$ contained by a Darboux chart and isotopic to the Legendrian unknot with Thurston–Bennequin invariant being equal to -2; see Figure 4 for such a knot. It is not hard to see that the resulting contact structure ξ' is overtwisted, hence there is a Legendrian link in (Y, ξ') such that contact $(+1)$–surgery on it gives S^3 with some contact structure. Since contact $(+1)$–surgery can be inverted by contact (-1)–surgery, the above argument shows that a sequence of contact (-1)–surgeries on *some* contact S^3 results in (Y, ξ). In the next section we will give a surgery diagram for any contact structure on S^3. (The verification that the examples given in Section 2 comprise a complete list of contact structures on S^3 is postponed until Section 3.) □

We close this section with a statement which will be used frequently, and a more detailed explanation of the phenomenon will be discussed in the next chapter.

THEOREM 2.8 (Eliashberg, Gompf). *If* $\mathbb{L} = \mathbb{L}^-$ *then the contact 3–manifold* $(Y_{\mathbb{L}}, \xi_{\mathbb{L}})$ *is Stein fillable.* □

Examples.

Contact structures on S^3 *and* $S^1 \times S^2$. By converting the contact surgery coefficients on the diagrams of Figures 4 and 5 to smooth surgery coefficients, it is quite easy to see that these diagrams rep-

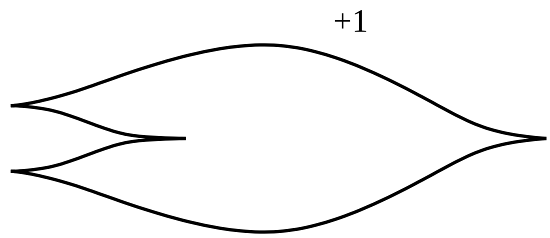

FIGURE 4. Contact structure ξ_1 on S^3

resent contact structures ξ_1 and ξ_{-1} on the 3–sphere S^3. By taking connected sums (which means simply to draw the diagrams next to each other) of n copies of Figures 4 and 5 we get sequences ξ_n and ξ_{-n} of contact structures on S^3. We define ξ_0 as the connected sum of ξ_1 and ξ_{-1}. As will be shown later, all these structures are overtwisted, and not homotopic to each other. In addition, all homotopy types of 2–plane fields are realized by one of the ξ_k ($k \in \mathbb{Z}$). Since the unique tight contact structure on S^3 can be given by the empty surgery diagram, we have the list of all contact structures on S^3 presented by surgery diagrams.

In a similar manner, we can draw diagrams providing overtwisted contact structures on $S^1 \times S^2$. (For details see [**3**].) It is a little bit more involved to find a diagram for the unique tight contact structure on $S^1 \times S^2$. A direct argument presented in [**3**] shows that contact (+1)–surgery on the Legendrian unknot shown by Figure 6 is tight. We will show tightness of this structure using contact Ozsváth–Szabó invariants.

In fact, using the above ideas now it is quite simple to find a diagram for all overtwisted contact structures on a 3–manifold Y given by a smooth rational surgery diagram. Details of this algorithm are given in [**3**].

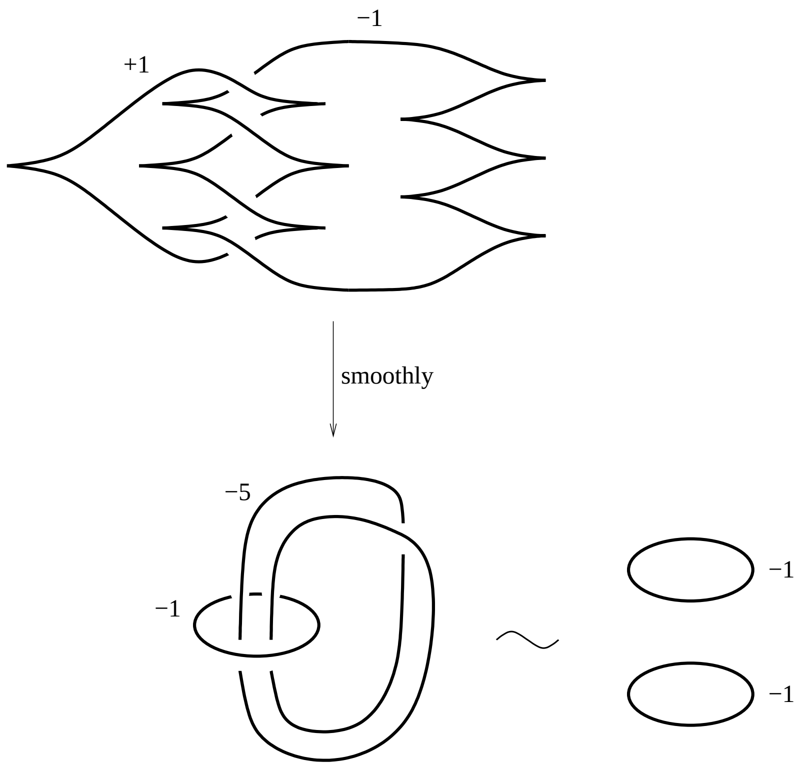

FIGURE 5. Contact structure ξ_{-1} on S^3

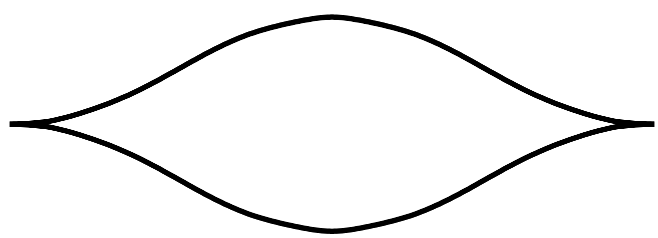

FIGURE 6. The Legendrian unknot

Contact structures on small Seifert fibered 3–manifolds. It is much harder to find all the *tight* structures on a given 3–manifold. In general this question is still open, but for some families of 3–manifolds we have a classification of tight structures. Here we restrict our attention to some special cases.

Let $L(p,q)$ denote the lens space we get by $(-\frac{p}{q})$–surgery along the unknot. It is not hard to see that this surgery is equivalent to a sequence of surgeries along a chain of unknots, all with integer surgery coefficients ≤ -2.

EXERCISE 2.9. Verify the above statement. (*Hint:* Apply the result of Exercise 2.2.)

Putting these unknots into Legendrian position and adding zig–zags to them we get surgery diagrams representing tight contact structures on $L(p, q)$. We can arrange that all the contact surgery coefficients are (-1), and then by appealing to Theorems 2.8 and 1.8 we conclude that the resulting structures are tight (in fact, Stein fillable). Notice that we have a freedom when putting the zig–zags on the two sides of the unknots; different choices will result in contact structures with different homotopy types. According to [**16**], these diagrams represent all tight contact structures on $L(p, q)$.

The above classification can be extended to a wider class of 3–manifolds of some *small Seifert 3–manifolds.* A Seifert fibered 3–manifold M is *small* if it fibers over S^2 with three singular fibers. Let $M = M(r_1, r_2, r_3)$ as given in Example 2.1 and define $e_0(M)$ as

$$e_0(M) = \sum_{i=1}^{3} [r_i].$$

It is not hard to see that $e_0(M)$ is an invariant of the Seifert fibered 3–manifold. (Notice that there might be many choices for r_i to present M as $M(r_1, r_2, r_3)$.) Also, if M has one or two singular fibers, then M is a lens space. For $e_0(M) \neq -1$ it is fairly easy to give surgery diagrams for tight contact structures on M, see [**12, 34**] and Exercise 2.10. In fact, for $e_0(M) \neq -2, -1$ these diagrams comprise a complete list of tight structures. In the case of $e_0(M) = -2$ there might be other structures for which we do not have the corresponding surgery diagram. If $e_0(M) = -1$, our understanding is less satisfactory. The difficulty lies in two facts: there are many small Seifert fibered spaces without any tight contact structures [**10, 21**], and even if there are tight structures, their tightness is more involved to prove since for many 3–manifolds the contact structures cannot be fillable [**20**]. We will return to these cases in Section 6.

EXERCISE 2.10. **(a)** Show that if $e_0(M) \geq 0$ then there are $r_1, r_2, r_3 \in \mathbb{Q}$ such that $M = M(r_1, r_2, r_3)$ and $r_i > 0$.
(b) Show that for any small Seifert fibered 3–manifold M with $e_0(M) = e$ there are rational numbers r_1, r_2, r_3 with $r_i \in (0, 1) \cap \mathbb{Q}$ such that M can be given by the surgery diagram of Figure 7.
(c) Verify that if $e_0(M) \leq -2$ then M can be presented by a surgery diagram of unknots along a tree with all framings ≤ -2.

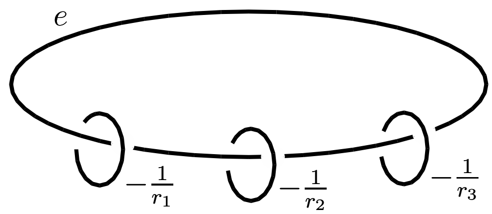

FIGURE 7. Another diagram for the small Seifert fibered 3–manifold M

3. 4–dimensional theory

Handle attachments, Kirby calculus. It is not hard to see that if the surgery coefficient of a particular knot is an integer, then the corresponding surgery can be realized by a 4–dimensional 2–handle attachment. More precisely, consider the 3–manifold Y with a knot $K \subset Y$. Fix a longitude for K and perform integer surgery along K. (Notice that the fact that a surgery coefficient is an integer does not depend on the chosen longitude, although the actual value of the surgery coefficient does.) Alternatively, consider the 4–manifold $Y \times [0,1]$ and attach a 2–handle $D^2 \times D^2$ along $\partial D^2 \times D^2$ to $Y \times \{1\}$ along K. Since $\partial D^2 \times \{0\}$ will map to K, in order to fix the gluing map we only need to specify a framing of K in Y, which (as we already remarked) is equivalent to fixing a longitude of K. Since $\partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2$, after this procedure the part $\partial D^2 \times D^2$ with its image will disappear from the 3–manifold (and sinks into the 4–manifold) while $D^2 \times \partial D^2$ appears on the new boundary. Therefore the handle attachment has the same effect as surgery along K; the surgery coefficient can be determined by the chosen framing, i.e. the chosen longitude. Therefore this alternative method works if and only if the surgery coefficient is an integer. The advantage we get is that we do not only have the surgered 3–manifold, but also a 4–dimensional cobordism between the original and the resulting 3–manifolds. As we will see, such a cobordism can be very conveniently used in many specific problems. Any rational surgery diagram can be easily converted into a surgery diagram (along a possibly different link) involving only integer coefficients using the slam–dunk operation as it is given in Exercise 2.2.

Suppose that the contact 3–manifold (Y, ξ) is given as $(Y_{\mathbb{L}}, \xi_{\mathbb{L}})$ for some Legendrian link $\mathbb{L} = \mathbb{L}^+ \cup \mathbb{L}^- \subset (S^3, \xi_{st})$. Since all the surgery coefficients in such diagrams are integers, we immediately see a 4–manifold X defined by the picture: the description of integer surgeries above provides a cobordism from S^3 to Y, which can be glued to the 4–disk D^4 to get the compact 4–manifold X with $\partial X = Y$.

First we would like to find some extra structure on X. The main theorem of the subject is the following result of Eliashberg:

THEOREM 3.1 (Eliashberg, Gompf; [**5**]). *Suppose that $\mathbb{L} = \mathbb{L}^-$ in the above situation. Then the resulting 4–manifold X admits a Stein structure inducing $\xi = \xi_{\mathbb{L}}$ on $Y = \partial X$. Consequently a surgery diagram involving only contact (-1)–surgeries gives Stein fillable, hence tight contact structure.* □

The idea of the proof of Eliashberg's theorem goes back to a result of Weinstein:

THEOREM 3.2 (Weinstein, [**33**]). *If (X, ω) is a weak symplectic filling of (Y, ξ) and $L \subset (Y, \xi)$ is a Legendrian knot then for the handle attachment $X' = X \cup H$ inducing contact (-1)–surgery along L the symplectic structure ω extends to a symplectic structure ω' on X'. Moreover, (X', ω') will be compatible with the resulting contact 3–manifold (Y', ξ').* □

COROLLARY 3.3. *If (Y', ξ') is given by contact (-1)–surgery on (Y, ξ) and (Y, ξ) is fillable then so is (Y', ξ').* □

REMARK 3.4. The original setup of Weinstein's handle attachment picture involves a Liouville vector field near the boundary, which exists only for strong fillings. In attaching a handle, though, this vector field is used only near the knot along which the attachment is carried out. Since the obstruction to the existence of a Liouville vector field v is cohomological, such v always exists on a neighborhood of the given knot. Therefore the general gluing scheme works for weak fillings as well.

Homotopy theory for contact surgery diagrams. The question of determining the homotopy type of a contact structure given by contact surgery diagram was first addressed by Gompf in the case when $\mathbb{L} = \mathbb{L}^-$ in [**14, 15**]. Below we sketch its extension to the general case.

First of all, note that since we use only 2–handles to build up the 4–manifold X, the result will be simply connected. Recall that spin^c structures on a simply connected 4–manifold X are in bijection with characteristic elements of $H^2(X; \mathbb{Z})$. (See the appendix of this section for more about spin^c structures.) Therefore the spin^c structure $\mathfrak{t}_\xi$ induced by the contact structure of the diagram can be specified by a characteristic cohomology class $c \in H^2(X; \mathbb{Z})$ through its corresponding spin^c structure $\mathfrak{s}$ which satisfies $\mathfrak{s}|_{\partial X} = \mathfrak{t}_\xi$. (It is easy to see that if X is simply connected then all spin^c structures from ∂X extend to X.) The definition of c is quite simple: consider the basis $\alpha_{K_1}, \dots, \alpha_{K_n}$ of $H_2(X;\mathbb{Z})$ induced by the 2–homologies corresponding to the surgery curves $K_1, \dots, K_n$. Then it can be shown that the class c which evaluates on α_{K_i} as $\mathrm{rot}(K_i)$ will satisfy the properties required above.

We sketch the proof (which relies on Gompf's result mentioned above) when $\mathrm{tb}(K_i) \neq 0$. For more details see [**3**]. Consider $L \subset (S^3, \xi_{st})$ and let L' denote its Legendrian push–off. By doing contact (-1)–surgery along L we get a Stein 4–manifold X_L such that $c_1(X_L)$ evaluates on the generator α_L of $H_2(X_L;\mathbb{Z})$ as $\mathrm{rot}(L)$. After performing the handle attachment corresponding to contact $(+1)$–surgery on L' we get a 4–manifold X. The complex structure on X_L will extend (for simple homotopy reasons) to a complex structure J on $X - \{P\}$, and $c = c_1(J)$ will further extend to an element $c \in H^2(X;\mathbb{Z})$. We know that c evaluates on α_L as $\mathrm{rot}(L)$; suppose that $\langle c, \alpha_{L'}\rangle = k$. Let p denote the d_3–invariant of the 2–plane field of complex tangencies along $S^3 = \partial\nu\{P\} \subset X$. Since (-1)–surgery on L and $(+1)$–surgery on L' gives (S^3, ξ_{st}) back, when computing the 3–dimensional invariant of (S^3, ξ_{st}) using X we get an equation involving k and p. By the same argument for n Legendrian push–offs of L and n Legendrian push–offs of L' this equation has the form

$$-\tfrac{1}{2} = \tfrac{1}{4}(n(k^2 - \mathrm{rot}^2(L)) - n^2\mathrm{tb}(L)(k - \mathrm{rot}(L))^2) + n(p - \tfrac{1}{2}) - \tfrac{1}{2},$$

implying that $k = \mathrm{rot}(L)$ once $\mathrm{tb}(L)$ is nonzero. The computation also shows that $p = \frac{1}{2}$, hence, provided $c_1(\xi_{\mathbb{L}})$ is torsion, the 3–dimensional invariant $d_3(\xi_{\mathbb{L}})$ can be computed as

$$d_3(\xi_{\mathbb{L}}) = \tfrac{1}{4}(c^2 - 3\sigma(X) - 2\chi(X)) + q,$$

where q denotes the number of components in $\mathbb{L}^+$.

Example 3.5. Let ξ_k ($k \in \mathbb{Z}$) denote the contact structure on S^3 defined in the previous chapter. Then, after a short calculation the above formula shows that $d_3(\xi_k) = k - \frac{1}{2}$, and hence these structures are all nonisotopic. If $k \neq 0$ then (since S^3 admits a unique tight contact structure with 3–dimensional invariant equal to $-\frac{1}{2}$) these are all overtwisted. For ξ_0 a direct argument provides overtwistedness. Similar computation applies to diagrams representing contact structures on $S^1 \times S^2$.

It is easy to see that if ξ is an oriented 2–plane field on S^3 then its 3–dimensional invariant is of the form $k - \frac{1}{2}$. This follows from the fact that for any 4–manifold X with $\partial X = S^3$ and almost–complex structure J the expression

$$d_3+\frac{1}{2}=\frac{1}{4}(c_1^2(X,J)-3\sigma(X)-2\chi(X))+\frac{1}{2}=$$

$$=\frac{1}{4}(c_1^2(X,J)-\sigma(X))-\frac{\sigma(X)+\chi(X)}{2}+\frac{1}{2}=\frac{1}{4}(c_1^2(X,J)-\sigma(X))-\frac{2b_2^+(X)-2b_1(X)}{2}$$

is an integer. This is obvious since $c_1(X,J)\in H^2(X;\mathbb{Z})$ is characteristic, hence $c_1^2(X,J)\equiv\sigma(X) \pmod 8$.

In conclusion, by Eliashberg's theorem we conclude that $\{\xi_k \mid k\in\mathbb{Z}\}$ together with the standard contact structure ξ_{st} (represented by the empty diagram) comprises a complete list of contact structures (up to isotopy) on the 3–sphere S^3. Notice that this observation concludes the proof of Theorem 2.5 from the previous chapter.

Embedding theorems. Let (X,ω) be a given symplectic filling of (Y,ξ). It turned out to be very useful to study embeddings of (X,ω) into closed 4–manifolds with some additional structures. In this section we will recall the basic results and some ramifications of this theory.

THEOREM 3.6 (Lisca–Matić, [**19**]). *If (X,ω) is a Stein filling of (Y,ξ) then there exists a closed minimal complex surface Z of general type such that X Kähler embeds into Z.* □

REMARK 3.7. The fact that a Stein filling can be Kähler embedded into a Kähler manifold can be verified much more easily. The important point of Theorem 3.6 is that the target manifold can be chosen to be a minimal surface of general type. This property is very important in applications concerning Seiberg–Witten theory.

A nice proof (relying on the theory of Lefschetz fibrations) for the weaker statement of embedding (X,ω) into a minimal symplectic 4–manifold was found by Akbulut and Ozbagci [**1**]. A corollary of Theorem 3.6 can be used to distinguish tight contact structures:

COROLLARY 3.8. *If J_1, J_2 are two Stein structures on a fixed 4–manifold X inducing contact structures ξ_1 and ξ_2 on $Y=\partial X$ then $c_1(J_1)\neq c_1(J_2)$ implies that ξ_1 and ξ_2 are not isotopic.* □

EXERCISE 3.9. Using Corollary 3.8 find lower bounds on the number of tight contact structures on small Seifert fibered manifolds with $e_0(M)\neq -1$. (*Hint*: See [**12, 34**].)

The above embedding theorem of Lisca and Matić was extended by Eliashberg [**6**] (see also Etnyre [**8**]) to weak fillings as follows:

THEOREM 3.10. *If (X,ω) is a weak symplectic filling of (Y,ξ) then it symplectically embeds into a closed symplectic 4–manifold U.*

Proof. Consider $\mathbb{L} = \mathbb{L}^+ \cup \mathbb{L}^- \subset (S^3, \xi_{st})$ such that $(Y_{\mathbb{L}}, \xi_{\mathbb{L}}) = (Y, \xi)$. Perform contact (-1)–surgeries along Legendrian push–offs of knots in $\mathbb{L}^+$, getting a contact 3–manifold (Y_1, ξ_1) with a weak symplectic filling (X_1, ω_1). Notice that (X, ω) symplectically embeds into (X_1, ω_1). Now consider Legendrian trefoils with tb=1 for each surgery curve linking the particular curve once (and not linking the others) and perform contact (-1)–surgeries on them. The resulting contact 3–manifold (Y_2, ξ_2) has $H_1(Y_2; \mathbb{Z}) = 0$ with a weak filling (X_2, ω_2) such that $(X, \omega) \subset (X_1, \omega_1) \subset (X_2, \omega_2)$. Since $b_1(Y_2) = 0$, the symplectic form ω_2 can be perturbed to a symplectic form $\tilde{\omega}_2$ near ∂X_2 such that $(X_2, \tilde{\omega}_2)$ becomes a strong filling of (Y_2, ξ_2). Notice that (Y_2, ξ_2) is given by a sequence of contact (-1)–surgeries, therefore it is Stein fillable with Stein filling (W, J). Apply the embedding of Theorem 3.6 to W, providing a closed complex surface Z containing W. Since both W and X_2 are strong fillings of the same contact 3–manifold, we can perform surgery along Y_2, providing a closed symplectic 4–manifold $U = (Z - W) \cup X_2$, into which (X, ω) symplectically embeds. □

This embedding theorem has far-reaching applications in low–dimensional topology: it served as the missing step in the proof of *Property P* given by Kronheimer and Mrowka [**17**], and completed a Heegaard Floer theoretic proof [**30**] of the lens space surgery theorem of Kronheimer, Mrowka, Ozsváth and Szabó [**18**]. See also [**32**].

Appendix: Spin^c structures on 3– and 4–manifolds.

Definition 3.11. A *spin^c structure* on an oriented 3–manifold Y is an equivalence class of nowhere zero vector fields, where v_1 and v_2 are equivalent if they are homotopic outside a ball $B^3 \subset Y$.

Alternatively, we can consider the same equivalence relation on oriented 2–plane fields in the tangent bundle TY. Therfore a 2–plane field automatically induces a spin^c structure. Notice that for a spin^c structure $\mathfrak{t} \in Spin^c(Y)$ the class $c_1(\mathfrak{t}) \in H^2(Y; \mathbb{Z})$ is defined by taking a representative of $\mathfrak{t}$ and considering it as a complex line bundle. It is easy to see that this definition is equivalent to the conventional definition of spin^c structures: An oriented 2–plane field reduces the structure group of TY from $SO(3)$ to $U(1)$, which admits a canonical lift to $U(2) = Spin^c(3)$. Conversely, a spin^c bundle $P \to Y$, through the canonical representation of $U(2)$ on $\mathbb{C}^2$, provides a $\mathbb{C}^2$–bundle $W \to Y$ (called the bundle of spinors), and a nowhere vanishing section $\phi \in \Gamma(W)$ gives rise to a section of unit length of TY. It is not hard to verify that $H^2(Y; \mathbb{Z})$ admits a free and transitive action

$$\mathfrak{t} \mapsto a^*(\mathfrak{t})$$

($\mathfrak{t} \in Spin^c(Y)$, $a \in H^2(Y;\mathbb{Z})$) on $Spin^c(Y)$, with the property $c_1(a^*(\mathfrak{t})) = c_1(\mathfrak{t}) + 2a$.

The action $v \mapsto -v$ of multiplication by (-1) on vector fields induces an involution

$$\mathfrak{J} = \mathfrak{J}_Y : Spin^c(Y) \to Spin^c(Y).$$

A straightforward argument shows that $c_1(\mathfrak{J}\mathfrak{t}) = -c_1(\mathfrak{t})$. The fixed points of this action satisfy $c_1(\mathfrak{t}) = 0$, and since $c_1(\mathfrak{t})$ is the obstruction to reducing the structure group of the principal $Spin^c(3)$ bundle to $Spin(3) = SU(2)$, we have

Corollary 3.12. *A spin^c structure $\mathfrak{t} \in Spin^c(Y)$ is a fixed point of the involution $\mathfrak{J}$ if and only if $\mathfrak{t}$ can be given by a spin structure.* $\square$

In a similar fashion we proceed for 4–manifolds. For a fixed 4–manifold X (with possibly nonempty boundary ∂X) let J_1, J_2 be two almost–complex structures defined on $X - \{x_1, \ldots, x_n\}$ with $\partial X \cap \{x_1, \ldots, x_n\} = \emptyset$. The two structures J_1 and J_2 are said to be homologous if there is a 1–dimensional submanifold C (with possibly nonempty boundary) containing x_i $(i = 1, \ldots, n)$ such that J_1 is homotopic to J_2 on $X - C$.

Definition 3.13. An equivalence class of homologous almost–complex structures on X is called a spin^c structure on X.

The first Chern class $c_1(\mathfrak{s})$ of $\mathfrak{s} \in Spin^c(X)$ is defined as the extension of $c_1(X - \{x_1, \ldots, x_n\}, J)$ through the points where J is undefined (for an almost–complex structure J representing $\mathfrak{s}$). Since J reduces the structure group $SO(4)$ of TX to $U(2)$ outside of a finite set, and $U(2)$ admits a canonical lift to

$$Spin^c(4) = \{(A, B) \in U(2) \times U(2) \mid \det A = \det B\},$$

we can easily see that our definition above is equivalent to the traditional definition of spin^c srtructures through lifting of cocycle structures of appropriate principal bundles. For one direction we also need that a spin^c structure uniquely extends through a point, and for the converse direction we need that a section $\phi \in \Gamma(W^+)$ of the positive spinor bundle provides an isomorphism of W^- and TX away from the zero set of ϕ. Since W^- is a complex 2–plane bundle, this construction provides the necessary almost–complex structure on TX. Notice that, in fact, an oriented 2–plane bundle on $X - \{x_1, \ldots, x_n\}$ already determines an almost–complex structure by defining J as rotation on the orthogonal plane. As in the 3–dimensional case, it is also quite easy to see that

$H^2(X;\mathbb{Z})$ admits a free, transitive action on $Spin^c(X)$. By considering the conjugate complex multiplication we get an involution

$$\mathfrak{J} = \mathfrak{J}_X : Spin^c(X) \to Spin^c(X)$$

with the property that $c_1(\mathfrak{J}\mathfrak{s}) = -c_1(\mathfrak{s})$. As in the 3–dimensional case, the spin^c structure $\mathfrak{s}$ is induced by a spin structure if and only if $\mathfrak{J}\mathfrak{s} = \mathfrak{s}$, equivalently if $c_1(\mathfrak{s}) = 0$.

If X is a manifold with boundary ∂X then the oriented 2–plane field of complex tangencies along the boundary ∂X provides a restriction map $r : Spin^c(X) \to Spin^c(\partial X)$.

4. Heegaard Floer theory

In this chapter we outline the basics of Heegaard Floer theory; we restrict ourselves to a short introduction, highlighting the aspects crucial for contact topological considerations. For a more detailed treatment see [**25, 26**] and the contributions [**31, 32**] in this volume.

Ozsváth–Szabó homologies of 3–manifolds. Elementary Morse theory shows that a closed, oriented 3–manifold Y admits a *Heegaard decomposition* $Y = U_1 \cup_{\Sigma_g} U_2$ into two solid genus–g handlebodies U_1 and U_2, glued together along a surface Σ_g of genus g. A solid genus–g handlebody with boundary Σ_g can be specified by g disjoint, simple closed curves $\alpha_1, \ldots, \alpha_g \subset \Sigma_g$ which are linearly independent in homology: attaching handles along α_i (together with a 3–ball) we recover the given handlebody. Therefore Y can be described by

$$(\Sigma_g, \{\alpha_i\}_{i=1}^g, \{\beta_j\}_{j=1}^g).$$

Consider the g^{th} symmetric power $Sym^g(\Sigma_g)$ and the g–dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \ldots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \ldots \times \beta_g$ in it. A symplectic structure on Σ_g gives rise to a symplectic structure on $Sym^g(\Sigma_g)$; let J be a compatible almost-complex structure. Furthermore, fix a point $z \in \Sigma_g$ distinct from all the α– and β–curves and consider the hypersurface $V_z = \{z\} \times Sym^{g-1}(\Sigma_g)$, which is disjoint from the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. For $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ let $\mathfrak{M}_{x,y}$ denote the moduli space of holomorphic maps $u : \Delta^2 \to Sym^g(\Sigma_g) - V_z$ from the unit disk $\Delta^2 \subset \mathbb{C}$ with the properties that $u(i) = x, u(-i) = y$ and the arc connecting i and $-i$ on $\partial\Delta^2$ is mapped into $\mathbb{T}_\alpha$ (resp. into $\mathbb{T}_\beta$) if the points on the arc have positive (resp. negative) real parts. The space $\mathfrak{M}_{x,y}$ admits an $\mathbb{R}$–action, let $\mathfrak{N}_{x,y}$ denote the result of the factorization by this action.

Consider $\widehat{CF}(Y) = \oplus_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}_2\langle x \rangle$ and define the map $\partial : \widehat{CF}(Y) \to \widehat{CF}(Y)$ by the matrix element $\langle \partial x, y \rangle$ (for $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$) to be zero if

$\dim \mathfrak{N}_{x,y} > 0$ and

$$\langle \partial x, y\rangle = \#\mathfrak{N}_{x,y} \qquad (mod\ 2)$$

if $\dim \mathfrak{N}_{x,y} = 0$.

REMARK 4.1. For the sake of simplicity above we used $\mathbb{Z}_2$–coefficients. The theory can be set up using $\mathbb{Z}$–coefficients, in which case a coherent choice of orientations of the various moduli spaces must be made. Such a choice exists, but since we will use only mod 2 invariants, we do not deal with the details of this subtlety. We also note that when $b_1(Y) \neq 0$, not every Heegaard diagram gives rise to a well-defined theory, since for $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ there might be infinitely many homotopy types of disks connecting them, each possibly containing holomorphic representatives. By choosing *admissible* Heegaard diagrams, this case can be ruled out, and so for such decompositions the boundary operator ∂ is defined and satisfies $\partial \circ \partial = 0$. For definition and details on admissibility see [**25**].

Standard theory of Floer homologies shows that $\partial \circ \partial = 0$, hence $(\widehat{CF}(Y), \partial)$ is a chain complex. We define $\widehat{HF}(Y)$ as the homology of this chain complex.

THEOREM 4.2 (Ozsváth–Szabó, [**25**]). *The Abelian group $\widehat{HF}(Y)$ is an invariant of the 3–manifold Y and is independent of the choices made throughout its definition.* □

It can be shown directly that by fixing the base point z, any intersection point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines a spin^c structure $\mathfrak{t}_x$, and ∂x can have components only with the same induced spin^c structure. Consequently the chain complex $(\widehat{CF}(Y), \partial)$ naturally splits as a direct sum $\oplus_{\mathfrak{t}\in Spin^c(Y)}(\widehat{CF}(Y, \mathfrak{t}), \partial)$, defining a splitting as

$$\widehat{HF}(Y) = \oplus_{\mathfrak{t}\in Spin^c(Y)}\widehat{HF}(Y, \mathfrak{t}).$$

As above, it has been proved [**25**] that the group $\widehat{HF}(Y, \mathfrak{t})$ is an invariant of the spin^c 3–manifold $(Y, \mathfrak{t})$.

For $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ inducing the same spin^c structure $\mathfrak{t}$ consider the formal dimension of the moduli space $\mathfrak{M}_{x,y}$, and take this number as the difference of the gradings of x and y, denoted by $gr(x, y)$. Of course, this number might depend on the chosen component of the moduli space $\mathfrak{M}_{x,y}$, i.e., on the homotopy type of the disk connecting x and y. As application of the appropriate index theorem shows, $gr(x, y)$ is well-defined only modulo $d(\mathfrak{t})$, where $d(\mathfrak{t})$ is the divisibility of the first Chern class $c_1(\mathfrak{t}) \in H^2(Y; \mathbb{Z})$ of the spin^c structure $\mathfrak{t}$. Consequently, if

$c_1(\mathfrak{t})$ is a torsion element, the above procedure provides a relative $\mathbb{Z}$–grading on $\widehat{CF}(Y)$, which descends to a relative $\mathbb{Z}$–grading on $\widehat{HF}(Y)$. By fixing the convention that the group $\widehat{HF}(S^3) = \mathbb{Z}_2$ is in degree 0, there is a lift of the above relative $\mathbb{Z}$–grading to an absolute $\mathbb{Q}$–grading (provided that $c_1(\mathfrak{t})$ is torsion).

LEMMA 4.3. *With the grading as given above, the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathfrak{t})$ with $c_1(\mathfrak{t})$ torsion splits as*

$$\widehat{HF}(Y, \mathfrak{t}) = \oplus_{d \in \mathbb{Q}} \widehat{HF}_d(Y, \mathfrak{t}).$$

The degree $d \in \mathbb{Q}$ is determined mod 1 by the spin^c structure $\mathfrak{t}$. Moreover, $\widehat{HF}_d(Y, \mathfrak{t})$ is isomorphic to $\widehat{HF}_d(Y, \mathfrak{J}\mathfrak{t})$ and to $\widehat{HF}_{-d}(-Y, \mathfrak{t})$. □

Suppose now that W is an oriented cobordism between the 3–manifolds Y_1 and Y_2. It is easy to see that W can be given as a sequence of 1–, 2– and 3–handle attachments. Since 1– and 3–handles can be attached essentially uniquely, the maps on the Ozsváth–Szabó homologies induced by those cobordisms follow a straightforward convention. If W is given by 2–handle attachments only, then W can be described by a Heegaard triple

$$(\Sigma_g, \{\alpha_i\}_{i=1}^g, \{\beta_j\}_{j=1}^g, \{\gamma_k\}_{k=1}^g),$$

and the map induced by such triple follows the same line of ideas as the definition of the homology groups, except now we count holomorphic triangles instead of holomorphic disks, see [**26**]. The case of general cobordism W now relies on the Morse theoretic argument of decomposing W into subcobordisms of the above types and composing the associated maps. In analogy with the definition of the 3–dimensional invariants, one can prove that the resulting map is independent of the choices involved; for more details see [**26**]. As in the 3–dimensional case, the maps split according to spin^c structures on the cobordisms. In the following F_W denotes the sum of the induced maps for all spin^c structures.

In addition, a spin^c cobordism $(W, \mathfrak{s})$ from $(Y_1, \mathfrak{t}_1)$ to $(Y_2, \mathfrak{t}_2)$, with $\mathfrak{t}_1, \mathfrak{t}_2$ torsion spin^c structures shifts the absolute $\mathbb{Q}$–grading by the rational number

$$\tfrac{1}{4}(c_1^2(\mathfrak{s}) - 3\sigma(W) - 2\chi(W)).$$

Basic properties. The most fundamental properties of these homology groups can be summarized as follows.

THEOREM 4.4 (Adjunction inequality for homologies, [**26**]). *Suppose that $\Sigma \subset Y$ is a closed oriented surface in a closed, oriented*

3–manifold Y and $\mathfrak{t} \in Spin^c(Y)$ is a given spin^c structure. The non-triviality of the Ozsváth–Szabó homology group $\widehat{HF}(Y, \mathfrak{t})$ implies that either $\Sigma = S^2$ and $\langle c_1(\mathfrak{t}), [\Sigma]\rangle = 0$, or $g(\Sigma) > 0$ and

$$|\langle c_1(\mathfrak{t}), [\Sigma]\rangle| \leq 2g(\Sigma) - 2.$$

□

THEOREM 4.5 (Adjunction inequality for maps, [**26**]). *If W is a 4-dimensional cobordism and $\Sigma \subset W$ is a closed oriented surface with positive genus in it then for $\mathfrak{s} \in Spin^c(W)$ the fact that $F_W \neq 0$ implies that $|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma]^2 \leq 2g(\Sigma) - 2$. In particular, if W contains a surface Σ with $[\Sigma]^2 > 2g(\Sigma) - 2 \geq 0$ then $F_{W,\mathfrak{s}} = 0$ for all spin^c structures $\mathfrak{s} \in Spin^c(W)$.* □

Finally, suppose that a 3–manifold Y and a knot $K \subset Y$ are given. Perform integer surgery along K, resulting in a 3–manifold Y_K and a cobordism X_1 from Y to Y_K. Consider a normal circle N to K and attach a 2–handle to Y_K along N with framing (-1). The resulting 3–manifold will be denoted by Y', while the cobordism is X_2. Repeat this last step, i.e., attach a 2–handle to Y' along a normal circle U of N with framing (-1). It is not hard to see that the resulting 3–manifold is diffeomorphic to Y; denote the last cobordism by X_3. The diagram below describes the situation.

$$\begin{tikzcd}
Y \arrow[rr, "X_1"] & & Y_K \arrow[dl, "X_2"] \\
& Y' \arrow[ul, "X_3"] &
\end{tikzcd}$$

This geometric situation induces a triangle on Ozsváth–Szabó homologies as depicted below.

$$\begin{tikzcd}
\widehat{HF}(Y) \arrow[rr, "F_{X_1}"] & & \widehat{HF}(Y_K) \arrow[dl, "F_{X_2}"] \\
& \widehat{HF}(Y') \arrow[ul, "F_{X_3}"] &
\end{tikzcd}$$

The central result for computing Ozsváth–Szabó homologies is the following

THEOREM 4.6 (Surgery exact triangle, [**26**]). *The triangle defined above for Ozsváth–Szabó homologies is exact.* □

For an elegant proof of the exactness of the surgery triangle see [**32**] in this volume.

5. Contact invariants

The most spectacular success of Ozsváth–Szabó homologies stems from its applications to knot theory and to contact topology. In the following we will discuss the definition and basic properties of the contact invariant defined in [**28**]. Applications will be given in the next chapter.

Open book decompositions and Giroux's theorem. The definition of the contact invariant $c(Y,\xi) \in \widehat{HF}(-Y, \mathfrak{t}_\xi)$ rests on a seminal result of Giroux, providing a close connection between open book decompositions and contact structures on a given 3–manifold Y. Here we restrict ourselves to an outline of this beautiful theory; for a more complete treatment the reader is advised to turn to [**9**] in this volume.

Suppose that $L \subset Y$ is a fibered link in Y, that is, the complement $Y - L$ fibers as $f\colon Y - L \to S^1$ over the circle S^1, and the fibers of f provide Seifert surfaces for L. In this case the pair (L, f) is an *open book decomposition* of Y. The fibers of f are the *pages*, while L is the binding of the open book decomposition. The monodromy of the fibration $f\colon Y - L \to S^1$ is called the *monodromy* of the open book decomposition (L, f). A contact structure ξ on Y is said to be *compatible* with an open book decomposition (L, f) on Y if L is transverse with respect to ξ and there is a contact 1–form α defining ξ such that the 2–form $d\alpha$ is a volume form on each page. In addition, we assume that the orientation of the binding as a transverse knot coincides with its orientation as the boundary of a page.

According to a classical theorem of Thurston and Winkelnkemper, for any open book decomposition there exists a contact structure compatible with it: by slightly perturbing the tangents of the pages and extending this plane field through the binding we get the desired contact structure. Giroux proved that the converse of this statement is also true, namely for any contact structure there is an open book decomposition compatible with it. The existence of this open book can, in fact, be deduced from a surgery diagram representing the given contact structure. In addition, a simple argument shows that if two contact structures are compatible with the same open book decomposition then they are isotopic. The converse of this correspondence is more subtle and requires a definition.

Definition 5.1. Suppose that an open book decomposition is given on Y with page F and monodromy φ. Let F' denote the surface we get by adding a 1–handle to F. The open book decomposition with page F' and monodromy $\varphi \circ t_a$ is called a *positive stabilization* of (F, φ) if

t_a is a right-handed Dehn twist along the simple closed curve $a \subset F'$ intersecting the cocore of the new 1–handle in a unique point.

With this definition in place, we can formulate the central result clarifying the relation between open book decompositions and compatible contact structures.

THEOREM 5.2 (Giroux, [**13**]). **(a)** *For a given open book decomposition of Y there is a compatible contact structure ξ on Y. Contact structures compatible with a fixed open book decomposition are isotopic.* **(b)** *For a contact structure ξ on Y there is a compatible open book decomposition of Y. Two open book decompositions compatible with a fixed contact structure admit a common positive stabilization.* □

Contact Ozsváth–Szabó invariants. The above result of Giroux shows that any invariant of the open book decomposition which is invariant under positive stabilization is a contact invariant. For simplicity let us assume that the binding of the given open book decomposition is connected. (This can always be achieved by sufficiently many positive stabilizations.) Perform 0–surgery along the binding, resulting in a fibered 3–manifold Y_B. The definition of the contact Ozsváth–Szabó invariant relies on the following lemma.

LEMMA 5.3. *Suppose that Y is a closed 3–manifold which fibers over S^1. If $\mathfrak{t}_{can}$ denotes the spin^c structure given by the tangents of the fibers of $Y \to S^1$, then $\widehat{HF}(Y, \mathfrak{t}_{can}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with a distinguished generator g. Moreover, $\widehat{HF}(Y, \mathfrak{t}) = 0$ for all spin^c structures distinct from $\mathfrak{t}_{can}$ satisfying $\langle c_1(\mathfrak{t}), F\rangle = \langle c_1(\mathfrak{t}_{can}), F\rangle$, where F denotes the homology class of the fiber of $Y \to S^1$.*

In specifying the distinguished element of $\widehat{HF}(Y, \mathfrak{t}_{can})$ one has to use another version of Ozsváth–Szabó homology groups (namely the group $HF^+(Y, \mathfrak{t}_{can})$) and the map connecting the two theories — we will not go into the details of the definition of g.

Proof (sketch). The argument rests on a simple application of the surgery exact triangle together with a sample computation. Let $\mathfrak{T} = \{\mathfrak{t} \in Spin^c(Y) \mid \langle c_1(\mathfrak{t}), F\rangle = \langle c_1(\mathfrak{t}_{can}), F\rangle\}$. In the first step we show that the group $\oplus_{\mathfrak{t}\in\mathfrak{T}}\widehat{HF}(Y, \mathfrak{t})$ does not depend on the monodromy of the fibration, just on the genus of the fiber. To this end, suppose that $Y_1, Y_2 \to S^1$ are two given genus–g fibrations. There is a cobordism between them which is a Lefschetz fibration over the annulus; by induction we may assume that the fibration has a unique singular fiber, that is, Y_2 is given as (-1)–surgery (with respect to the fiber framing) along the nonseparating vanishing cycle of the singular fiber. Writing down the exact triangle, it is easy to see that the third group is zero, since 0–surgery (with respect to the fiber framing) reduces the genus of the fiber, hence the adjunction inequality implies the result. The sample computation can be carried out for the result of the 0–surgery along the $(2, 2g+1)$ torus knot. (A related exact triangle reduces this computation to simple algebra.) $\square$

Recall that $g \in \widehat{HF}(Y_B, \mathfrak{t}_{can}) \cong \widehat{HF}(-Y_B, \mathfrak{t}_{can})$ denotes the distinguished generator of this homology group. When turning the cobordism of the above 0–surgery upside down, we get a cobordism W from $-Y_B$ to $-Y$. The contact invariant $c(Y, \xi)$ of a contact 3–manifold is defined by $F_W(g) \in \widehat{HF}(-Y)$. The proof of the fact that the resulting element is an invariant of the contact structure (and not only the open book decomposition) proceeds in two steps. First we show the following:

Proposition 5.4 (Ozsváth–Szabó, [**28**] Lemma 4.4). *Suppose that an open book decomposition on Y with monodromy φ is given, and denote the element corresponding to this decomposition via the above recipe by $c(\varphi) \in \widehat{HF}(-Y)$. Then for every positive integer $h \in \mathbb{N}$ there is an element $c(\varphi, h) \in \widehat{HF}(-Y)$ such that $c(\varphi, h) = c(\varphi')$ for any monodromy φ' we get by applying $2h$ positive stabilizations to the given open book decomposition.* $\square$

The proof of the proposition follows the same line of argument outlined in the proof of Lemma 5.3 above. Using induction, it would be sufficient to show that $c(\varphi) = c(\varphi, 1)$. This identity is, however, hard to deal with directly, since the two open book decompositions correspond to different genera, hence the fibrations we get by the 0–surgeries are not connected by any natural cobordism. Therefore one has to use more sophisticated tools in proving that the invariant $c(\varphi)$ is an invariant of the compatible contact structure, rather than only the open book decomposition.

This second step of the proof of invariance relies on the knot invariants introduced in [**29**]. Ozsváth and Szabó noticed that the knot group $\widehat{HFK}(Y, L, \mathfrak{t})$ of the binding (with an appropriate relative spin^c structure determined by the genus of the page) is cyclic, and the image of its generator in $\widehat{HF}(Y)$ can be easly seen to be invariant under simple positive stabilizations of the open book decomposition. Finally, an explicit computation with appropriate Heegaard diagrams show that this knot invariant is equal to the element $c(\varphi)$ defined above. This last step verifies that our definition provides a contact invariant.

Basic properties. Recall that the Ozsváth–Szabó homology groups split as a direct sum $\widehat{HF}(Y) = \oplus_{(\mathfrak{t},d)\in\mathfrak{P}}\widehat{HF}_d(Y,\mathfrak{t})$ where $\mathfrak{P}$ is the set of homotopy types of oriented 2–plane fields on Y and the pair $(\mathfrak{t},d)$ stands for the spin^c structure and the 3–dimensional invariant determined by a given oriented 2–plane field. The first property of the contact invariant is that it is an element of the summand corresponding to the 2–plane field of the contact structure:

LEMMA 5.5. *For a contact 3–manifold (Y,ξ) the contact Ozsváth–Szabó invariant $c(Y,\xi)$ is an element of $\widehat{HF}_{-d_\xi}(-Y,\mathfrak{t}_\xi)$.* □

The next property provides a way for computing the invariant for contact structures given by contact surgery diagrams.

THEOREM 5.6. *Suppose that (Y_2,ξ_2) is given as contact $(+1)$–surgery along the Legendrian knot $L\subset(Y_1,\xi_1)$; the corresponding cobordism is denoted by X. Then*

$$F_{-X}(c(Y_1,\xi_1)) = c(Y_2,\xi_2).$$

Proof. Fix an open book decomposition of Y which contains L in a page such that the contact framing and the surface framing on L coincide. A simple modification of the argument given in the proof of Proposition 5.4 now provides the argument. □

Using the definition, it can be shown that

LEMMA 5.7. *The contact invariant $c(S^3,\xi_{st})\in\widehat{HF}(S^3)=\mathbb{Z}_2$ of the standard contact 3–sphere is nonzero.*

Proof. Consider the standard open book decomposition on S^3 with the trivial knot as binding and apply the definition. □

From the transformation rule of Theorem 5.6 the following important vanishing and nonvanishing results can be easily deduced.

PROPOSITION 5.8. *If (Y,ξ) is overtwisted then $c(Y,\xi)=0$.*

Proof. It is not hard to see from the classification of overtwisted contact structures that there is a contact structure ξ' on Y such that (Y,ξ) is given as contact $(+1)$–surgery along the Legendrian knot of Figure 4, located in a Darboux chart of (Y,ξ'). Since the cobordism corresponding to this surgery contains a sphere of square (-1), after reversing orientation it is clear that $c(Y,\xi)=F_{-X}(c(Y,\xi'))=0$ independent of the value of $c(Y,\xi')$. □

PROPOSITION 5.9. *If (Y,ξ) is Stein fillable then $c(Y,\xi)\neq 0$.*

Proof. Using Eliashberg's theorem it can be shown that any Stein fillable contact 3–manifold can be given by a sequence of contact (-1)–surgeries on one of the contact 3–manifolds η_k, where η_k is a contact structure on $\#_k S^1 \times S^2$ $(k \geq 0)$ given by doing contact $(+1)$–surgery along the k–component Legendrian unlink. It can be shown that $c(\#_k S^1 \times S^2, \eta_k) \neq 0$ (see Exercise 5.10), and since it is given by a sequence of contact $(+1)$–surgeries on (Y, ξ), we get that for some cobordism W the equation $F_W(c(Y, \xi)) = c(\#_k S^1 \times S^2, \eta_k)$ holds, implying that $c(Y, \xi) \neq 0$. □

Exercise 5.10. Using induction on k verify that the contact invariant $c(\#_k S^1 \times S^2, \eta_k)$ is nonzero.

6. Applications

After having discussed the basic properties of the contact Ozsváth–Szabó invariants, we present some results relying on these notions.

Surgery along knots in S^3. First we examine the problem of the existence of tight contact structures on 3–manifolds of the form $Y = S^3_r(K)$, i.e., Y can be given by a single Dehn surgery on S^3. Let us recall that the *maximal Thurston–Bennequin number* $TB(K)$ of a knot $K \subset S^3$ is defined by

$$\max\{tb(L) \mid L \text{ is smoothly isotopic to } K \text{ and Legendrian in } (S^3, \xi_{st})\}.$$

The *slice–genus* (or 4–ball genus) $g_s(K)$ of $K \subset S^3$ is by definition

$$\max\{g(F) \mid F \subset D^4, \partial F = K \subset S^3\}.$$

Using gauge theory it has been proved that $TB(K) \leq 2g_s(K) - 1$.

Theorem 6.1. *If $TB(K) = 2g_s(K) - 1 > 0$ is satisfied for a knot K then $S^3_r(K)$ admits a positive tight contact structure for any $r \neq TB(K)$.*

Notice that if K is the (p, q) torus knot $T_{(p,q)}$, then for $p, q \geq 2$ and relative prime it has $TB(T_{(p,q)}) = pq - p - q$, which is equal to $2g_s(T_{(p,q)}) - 1$, hence those knots satisfy the assumptions of the theorem. For example, the right-handed trefoil knot T depicted in Figure 8 satisfies the assumptions. (For a more detailed and simpler proof of Theorem 6.1 for $T =$ the trefoil knot, see the contribution [**22**] in this volume.) In fact, any nontrivial algebraic knot does the same, and there are many other knots with this property. For example, if for the knots K_1, K_2 we have $TB(K_i) = 2g_s(K_i) - 1$ then the same equation holds for their connected sum $K_1 \# K_2$.

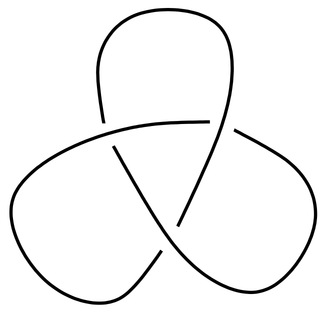

FIGURE 8. The right-handed trefoil knot

Proof. Let L be a Legendrian knot smoothly isotopic to K with $\mathrm{tb}(L) = TB(K)$. Let $k \geq 0$ be a fixed integer and consider $L_1, \ldots, L_k$, L_{k+1} Legendrian push–offs of L, while $C_1, \ldots, C_t$ is a chain of Legendrian unknots linked to L_{k+1}. Fix a rational number $r \neq TB(K)$.

LEMMA 6.2. *For any rational number $r \neq TB(K)$ there is an integer $k \geq 0$ and suitable stabilizations of $L_{k+1}, C_1, \ldots, C_t$, such that a sequence of contact $(+1)$–surgeries on $L_1, \ldots, L_k$ and contact (-1)–surgeries on the stabilized $L_{k+1}, C_1, \ldots, C_t$ yields a contact 3–manifold (Y, ξ) such that Y is diffeomorphic to $S^3_r(K)$.*

Proof (sketch). Let $r' = r - TB(K) \neq 0$. If $r' < 0$ then take $k = 0$ and the stabilizations are directed by the continued fraction coefficients of r'. In case $r' = \frac{p'}{q'} > 0$, take $k > 0$ satisfying $q' - kp' < 0$ and repeat the above recipe for $\frac{p'}{q'-kp'}$. For a more detailed description of the algorithm see [**3, 20**]. □

Having the above lemma at hand we have candidate contact structures on $S^3_r(K)$; in the following we will use contact Ozsváth–Szabó invariants to prove their tightness. Let (Y_k, ξ_k) denote the contact 3–manifold we get by performing only the $(+1)$–surgeries, $k \geq 0$; in particular, $(Y_0, \xi_0) = (S^3, \xi_{st})$. Since contact (-1)–surgery on a contact 3–manifold with nonvanishing contact Ozsváth–Szabó invariants is tight, $c(Y_k, \xi_k) \neq 0$ will imply that all the contact structures defined by Lemma 6.2 are tight, concluding the proof of Theorem 6.1. The fact $c(Y_k, \xi_k) \neq 0$ can be proved by induction — this is the step where the assumption $TB(K) = 2g_s(K) - 1$ is used. To start the induction, we notice that for $k = 0$ the statement clearly holds. Since (Y_{k+1}, ξ_{k+1}) is given by contact $(+1)$–surgery along a Legendrian knot in (Y_k, ξ_k), for the corresponding cobordism X_k we have

$$F_{-X_k}(c(Y_k, \xi_k)) = c(Y_{k+1}, \xi_{k+1}).$$

Therefore induction together with the injectivity of F_{-X_k} implies the result. This latter claim of injectivity can be proved by applying the surgery exact triangle for the cobordism $-X_k$.

For the sake of simplicity we sketch the proof of this last step for the case when K is the right-handed trefoil knot depicted by Figure 8 (the general case follows a very similar pattern). In this case the surgery triangle induced by the handle attachment along the $(k+1)$st Legendrian trefoil is given by Figure 9. If V_k denotes the cobordism from Y

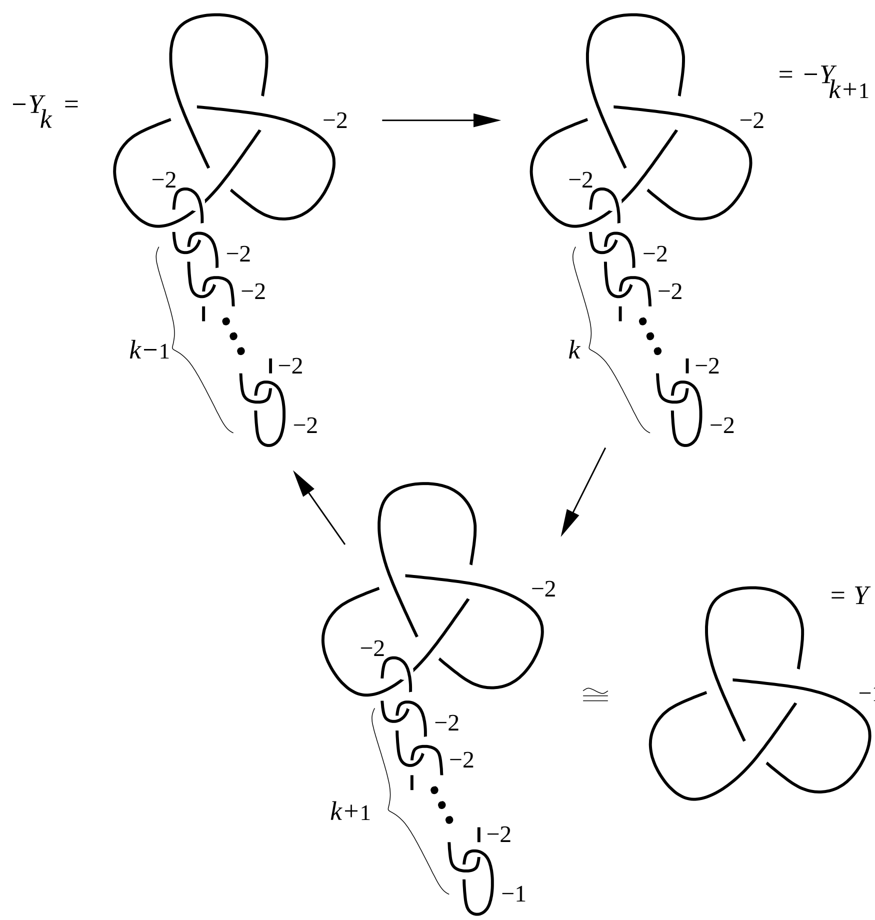

FIGURE 9. The exact triangel induced by $-X_k$

to $-Y_k$, then a simple geometric arguments shows

PROPOSITION 6.3. *The 4–manifold V_k contains a torus of self–intersection $k+1$.* ☐

Now the proof of Theorem 6.1 can be easily completed for the special case when $K = T$: By the adjunction formula for cobordisms, Proposition 6.3 implies that the induced homomorphism F_{V_k} is trivial, hence the injectivity of F_{-X_k} follows from the exactness of the triangle. This concludes the proof of Theorem 6.1. $\square$

EXERCISE 6.4. Find a genus–$g_s(L)$ surface $\Sigma \subset V_0$ with $[\Sigma]^2 = \mathrm{tb}(L)$ for general L satisfying $\mathrm{tb}(L) = 2g_s(L) - 1$. Let t denote $\mathrm{tb}(L)$. For $k > 0$ find a surface of genus $\frac{1}{2}(t(t-1)k + t + 1)$ with self–intersection $t^2k + t$ in V_k. (For the solution see [**20**].)

Notice that the above result deals only with surgeries satisfying $r \neq TB(K)$. This assumption plays an important role in defining the candidate tight contact structure; for $r = TB(K)$ the previous strategy would provide an overtwisted structure. It seems to be a more subtle question to understand what happens on the 3–manifold $S^3_{TB(K)}(K)$. Below we give a partial answer to this question, concentrating on some particular families of torus knots.

A sample computation. Using convex surface theory it can be proved that the oriented 3–manifolds $S^3_{2n-1}(T_{(2,2n+1)})$ admit no positive tight contact structures once $n \geq 1$ [**21**]. It is natural to ask what happens with other knots for which Theorem 6.1 applies, when we perform the *critical* surgery with coefficient $TB(K)$. Below we show that in one sample case the corresponding 3–manifold admits a positive tight contact structure. This computation generalizes to a wider family of knots, see [**21**].

PROPOSITION 6.5. *The 3–manifold $Y = S^3_5(T_{(3,4)})$ admits a tight contact structure.*

The proof will obviously follow from

THEOREM 6.6. *The contact structure ξ defined by the contact surgery diagram of Figure 10 on Y is tight.*

Proof. Consider the contact structure ξ' given by the diagram of Figure 10 after deleting one of the $(+1)$–framed surgery curves. The underlying 3–manifold will be denoted by S. It is easy to see that $c(S, \xi') \neq 0$, since, according to the diagram, ξ' is given by a sequence of (-1)–surgeries on the tight $S^1 \times S^2$. Let X denote the cobordism defined by the second contact $(+1)$–surgery. Our aim is to show that $c(S, \xi')$ is not in $\ker F_{-X}$, i.e.,

$$c(Y, \xi) = F_{-X}(c(S, \xi')) \neq 0.$$

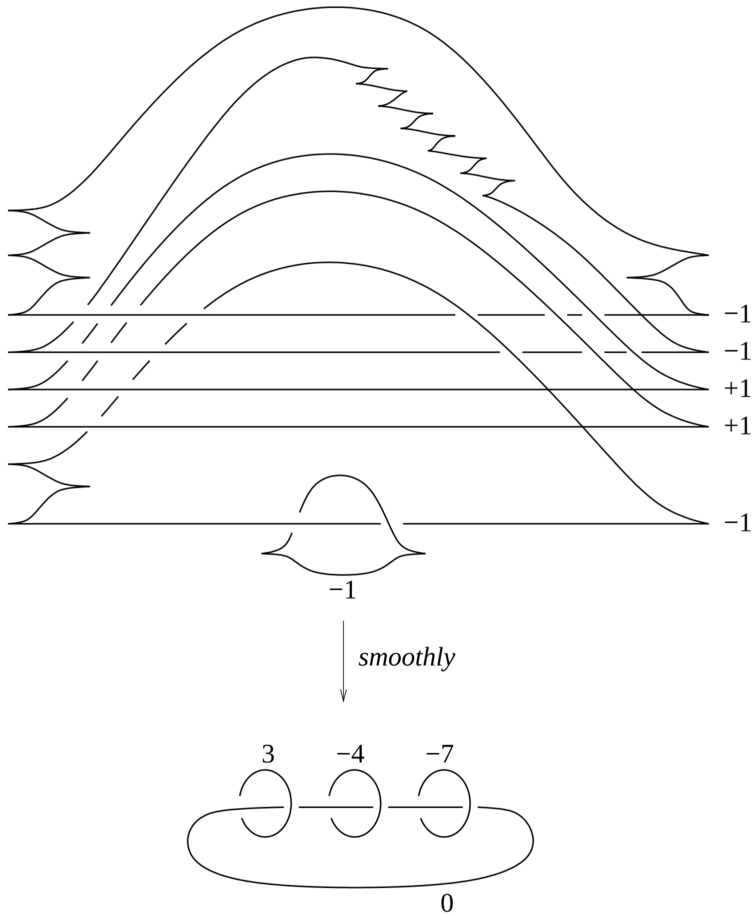

FIGURE 10. Tight contact structure on $Y = M(-\frac{1}{2}, \frac{1}{4}, \frac{1}{7}) = S^3_5(T_{(3,4)})$

In order to analyze the map F_{-X}, consider the exact triangle defined by the cobordism $-X$. Here we will follow the convention of denoting the 3–manifolds by solid surgery curves, while the cobordisms are denoted by dashed curves.

It is not hard to verify that the 3–manifolds S, Y and L are all L–spaces, that is, $\dim \widehat{HF}(S) = |H_1(S;\mathbb{Z})| = 89$, $\dim \widehat{HF}(Y) = 5$ and $\dim \widehat{HF}(L;\mathbb{Z}) = 84$. In particular, for a given spin^c structure $\mathfrak{t}$ (on any of the above three 3–manifolds) the corresponding Ozsváth–Szabó homology group (with $\mathbb{Z}_2$–coefficients) admits a unique nontrivial element $a_{\mathfrak{t}}$, which we will denote by the spin^c structure itself. Let U denote the cobordism given by attaching a 2–handle along the dashed curve with framing 0 in Figure 11 and V the third cobordism of the same figure.

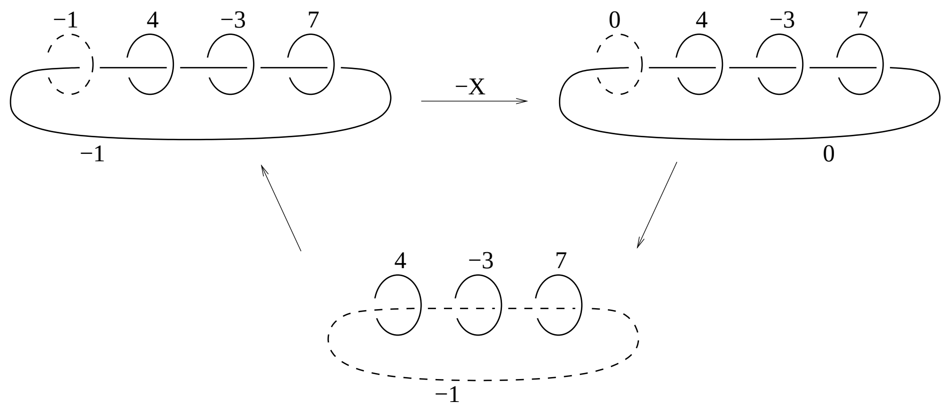

FIGURE 11. The exact triangel induced by $-X$

By exactness we get that $F_U = 0$, therefore F_V is injective and F_{-X} is surjective. The 3–manifold L is the connected sum of three lens spaces, more precisely $L = L(7,6)\#L(4,3)\#L(3,1)$. It admits two spin structures $\mathfrak{t}_1$ and $\mathfrak{t}_2$ and a simple geometric argument shows that exactly one of them (say $\mathfrak{t}_1$) extends to V as a spin structure and $\mathfrak{t}_2$ extends to U as a spin structure. The crucial step in the argument is the following observation:

PROPOSITION 6.7. *If* $\mathfrak{t}_e$ *denotes the unique spin structure on* Y *then for the gradings of the corresponding Ozsváth–Szabó homology elements we have*

$$gr(\mathfrak{t}_e) - gr(\mathfrak{t}_2) = \tfrac{1}{4}.$$

□

REMARK 6.8. There are several ways to prove this proposition. For example, we can directly compute the gradings of the two elements: For lens spaces the gradings are fairly easy to determine, for the small Seifert 3–manifold Y it is a little more complicated, but can be done using [**27**] or [**23**]. A less explicit, but possibly shorter argument considers the triangle induced by the cobordism $-U$ between L and $-Y$, and shows that the image of the spin structure $\mathfrak{t}_2$ has $\mathfrak{t}_e$ as nonzero component, from which it is a simple task to deduce the above proposition.

Since $\mathfrak{t}_2$ is self–conjugate under the $\mathbb{Z}_2$ action induced by conjugating the spin^c structures, its image $F_V(\mathfrak{t}_2)$ decomposes as $a + \mathfrak{J}a$ for some $a \in \widehat{HF}(-S)$. Injectivity of F_V implies that a and $\mathfrak{J}a$ are not in $\ker F_{-X}$. In a way similar to the proof of Proposition 6.7 it can be checked that $\langle F_{-X}(a), \mathfrak{t}_e\rangle = 1$, hence a has a homogeneous component

a_1 with the same property. By determining the spin^c structure of a_1 and comparing it to $\mathfrak{t}_{\xi'}$ we will conclude that the spin^c structures coincide, hence $a_1 = c(S, \xi')$, implying $F_{-X}(c(S, \xi')) \neq 0$. From Proposition 6.7 the determination of the spin^c structure of a_1 is a simple task: the degree shift between $\mathfrak{t}_2$ and a_1 is at most $\frac{1}{4}(-\frac{84k^2}{89}+1)$ and $k^2 > 0$ since $\mathfrak{t}_2$ does not extend as a spin structure to this cobordism. Similarly the degree shift between a_1 and $\mathfrak{t}_e$ is $\frac{1}{4}(-\frac{5l^2}{89}+1)$ and $l^2 > 0$ since there is no spin structure on the cobordism $-X$. Since $gr(\mathfrak{t}_e) - gr(\mathfrak{t}_2) = \frac{1}{4}$, it follows that $k^2 + l^2 = 2$, hence $k = \pm 1$, which specifies the spin^c structure of a_1 in terms of the spin structure $\mathfrak{t}_2$, which is easy to describe. Now a simple homotopy–theoretic computation verifies that $\mathfrak{t}_{a_1} = \mathfrak{t}_{\xi'}$ or $\mathfrak{t}_{a_1} = \mathfrak{J}\mathfrak{t}_{\xi'}$, concluding the proof. For details of the steps only sketched above the reader is advised to turn to [**21**]. □

Fillability. As it turns out, many of the tight contact structures found by Theorem 6.1 are nonfillable. In fact

Theorem 6.9. *The manifold $S^3_r(T_{(p,pn+1)})$ with $p, n \in \mathbb{N}$ and $r \in [p^2n - pn - 1, p^2n + p - 1)$ supports no fillable contact structure.*

Proof. The proof proceeds roughly as follows. First, using the surgery exact triangle, the fact that $(pq-1)$–surgery on $T_{(p,q)}$ is a lens space, and the adjunction formula for cobordisms, one can show that the 3–manifolds encountered above are all L–spaces. According to [**30**] a weak symplectic filling of an L–space has vanishing b_2^+ invariant. Kirby calculus and some elementary algebra shows that the manifolds encountered above can be given as boundaries of positive definite 4–manifolds with intersection forms which do not embed into any diagonal definite lattice. By reversing the orientation of these 4–manifolds and gluing them to potential fillings we end up with a closed, negative definite 4–manifold with nonstandard intersection form, contradicting Donaldson's famous diagonalizability theorem. This shows that no filling of the above manifolds can exist. □

By analyzing the freedom of putting stabilizations on the Legendrian knots C_i we get

Corollary 6.10. *For any $n \in \mathbb{N}$ there is a rational homology sphere M_n which carries at least n pairwise nonisomorphic tight contact structures, none of them fillable.* □

We conclude this chapter by showing examples where the manifold carries both fillable and nonfillable tight contact structures. To this

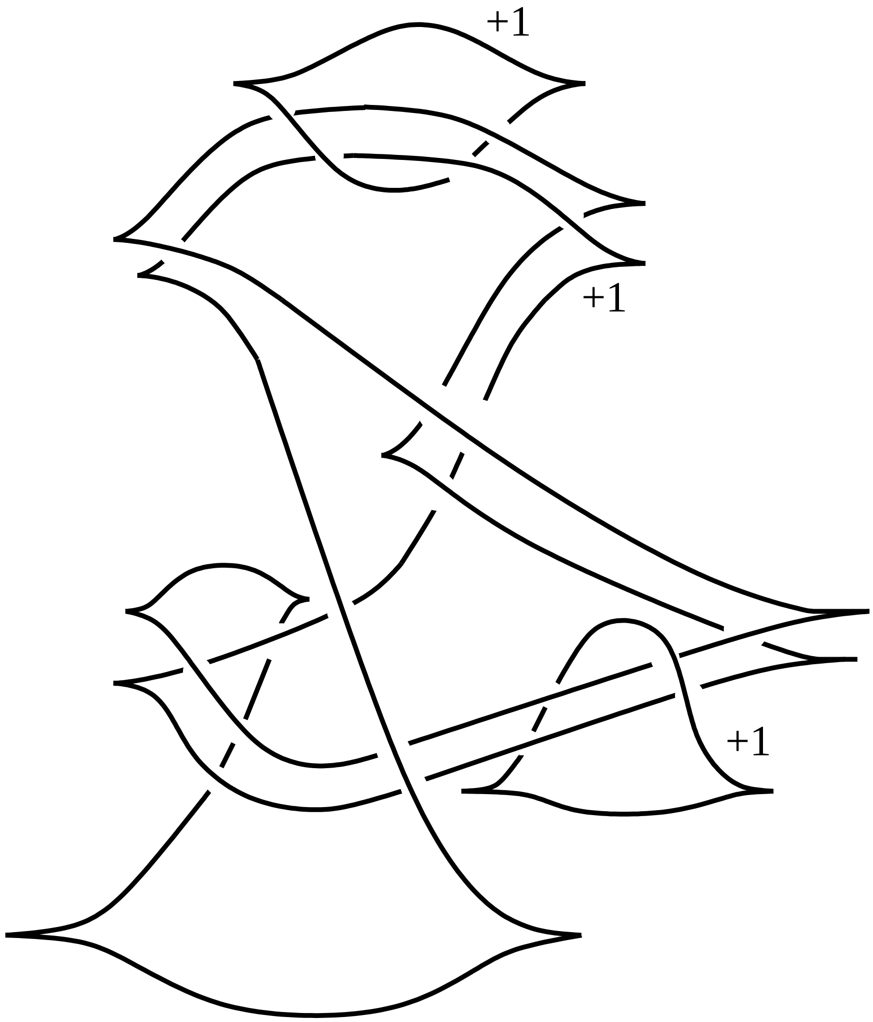

FIGURE 12. Nonfillable tight contact circle bundle

end, notice that by Theorem 6.1 contact (+1)–surgery on the Legendrian trefoil knot with Thruston–Bennequin invariant 1 and rotation number 0 provides a tight, nonfillable contact 3–manifold. Choosing a particular such knot (as is given by the Legendrian trefoil of Figure 12) and doing two contact (+1)–surgeries along the Legendrian unknots we get a contact structure which is still not fillable. Its tightness can be proved by viewing it as contact (+1)–surgery on the tight contact $S^1 \times S^2 \# S^1 \times S^2$. The analysis of the induced map shows its injectivity as before, hence the contact Ozsváth–Szabó invariant of the contact structure of Figure 12 is nonzero, conluding the proof of tightness.

It is not hard to see that the above contact structure is defined on the circle bundle over the torus with Euler number equal to 2. Stein fillable structures on this 3–manifold were given by Gompf in [**14**]. From the above example appropriate contact (-1)–surgeries provide a family of tight, nonfillable structures on many Seifert fibered 3–manifolds over T^2. A small modification of the argument (by starting with the connected sum of n copies of the trefoil) extends to Seifert fibered 3–manifolds over higher genus surfaces.

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Rényi Institute, Budapest and Institute for Advanced Study, Princeton

E-mail address: `stipsicz@math-inst.hu` and `stipsicz@math.ias.edu`