

A COMPACTIFICATION OF THE SPACE OF MAPS FROM CURVES

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ABSTRACT. We construct a new compactification of the moduli space of maps from pointed nonsingular complex projective stable curves to a nonsingular complex projective variety with prescribed ramification indices at the points. It is shown to be a proper DM-stack equipped with a natural virtual fundamental class.

1. INTRODUCTION

1.1. Let $\mu = (\mu_1, \dots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$. In this paper we construct a new compactification $\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$ of the space of all maps f from genus g , n -pointed nonsingular complex projective curves (C, p_1, \dots, p_n) to a nonsingular complex projective variety X representing class $\beta \in H_2(X, \mathbb{Z})$ such that:

- f is unramified everywhere possibly except at p_1, \dots, p_n where the ramification indices of f are μ_1, \dots, μ_n respectively.
- $f(p_i)$, $i = 1, \dots, n$, are pairwise distinct.

The boundary consists of suitable maps, still keeping the conditions in a sense, from n -pointed prestable genus g curves to a Fulton-MacPherson degeneration space of X . Here by a Fulton-MacPherson degeneration space we mean a fiber of the “universal” family $X[m]^+ \rightarrow X[m]$ of the Fulton-MacPherson configuration space of m distinct labeled points in X , for some integer m (Definition 2.1.1). The elements in the moduli space are called stable ramified maps (Definition 3.1.1). They can be considered as stable log-unramified maps. The stable ramified maps do not have any domain component contracted under the maps. We show that the moduli space $\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$ is a proper Deligne-Mumford stack over \mathbb{C} , carrying a natural virtual fundamental class (Theorem 4.2.4 and Section 5). Thus, we define ramified Gromov-Witten invariants (Section 5). The construction given here can be generalized to for example, the relative case, that is, one can combine our construction with J. Li’s construction of the moduli space of stable relative maps. The details will appear elsewhere (In [KS], a generalization of Fulton-MacPherson configuration spaces is given, in particular). The compactification (including its generalization to the relative case) incorporates the four known compactifications in the algebro-geometric category: Fulton-MacPherson’s compactification of the configuration space; Kontsevich’s stable map compactifications; Harris-Mumford’s compactification of

Hurwitz scheme, and Li's compactification of stable relative maps. All of the four will play key roles in our compactification.

This paper is motivated by the third author's talks on a joint work [FO] with Fukaya (the talks were given, for example, at KIAS in 2004 and at Fields Institute in 2005). Within the realm of symplectic geometry, they treated a compactification of the moduli space of maps from pointed nonsingular rational curves to \mathbb{P}^r with prescribed ramification indices at markings.

The main advantage of the algebro-geometric method exploited in this paper is the systematic use of the wonderful property of the "universal" families of Fulton-MacPherson spaces [FM, L.Li], the notion of (log) admissible maps [HM, Mo, We, J.Li], the deformation theory for such maps [J.Li], as well as the now-standard tools in Gromov-Witten theory in the algebro-geometric category [BF, FP, Kon, Kr, LT].

1.2. In relation to Morse homotopy, Chern-Simons theory and the open string theory on the cotangent bundle, Fukaya and Oh [FOh1] studied degeneration of disc moduli space that defines the A_∞ -category of Lagrangian graphs to the moduli space of *graph flows* in the Morse homotopy theory, and proved that the two moduli spaces are diffeomorphic in the semi-classical limit. One crucial reason why we get this diffeomorphism is that there do not exist disc (or sphere) bubbles at all and so there is no quantum contribution in that case. Then they proposed to study similar degeneration problems [Fu], [Oh1] in the presence of disc or sphere bubbles by refining the methods used in [FOh1]. In the presence of bubbles, they expected that the limit configurations will be the union of holomorphic discs and spheres connected by gradient trajectories of given Morse functions [Fu, Oh1]. More recently, this kind of convergence result has some interesting application to the geometry of Hamiltonian groups [Oh2] and the details of the convergence proof, at least in the Hausdorff topology, have indeed been written down by Oh in the paper [Oh3].

Besides the analysis of [FOh1], the essential analysis needed for the study of degeneration beyond the Hausdorff topology in the presence of quantum contribution lies in the area near the joins of the "thick" parts (i.e., the disc or spheres) and the "thin" parts (i.e., the part of gradient trajectories). It turns out that the study of this problem is a central problem in understanding further regularity property of the stable map type moduli spaces beyond the construction of fundamental cycles. One should note that although the stable map moduli space in general has a fundamental cycle, the topology of the moduli space of stable maps in the existing literature [FOn], [LT] has little regularity near the boundary of strata. This is partly because stable map compactification regularizes the *domain* by adding ghost bubble components every time more than two irreducible components merge at point in the image, but does not require any regularity for the image of the maps at the intersections other than the Hausdorff continuity (see [FOn] for the explicit description of the topology). This leads to loss of information of the

higher order behavior of the *maps* in the limit which is crucial to understand the kind of degeneration needed in the problem mentioned above or [FOh1].

This led Fukaya and Oh to study a finer compactification of holomorphic maps in general which will contain higher order information at the intersection of several components in the limit and which remains to be Hausdorff. In [FOh2], Fukaya and Oh have described various geometric properties of this enhanced compactification. Since the necessary analysis needed to construct the compactification and the Kuranishi structure on it in the sense of [FOn] has not been developed in the context of *pseudo-holomorphic* curves in the existing literature, this analysis itself has to be established. The analysis needed to establish the gluing problem and the transversality problem turns out to require the kind of analysis that is needed to construct the symplectic field theory [EGH] in the context of *asymptotically cylindrical ends*, not just in the context of *cylindrical ends* as in [EGH]. This is a part of the contents of Fukaya and Oh's paper [FOn2] in preparation.

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2. STACKS OF FULTON-MACPHERSON DEGENERATION SPACES

2.1. FM degeneration spaces. Let X be a nonsingular complex algebraic variety with dimension r . Denote by $X[n]$ the Fulton-MacPherson configuration space of n distinct labeled points in X . We refer to the paper [FM] for constructions and basic properties of the configuration space. The space $X[n]$ has the “universal family”

$$\pi_{X[n]} : X[n]^+ \rightarrow X[n]$$

with disjoint sections

$$\sigma_i : X[n] \rightarrow X[n]^+$$

for $i = 1, \dots, n$. In fact, the space $X[n]^+$ is an iterated blowup of $X[n] \times X$ so that there are two natural projections, $\pi_{X[n]}$ as mentioned above and

$$\pi_X : X[n]^+ \rightarrow X$$

from the second projection.

Definition 2.1.1. We call the data $(\pi_{\mathcal{W}/S} : \mathcal{W} \rightarrow S, \pi_{\mathcal{W}/X} : \mathcal{W} \rightarrow X)$ a Fulton-MacPherson degeneration space of X over a \mathbb{C} -scheme S (for short, an FM degeneration space) if:

- \mathcal{W} is an algebraic space over S (by $\pi_{\mathcal{W}/S}$), and $\pi_{\mathcal{W}/X}$ is a morphism over \mathbb{C} .
- there is an étale cover $T \rightarrow S$.

- for some integer $n > 0$, there is a fiber square

$$\begin{array}{ccc} \mathcal{W}|_T & \longrightarrow & X[n]^+ \\ \pi_{\mathcal{W}/S}|_T \downarrow & & \downarrow \pi_{X[n]} \\ T & \longrightarrow & X[n] \end{array}$$

satisfying that the restriction of $\pi_{\mathcal{W}/X}$ to $\mathcal{W}|_T$ coincides with the composite $\mathcal{W}|_T \rightarrow X[n]^+ \rightarrow X$.

Furthermore, when S is $\text{Spec}\mathbb{C}$, we will simply say that \mathcal{W} is a FM degeneration space of X . When n is specified, call \mathcal{W} a level- n FM degeneration space.

Notation: If $h : T \rightarrow S$ is a map of \mathbb{C} -schemes and $(\pi_{\mathcal{W}/S} : \mathcal{W} \rightarrow S, \pi_{\mathcal{W}/X} : \mathcal{W} \rightarrow X)$ is a FM degeneration space, denote by $h^*\mathcal{W}$ the fiber product $T \times_S \mathcal{W}$. Of course it is a FM degeneration space. E.g., if h is a map from a \mathbb{C} -scheme T to $X[n]$, then we have the FM degeneration space $h^*X[n]^+$.

Example 2.1.2. Let $S = \text{Spec}\mathbb{C}[[t]]$ and $g : S \rightarrow X^2$ be a map. Suppose that only the closed point p of S hits the diagonal Δ of X^2 under the map g . Hence there is the unique lift $\tilde{g} : S \rightarrow X[2] := \text{Bl}_\Delta X^2$. Let k be the intersection number of S and the exceptional divisor of $X[2]$, and let $q = g(p)$. Since $X[2]^+$ is the blowup of $X[2] \times X$ along the proper transform of the small diagonal $\Delta_{\{1,2,3\}}$ of X^3 , we see by direct computation that $\tilde{g}^*X[2]^+$ is isomorphic, as an FM degeneration space over S , to the pullback of the blowup $\text{Bl}_{(p,q)}S \times X$ of $S \times X$ under the base change

$$\begin{array}{c} S \rightarrow S \\ t \mapsto at^k + o(t^k) \end{array}$$

for some nonzero $a \in \mathbb{C}$.

One of goals of section 2 is to construct the Artin stack which parameterizes FM degeneration spaces of X . To do so, we will recall some elementary properties of (the constructions of) $X[n]$, $X[n]^+$, and $\pi_{X[n]}$ following [FM]. Using a variation of FM configuration spaces, we will express the stack as the stackification of a groupoid scheme. After that, it will be straightforward to see that the stack is algebraic by a proposition in [LM].

2.2. Basic properties of $\pi_{X[n]} : X[n]^+ \rightarrow X[n]$. In [FM], the space $X[n]$ is constructed by an iterated blowup of X^n along nonsingular subvarieties Δ_I for all subset $I \subset N := \{1, 2, \dots, n\}$ with $|I| \geq 2$, where Δ_I denotes the proper transform of the diagonal

$$\{\mathbf{x} \in X^n \mid x_i = x_j, \forall i, j \in I\}.$$

The blowup order is suitably taken. For convenience, we denote also by Δ_I the proper transform of the exceptional divisor of the center Δ_I . This notation causes no confusion if it is clear to which space Δ_I belongs.

Notation: Let V_1 be a blowup of a nonsingular complex variety V_0 along a nonsingular closed subvariety Z . If Y is an irreducible subvariety of V_0 , we will use the same notation Y to denote

- the total transform of Y , if $Y \subset Z$;
- the proper transform of Y , otherwise.

The “universal” family $X[n]^+$ is constructed by an iterated blowup

$$X[n]^+ := Y_{n-1} \rightarrow Y_{n-2} \rightarrow \cdots \rightarrow Y_0 := X[n] \times X.$$

The intermediate space Y_{k+1} is the blowup of Y_k along all disjoint nonsingular subvarieties Δ_{I^+} such that $I \subset N$ and $|I| = n - k$, where $I^+ := I \cup \{n+1\}$.

2.3. A local description of $\pi_{X[n]}$. Note that in any stage Y_k , Δ_{I^+} is the transversal intersection $\Delta_I \cap \Delta_{a^+}$ if $I \subset N$, $|I| \leq n - k$ and $a \in I \neq \{a\}$, where $a^+ := \{a, n+1\}$. Also note that in each Y_k , Δ_{a^+} is transversal to every fiber of the natural projection $Y_k \rightarrow X[n]$. These observations provide a local description of the projection $\pi_{X[n]} : X[n]^+ \rightarrow X[n]$ as following. If p is not a smooth point of the projection $\pi_{X[n]}$, then with respect to suitable coordinates it induces a map on completed local rings of the form

$$\begin{aligned} \widehat{\mathcal{O}}_{\pi_{X[n]}(p)} &\cong \mathbb{C}[[t_1, \dots, t_{rn}]] \rightarrow \widehat{\mathcal{O}}_p \cong \mathbb{C}[[t_1, \dots, t_{rn}, z_1, \dots, z_{r+1}]] / (z_1 z_2 - t_1) \\ & t_i \mapsto t_i. \end{aligned}$$

Projective Tangent Bundle Map. Let W be a FM degeneration space of X over \mathbb{C} . If p is a smooth point of W , then by the local description of $\pi_{X[n]}$ we obtain a natural identification

$$\mathbb{P}T_p W \cong \mathbb{P}T_{\pi_{W/X}(p)} X$$

of the projectivizations of tangent spaces. This identification does not depend on the choices of the isomorphisms (fixing X) between W and fibers of the universal family $\pi_{X[n]} : X[n]^+ \rightarrow X[n]$.

Forgetting Markings. Since only the labeling of points matters, we use notation $X[M]$ for $X[m]$ if M is a set of order m . For example, often $X[N]$ will be used instead of $X[n]$, where $N = \{1, \dots, n\}$. There is another useful construction of $X[N]$ as following. The space $X[N]$ coincides with the closure of $X^n \setminus \cup_I \Delta_I$ in

$$X^n \times \prod_{I \subset N, |I| \geq 2} \text{Bl}_{\Delta_I} X^I.$$

Similarly, $X[N]^+$ is the closure of

$$X[N] \times X \setminus \bigcup_{I \subset N, |I| \geq 2} \Delta_{I^+}$$

in

$$(X[N] \times X) \times \prod_{I \subset N, |I| \geq 2} \text{Bl}_{\Delta_{I^+}} (X[N] \times X),$$

where these Δ_{I^+} are of codimension $n + 1$.

Let $N+1 = N \cup \{n+1\}$. The closure constructions show that by forgetting a point labeled by say, $n + 1$, there is a natural commuting diagram

$$\begin{array}{ccc} X[N+1]^+ & \xrightarrow{\pi_+} & X[N]^+ \\ \downarrow \pi_{X[N+1]} & & \downarrow \pi_{X[N]} \\ X[N+1] & \longrightarrow & X[N] \end{array}$$

and an induced map $\tilde{\pi}_+ : X[N+1]^+ \rightarrow X[N+1] \times_{X[N]} X[N]^+$.

Lemma 2.3.1. *The map $\tilde{\pi}_+$ is an isomorphism over the open locus of $X[N+1]$ where $\pi_+ \circ \sigma_{n+1}$ meets neither any σ_i , $i = 1, \dots, n$ nor the relative singular locus of $X[N]^+/X[N]$.*

Proof. Straightforward. \square

Remark 2.3.2. In general, if the forgetting σ_{n+1} makes a stable degeneration of X unstable (see pages 194 and 195 in [FM] for the definition of the stable degeneration), then the forgetting mapping contracts the nonstable component in the obvious way.

Operation. Consider a commuting diagram

$$\begin{array}{ccc} & & X[n+1] \\ & & \downarrow \\ & & X[n]^+ \\ S \nearrow^h & & \downarrow \\ & & X[n] \\ S \xrightarrow{g} & & \end{array}$$

such that the image of h meets neither the relative singular locus of $X[n]^+/X[n]$ nor sections $\sigma_i \circ g$, $i \in N$. Then there is a unique lift $\tilde{h} : S \rightarrow X[n+1]$ of h since $X[n+1]$ is the blowup of $X[n]^+$ along the images of sections σ_i . Note that S -scheme $g^*X[n]^+$ is canonically isomorphic to S -scheme $\tilde{h}^*X[n+1]^+$ preserving σ_i , $i = 1, \dots, n$, due to Lemma 2.3.1. The iterated version of the operation will be a key tool in section 4.

2.4. Trees of FM degenerations. Note that a FM degeneration space of $X := \mathbb{P}_{\mathbf{k}}^1$ (without markings) over $\mathbf{k} := \mathbb{C}$ is a connected genus 0 nodal curve with a distinguished component $\mathbb{P}_{\mathbf{k}}^1$. In that case there is a natural dual tree graph corresponding to the prestable curve. Likewise, for a FM degeneration space W of any nonsingular variety X , we associate a tree whose vertices (resp. edges) one-to-one correspond to components of W (resp. W^{sing}). The distinguished vertex corresponding to the component, a blowup of X , is called the *root* of the tree. Every tree of a FM degeneration has one root. For a given vertex, the number of edges of the minimal chain connecting the root and the vertex will be called the *level* of the corresponding component

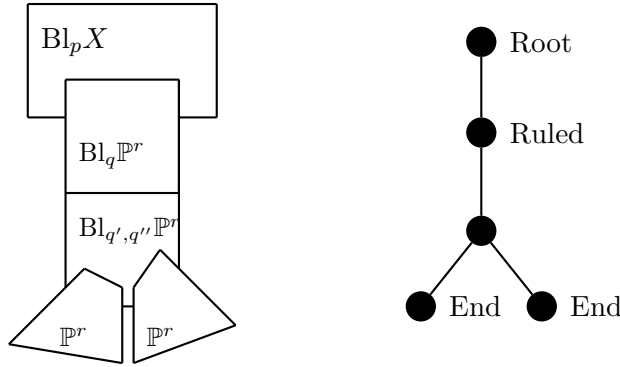


FIGURE 1. An FM degeneration space and its corresponding tree graph.

of W . Call a component of W an *end* if it is not the root component and the valance of its vertex is 1. A component of W is called *ruled* if its vertex has valance 2. The descendants of a component with associated vertex v are by definition components whose minimal associated chains contain the vertex v and its level is strictly larger than the level of v . For each non-root component Y of W , denote by $D_+(Y)$ the divisor of Y corresponding to the edge which connects the vertex of Y to the one level-down vertex. The $+$ sign in $D_+(Y)$ is explained by the fact that the intersection number of any curve on Y and the divisor is strictly positive unless the curve lies also on the one level-up component. Here the intersection number is taken in the nonsingular variety Y , not in the (maybe singular) variety W . Similarly define $D_-(Y)$ when Y is a ruled component. Non-root components of W will be also called *screens*.

Lemma 2.4.1. *Let $f_i : S \rightarrow X[n]$ be a morphism from a scheme S over \mathbb{C} . Suppose that there is an isomorphism between $f_i^* X[n]^+$, $i=1,2$, fixing X and preserving the n induced sections. Then $f_1 = f_2$.*

Proof. This can be seen by the universal property of Theorem 4 in [FM] and the natural identification $\mathcal{I}_{\Delta_J}|_x \cong ((T_x X)^J / T_x)^*$, where \mathcal{I}_{Δ_J} is the ideal sheaf of diagonal Δ_J of X^J and $x \in X = \Delta_J \subset X^J$, for $J \subset N$, $|J| \geq 2$. (See page 195 of [FM] for a friendly explanation of the universal property and its relation with screens.) Precisely speaking, étale locally, we can express the data $f^* \mathcal{I}_{\Delta_J} \rightarrow \mathcal{L}_J$ by sending $x_{i,j} - x_{i',j}$ to $\sigma_{i,j} - \sigma_{i',j}$ for $i, i' \in J \subset N$, $j = 1, \dots, r$ where $x_{i,j}$ are (the copy of) coordinates of the i -th component X of X^n ; and $\sigma_{i,j}$, $j = 1, \dots, r$ are fiberwise direction coordinates of $X[J]^+$, pulled back via the composite of the sections σ_i with the forgetting map

$$\tilde{\pi}_+ : X[N]^+ \rightarrow X[N] \times_{X[J]} X[J]^+.$$

We are thus reduced to showing that the given isomorphism induces an isomorphism between the $f_i^*(X[N] \times_{X[J]} X[J]^+)$. For this assertion, we

are immediately reduced to the case S is Noetherian, in which case it follows quickly from the claim that for any morphism $S \rightarrow X[N]$ we have $(1_S \times \tilde{\pi}_+)_* \mathcal{O}_{S \times_{X[N]} X[N]^+} \cong \mathcal{O}_{S \times_{X[J]} X[J]^+}$, and $R^p(1_S \times \tilde{\pi}_+)_* \mathcal{O}_{S \times_{X[N]} X[N]^+} = 0$ for $p > 0$. For the claim, it suffices to treat the case $J = \{1, \dots, n-1\}$, and using that $\tilde{\pi}_+$ is a proper morphism of flat Noetherian $X[N]$ -schemes, we are reduced to the case $S = \text{Spec} \mathbb{C}$ by a standard cohomology and base change argument as in [Mu] §II.5. Then $1_S \times \tilde{\pi}_+$ is (when not an isomorphism) a proper morphism of FM degeneration spaces contracting a ruled component or an end component. In each case the argument proceeds via the Theorem on Formal Functions: for the direct image, by a computation in local coordinates, and for the higher direct images, as in the proof of Theorem I.9.1(ii) in [BHPV]. \square

2.5. Automorphisms. Let W be a FM degeneration of X over $\text{Spec} \mathbf{k}$. It is a scheme over X . Define an automorphism of W/X to be an isomorphism ϕ of W fixing X , that is, $\pi_{W/X} \circ \phi = \pi_{W/X}$. Since there are no markings in W unlike an n -pointed stable degeneration [FM], note that the collection $\text{Aut}(W, X)$ of automorphisms becomes a nontrivial group if there is an end component. For example the group is $\mathbf{k}^r \rtimes \mathbf{k}^\times$ if the tree of W has only two vertices. Note that each end (resp. ruled) component Y is (non-canonically) isomorphic to $\mathbb{P}_{\mathbf{k}}^r$ (resp. the blowup of $\mathbb{P}_{\mathbf{k}}^r$ at a point) with a marked divisor $D_+(Y)$ (resp. $D_\pm(Y)$). The automorphism group of an end (resp. ruled) Y fixing $D_+(Y)$ (resp. $D_\pm(Y)$) is a subgroup, isomorphic to $\mathbf{k}^r \rtimes \mathbf{k}^\times$ (resp. \mathbf{k}^\times), of $\text{Aut}(W, X)$. So, for general (W, X) , we see that

$$\text{Aut}(W, X) \cong \left(\prod_{W_i : \text{ruled}} \mathbf{k}^\times \right) \times \left(\prod_{W_i : \text{ends}} \mathbf{k}^r \rtimes \mathbf{k}^\times \right)$$

In general, for a degeneration space \mathcal{W} of X over S , one may use relative Hilbert schemes (for example, see [Kol]) to conclude that $\text{Aut}(\mathcal{W}/S, X)$ is a group scheme over S with the base change property.

2.6. Spaces $X[n : m]$ and $X[n : m]^+$. In this subsection we define a compactification $X[n, m]$ of the configuration space of pairs of black colored n ordered points in X and red colored m ordered points in X . Let $N + M$ denote $\{1_b, \dots, n_b\} \cup \{1_r, \dots, m_r\}$, a collection of n “black numbers” and m “red numbers”. We will call the labels of N black and the labels of M red. Then we want compactify

$$\left(X^N \setminus \bigcup_{B \subset N : |B|=2} \Delta_B \right) \times \left(X^M \setminus \bigcup_{|R| \subset M : |R|=2} \Delta_R \right)$$

allowing red points to collide with black points but not allowing any two points with the same color to coincide. A “universal family” can be constructed by an iterated blowup of $X[n, m] \times X$ along smooth centers. The

centers will be proper transforms of suitable diagonals in the morphism $X[n, m] \times X \rightarrow X^n \times X^m \times X$. Those diagonals are

- Δ_{I^+} with $|I_{\text{black}}| |I_{\text{red}}| \geq 2$, $I_{\text{black}} := I \cap N$ and $I_{\text{red}} := I \cap M$;
- Δ_{B^+} with $|B| \geq 2$, $B \subset N$;
- Δ_{R^+} with $|R| \geq 2$, $R \subset M$,

where $A^+ = A \cup \{n + m + 1\}$.

A detail of the construction is as follows. First, let $X[1, 1] = X^2$. $X[n, m]$ and its “universal family” will be defined inductively. Start with $X[n, m] \times X$ and blow it up along Δ_{I^+} and then Δ_{B^+} and Δ_{R^+} . Any order of the blowups along centers of each type will give the same outcome as far as the blowup along $\Delta_{I_1^+}$ is taken before the blowup along $\Delta_{I_2^+}$ whenever $|I_1| > |I_2|$. Denote by $X[n, m]^+$ the result of the blowups. Then $\Delta_{i_b^+}$ and $\Delta_{i_r^+}$ provide sections σ_{i_b} and σ_{i_r} of $X[n, m]^+ \rightarrow X[n, m]$. Now define $X[n, m+1]$ to be the blowup of $X[n, m]^+$ along all $\Delta_{\{b,r\}^+}$ and then all Δ_{r^+} . We mark the last label as $(m+1)_r$. Similarly, define $X[n+1, m]$ to be the blowup of $X[n, m]^+$ along all $\Delta_{\{b,r\}^+}$ and then all Δ_{b^+} . We mark the last label as $(n+1)_b$. In fact, $X[n, m]$ is the closure of

$$X^{n+m} \setminus \bigcup_{\Delta} \Delta$$

in

$$X^{n+m} \times \prod_{\Delta} \text{Bl}_{\Delta} X^{n+m}$$

where Δ runs for all diagonals except the diagonals of type $\Delta_{b,r}$, $b \in N$ and $r \in M$. The justification of the claims above can be seen directly modifying the arguments in [FM] or L. Li’s general approach to the wonderful compactification [L.Li].

Define $X[n : m]$ to be the maximal open subset of $X[n, m]$ such that the restriction of the universal family $X[n, m]^+$ to the subset is still stable after forgetting red markings but keeping the black markings and vice versa (i.e., forgetting black markings but keeping the red markings). Here “stable” means by definition that every fiber has only the trivial automorphism fixing X and the remained marked points. Denote by $X[n : m]^+$ the restriction of the fibration $X[n, m]^+ \rightarrow X[n, m]$ to $X[n : m]$.

Lemma 2.6.1. *The image of a map $g : S \rightarrow X[n_1, n_2]$ is in $X[n_1 : n_2]$ if and only if $g^* X[n_1 : n_2]^+$ is isomorphic to both $g_i^* X[n_i]^+$ preserving n_i -sections and fixing X , for $i = 1, 2$, where g_i is the composite $S \xrightarrow{g} X[n_1, n_2] \rightarrow X[n_i]$.*

Proof. For $i = 1, 2$, there is the naturally induced map

$$h_i : X[n_1, n_2]^+ \rightarrow X[n_1, n_2] \times_{X[n_i]} X[n_i]^+$$

over $X[n_1, n_2]$. Then $X[n_1 : n_2]$ is the maximal open subset of $X[n_1, n_2]$ over which h_1 and h_2 are isomorphisms since the isomorphism condition is an open condition. \square

Corollary 2.6.2. *Consider $g_1 : S \rightarrow X[N_1]$ with extra sections $\sigma_{i_2}, i_2 \in N_2$, of $g_1^*X[N_1]^+$ such that these extra sections meet:*

- *neither each other*
- *nor the relative singular locus of $g_1^*X[N_1]^+/S$.*

Then it induces a unique map

$$g : S \rightarrow X[N_1, N_2]$$

such that canonically,

$$g^*X[N_1, N_2]^+ \cong g_1^*X[N_1]^+$$

preserving N_1 labeled sections and fixing X ; and the extra sections coincide the sections from the second label N_2 .

*Furthermore if each geometric fiber of $g^*X[N_1]$ is stable with respect to markings by $\sigma_{i_2}, i_2 \in N_2$, then canonically*

$$g^*X[N_1, N_2]^+ \cong g_2^*X[N_2]^+$$

preserving N_2 labeled sections and fixing X , where g_2 is the natural composite $S \xrightarrow{g} X[N_1, N_2] \rightarrow X[N_2]$.

2.7. Definition of $\mathfrak{X}[n]$. We define an Artin stack $\mathfrak{X}[n]$ which parameterizes level- n FM degeneration spaces as following. Define $\mathfrak{X}[n]$ as the stackification

$$[X[n : n] \rightrightarrows X[n]]$$

of the prestack associated to a groupoid scheme $X[n : n] \rightrightarrows X[n]$. The groupoid scheme is equipped with obvious maps s and t , diagonal $e : X[n] \rightarrow X[n : n]$, the obvious ‘‘composite’’ $m : X[n : n] \times_s X[n : n] \rightarrow X[n : n]$, and the exchange map $i : X[n : n] \rightarrow X[n : n]$. The stack is algebraic and smooth by Proposition 4.3.1 in [LM]. The conditions in the proposition can be easily checked. The conditions are that s and t are smooth and the relative diagonal $(s, t) : X[n : n] \rightarrow X[n] \times X[n]$ is separated and quasi-compact.

3. THE STACK OF STABLE RAMIFIED MAPS

3.1. Stable ramified maps. We introduce a generalized notion of stable maps. Recall the basic definitions/properties of stable maps (for example see [FP]). From now on, denote by X a nonsingular projective variety over \mathbb{C} . Let $NE_1(X) \subset H_2(X, \mathbb{Z})/\text{torsion}$ denote the semigroup of effective curve classes. Given $\beta \in NE_1(X)$, $g, n \in \mathbb{Z}_{\geq 0}$, and $\mu = (\mu_1, \dots, \mu_n)$, $\mu_i \in \mathbb{Z}_{\geq 1}$, we define:

Definition 3.1.1. *The data*

$$(C, \{p_1, \dots, p_n\}, \pi_{W/X} : W \rightarrow X, f : C \rightarrow W)$$

is called a stable map with μ -ramification from an n -pointed, genus g curve to an FM degeneration space W of X , representing class β (for short, a (g, β, μ) stable ramified map) if:

- (C, p_1, \dots, p_n) is a genus g prestable curve equipped with n ordered smooth points p_1, \dots, p_n .
- $\pi_{W/X} : W \rightarrow X$ is a FM degeneration of X over \mathbb{C} .
- The pushforward $(\pi_X \circ f)_*[C]$ of the fundamental class $[C]$ is β .
- The following four conditions are satisfied:

(1) Prescribed Ramification Index Condition:

- $C^{\text{sm}} = f^{-1}(W^{\text{sm}})$.
- $f|_{C^{\text{sm}}}$ is unramified everywhere possibly except at p_i .
- At p_i the ramification index

$$\text{length}(\mathfrak{m}_{p_i}/\mathfrak{m}_{f(p_i)}\mathcal{O}_{p_i}) + 1$$

of f is exactly μ_i .

(2) Distinct Points Condition: $f(p_i)$, $i = 1, \dots, n$ are pairwise distinct, smooth points of W .

(3) Admissibility Condition: At every nodal point p of C there is an identification

$$\hat{f}^* : \hat{\mathcal{O}}_{f(p)} \cong \mathbb{C}[[z_1, \dots, z_{r+1}]]/(z_1 z_2) \rightarrow \hat{\mathcal{O}}_p \cong \mathbb{C}[[x, y]]/(xy)$$

such that $\hat{f}^* z_1 = x^m$, $\hat{f}^* z_2 = y^m$ for some positive integer m .

(4) Stability Condition:

- For each ruled component W_r of W there is either an image of a marking in W_r (that is, $f(p_i) \in W_r$ for some i) or a nonlinear image $f(D) \subset W_r$ of an irreducible component D of C .
- For each end component $W_e \cong \mathbb{P}^r$ of W , there are either images of two distinct markings in W_e or a nonlinear image $f(D) \subset W_e$ of an irreducible component D of C .

Lemma 3.1.2. *Let $f : C \rightarrow W$ be a (g, β, μ) stable ramified map. Then, for each non-root component W_i , either, there are two distinct markings p_i and p_j such that their images $f(p_i)$ and $f(p_j)$ lie on a descendant of W_i or W_i itself, or the intersection number*

$$f_*([f^{-1}(W_i)]) \cdot D_+(W_i)$$

is greater than or equal to 2, where the intersection number is taken on the smooth component W_i .

Proof. If W_i does not contain two distinct markings, then we consider one of the ends in the descendants or W_i itself. It must contain a nonlinear image. \square

Lemma 3.1.3. *Let W be a target of a (g, β, μ) stable ramified map $f : C \rightarrow W$. Then the number of components of W is bounded above by an integer depending only on X , β , g and n . So is the number of components of C .*

Proof. First assume that $W = X$ is the projective line \mathbb{P}^1 , C is a nodal curve not necessarily connected, and $f : C \rightarrow W$ is a degree d arbitrary map, not contracting any irreducible component. Then the number of all

nodal points of C + the number of all ramification points of f + the number of all marked points is less than or equal to $d^2 + (g-1)d + n$. Due to Lemma 3.1.2, the images of those points under f are all possible points where the next level screens can be attached.

Now we come back to the general case that W is a FM-degeneration space of a nonsingular projective variety X . Choose an embedding $X \hookrightarrow \mathbb{P}^N$. Let W_k be the set of all level- k screens and C_k be the preimage of W_k under f . By considering a generic projection to \mathbb{P}^1 such that the composite $W \rightarrow X \dashrightarrow \mathbb{P}^1$ does not contract any component of C_0 , we see that $|W_1| \leq d^2 + (g-1)d + n \leq (d+g+n)^2$, where d is the degree of β . Note that the total degree of C in *each* level-one screen (as in \mathbb{P}^r after blowdown) is less than or equal to d . By the Admissibility Condition and the Stability Condition, the total degree + the number of marked points of C in *each* level-two screen is less than or equal to $d + n - 1$. By the same reasoning, the total degree + the number of marked points of C in a level- k screen is less than or equal to $d + n - k + 1$. Therefore the level of any screen can be at most $d + n$. This fact combined with the bound $|W_k| \leq (d + g + n)^2 \prod_{0 \leq i \leq k-2} (d + g + n - i)^2$ leads to a bound on the total number of components of W . Each component supports the image of at most d components of C , hence this bound times d is a bound on the number of components of C . \square

3.2. The families of stable ramified maps. Given $\beta \in NE_1(X)$, $g, n \in \mathbb{Z}_{\geq 0}$, and $\mu = (\mu_1, \dots, \mu_n)$ where $\mu_i \in \mathbb{Z}_{\geq 1}$, we define a family version of stable ramified maps.

Definition 3.2.1. *The data*

$$(\pi : \mathcal{C} \rightarrow S, \{p_1, \dots, p_n\}, \pi_{\mathcal{W}/S} : \mathcal{W} \rightarrow S, \pi_{\mathcal{W}/X} : \mathcal{W} \rightarrow X, f : \mathcal{C} \rightarrow \mathcal{W})$$

is called an S -family of stable maps with μ -ramification from n -pointed, genus g curves to an FM degeneration space \mathcal{W} of X , representing class β if:

- $(\pi : \mathcal{C} \rightarrow S, \{p_1, \dots, p_n\})$ is a family of genus g prestable curves over S , equipped with n ordered sections p_i .
- $(\pi_{\mathcal{W}/S} : \mathcal{W} \rightarrow S, \pi_{\mathcal{W}/X} : \mathcal{W} \rightarrow X)$ is a FM degeneration of X over S .
- The data forms a commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{f} & \mathcal{W} & \xrightarrow{\pi_X} & X \\ & \searrow \pi & \swarrow \pi_{\mathcal{W}/S} & & \\ & & S & & \end{array}$$

such that over each geometric point of S , it is a (g, β, μ) stable ramified map.

- **Admissibility Condition:** For any point of $t \in S$, if p is a nodal point of C_t , then \hat{f}^* can be expressed as:

$$\widehat{\mathcal{O}}_{f(p)} \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[z_1, \dots, z_{r+1}]]/(z_1 z_2 - s) \xrightarrow{\hat{f}^*} \widehat{\mathcal{O}}_p \cong \widehat{\mathcal{O}}_{\pi_S(p)}[[x, y]]/(xy - s')$$

with $\hat{f}^* z_1 = \alpha_1 x^m$, $\hat{f}^* z_2 = \alpha_2 y^m$, no restriction on z_i for $i > 2$, α_i being units in $\widehat{\mathcal{O}}_{\pi_S(p)}$, $s, s' \in \widehat{\mathcal{O}}_{\pi_S(p)}$, and m being a positive integer.

Remark 3.2.2. In Definition 3.1.1, conditions (2) and (4) are open conditions. Admissibility Condition, which is called Predeformability Condition in [J.Li], was introduced and studied by J. Li in his construction of stable relative map and relative Gromov-Witten invariants. When the target is one-dimensional, the notion of admissibility was introduced in [HM] and well studied in [Mo] by log structures. As explained at §3.8 in [Mo], we may let $\alpha_1 = \alpha_2 = 1$.

From now on we use the abbreviation of the imposed condition on stable ramified maps. For example, PRIC stands for Prescribed Ramification Index Condition.

Lemma 3.2.3. (Tangent Line Map Condition) *Let f be a family of stable ramified maps as in Definition 3.2.1. PRIC and AC together imply that there is a natural extension*

$$\mathbb{P}(Tf) : \mathcal{C} \rightarrow \mathbb{P}(TX)$$

of the projectivization of the induced map between tangent bundles $T(\mathcal{C}/S)^{\text{sm}} \rightarrow TX$.

Proof. This is a local property. PRIC (resp. AC) will imply that $\mathbb{P}(Tf)$ is well-defined at smooth points (resp. at singular points) of $\pi : \mathcal{C} \rightarrow S$. First, at a smooth point p of \mathcal{C}/S , let x be a S -relative uniformizing parameter. Since the image of p is a smooth point q , we have also t_1, \dots, t_r , S -relative uniformizing parameter at q . The ideal $(f^* dt_1, \dots, f^* dt_r)$ of the stalk of $\Omega_{\mathcal{C}/S}$ at p (here d is S -relative) is generated by $x^m dx$ if m is the ramification index of f at p . Hence

$$\left[\frac{f^* dt_1}{x^m dx}, \dots, \frac{f^* dt_r}{x^m dx} \right]$$

is regular at p since there is no jump of the ramification indices at p . In other words,

$$f^* \Omega_{\mathcal{W}} \rightarrow \Omega_{\mathcal{C}/S}(-\sum m_i p_i)$$

is surjective on the smooth locus of \mathcal{C}/S .

At a singular point of \mathcal{C}/S , the map f satisfies AC:

$$R[[z_1, \dots, z_{r+1}]]/(z_1 z_2 - s) \rightarrow R[[x, y]]/(xy - s')$$

sending $z_1 \mapsto \alpha x^m$, $z_2 \mapsto \beta y^m$, where $R = \widehat{\mathcal{O}}_{\pi(p)}$. Using the local description of $\pi_{X[n]}$ given in section 2.3, we can find the description of f as

$$x \mapsto \alpha(x^m, x^m z_3, \dots, x^m z_{r+1})$$

where the latter is given in the “coordinate” system $(z_1, z_1 z_3, \dots, z_1 z_{r+1})$ of X . Hence the derivative $\mathbb{P}Tf$ of the map is

$$[m, mz_3 + x \frac{\partial}{\partial x} z_3, \dots, mz_{r+1} + x \frac{\partial}{\partial x} z_{r+1}].$$

Hence the rational map $\mathbb{P}Tf : \mathcal{C} \dashrightarrow \mathbb{P}TX$ is well-defined also at the nodal point since $x \frac{\partial}{\partial x} = -y \frac{\partial}{\partial y}$ from $xdy + ydx = 0$. \square

3.3. Construction of $\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$. Define a category

$$\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$$

of μ -ramified admissible stable maps to X : It is a CFG over the étale site (Sch/\mathbb{C}) . A morphism is a commutative diagram

$$\begin{array}{ccccc} \mathcal{C}' & \xrightarrow{\quad} & \mathcal{W}' & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow = \\ & & S' & & X \\ & \swarrow & \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\quad} & \mathcal{W} & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & & \end{array}$$

where two side squares with the edge $S' \rightarrow S$ are fiber products. It is straightforward to see that this CFG is a stack.

Definition 3.3.1. An S -family of stable μ -ramified maps to a fixed target $X[l]^+/X[l]$ from n -pointed genus g curves is a stable map $(\mathcal{C} \rightarrow S, \{p_1, \dots, p_n\}, f : \mathcal{C} \rightarrow X[l]^+)$ with a commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & X[l]^+ \\ \downarrow & & \downarrow \\ S & \longrightarrow & X[l] \end{array}$$

satisfying properties:

- f is a fiberwise map.
- $f^{-1}((X[l]^+/X[l])^{\text{sing}}) = (\mathcal{C}/S)^{\text{sing}}$.
- *PRIC*.
- *DPC*.
- *AC*.
- *SC*.

Let β be a curve class in $H_2(X, \mathbb{Z})$ and denote by also β , the induced class in $H_2(X[l]^+, \mathbb{Z})$ using any canonical inclusion $X \subset X[l]^+$ as a general fiber of $X[l]^+ \rightarrow X[l]$.

Proposition 3.3.2. *The stack $\overline{\mathcal{M}}_{g,n}(X[l]^+/X[l], \beta)^\mu$ of stable μ -ramified maps to a fixed target $X[l]^+/X[l]$ from n -pointed genus g curves, representing class β , is a DM-stack over (Sch/\mathbb{C}) .*

Proof. Since the moduli of stable maps to $X[l]^+$ is a DM-stack, it is enough to show that: Given a family f of stable maps over S , representing class β , there is a subscheme Z of S such that: for any $T \rightarrow S$ such that $f|_T$ is a T -family of stable ramified maps (with fixed target $X[l]^+/X[l]$ if and only if $T \rightarrow S$ factors through $Z \rightarrow S$).

First, take the maximal open locus S_1 of S where f does not send any component of any geometric fiber of \mathcal{C} to the relative singular locus of $X[l]^+/X[l]$; and furthermore DPC and NC are satisfied. Then there is a natural closed subscheme S_2 of S_1 representing a functor of admissible stable maps. This can be shown by the proof of Theorem 2.11 in [J.Li] (or one may use §3C in [Mo]). Now take the maximal open locus S_3 of S_2 where $f^*\Omega_{X[l]^+/X[l]}^\dagger \rightarrow \Omega_{\mathcal{C}_{S_3/S_3}}^\dagger$ is surjective, where $\Omega_{X[l]^+/X[l]}^\dagger$ and $\Omega_{\mathcal{C}_{S_3/S_3}}^\dagger$ are the sheaves of relative log differentials induced from the log structures of the boundary divisors. The boundary of $X[l]^+$ consists of divisors Δ_T where $T \subset \{1, 2, \dots, l+1\}$, $|T| \geq 2$, $T \neq i^+$ ($i = 1, \dots, l$); the boundary of $X[l]$ consists of Δ_T , $T \subset \{1, 2, \dots, l\}$, $|T| \geq 2$; and the log structure of \mathcal{C}_{S_3} (resp. S_3) are given by pulling back of the canonical log structure of the moduli of $n+1$ -pointed (resp. n -pointed) prestable curves ([Mo]). In order to take care of PRIC, introduce relative uniformizing parameters z_j , $j = 1, \dots, r$ of $\mathcal{O}_{f(p_i(t))}$ at $f(p_i(t))$ where t is a point of S_3 ; and let z be a relative parameter of $\mathcal{O}_{p_i(t)}$ at $p_i(t)$. Then, take the closed subscheme S_4 of S_3 by equations $a_0 = 0, \dots, a_{\mu_i} = 0$, for all i, j , where $f^*(z_j) = a_{0,j} + a_{1,j}z + \dots \in \mathcal{O}_{p_i}$, $a_{k,j} \in \mathcal{O}_t$. At last, take the maximal open subscheme $Z := S_5$ where restricted to every geometric fiber \mathcal{C}_t , f has the ramification order exactly μ_i at $p_i(t)$. \square

Corollary 3.3.3. *The stack $\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$ is a DM-stack.*

Proof. We apply Theorem 4.21 in [DM]. We need to check conditions:

(1) The diagonal map of the stack is representable, quasi-compact, separated and unramified. The last part follows from SC.

(2) There is a scheme U and a smooth surjective map $U \rightarrow \overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$. To prove it, use the map

$$\overline{\mathcal{M}}_{g,n}(X[l]^+/X[l], \beta)^\mu \rightarrow \overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$$

for large enough l , and apply Lemma 3.1.3 and Proposition 3.3.2. \square

4. SEPARATEDNESS AND PROPERNESS

4.1. Separatedness.

Proposition 4.1.1. *The moduli stack $\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)$ is separated over \mathbb{C} .*

Proof. We will show that the valuative criterion for separateness is satisfied. Let R be a discrete valuation ring and K be its quotient field. Assume that K is a function field of dimension 1 over \mathbb{C} . For $i = 1, 2$, consider two stable ramified maps with n sections:

$$\begin{array}{ccccccc} \mathcal{C}_i & \xrightarrow{f_i} & \mathcal{W}_i & \xrightarrow{a_i} & X[n_i]^+ & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Spec}R & \xlongequal{\quad} & \text{Spec}R & \xrightarrow{g_i} & X[n_i] & & \end{array}$$

which are equivalent over K and

$$\mathcal{W}_i \cong g_i^* X[n_i]^+.$$

By the assumption, two pairs of K -schemes

$$(\mathcal{C}_1|_K, g_1|_K^* X[n_1]^+), \quad (\mathcal{C}_2|_K, g_2|_K^* X[n_2]^+)$$

are isomorphic by a pair (ϕ, φ) of isomorphisms over K such that:

$$\varphi \circ f_1|_K = f_2|_K \circ \phi,$$

φ fixes X and ϕ preserves the n sections of \mathcal{C}_i . By the uniqueness of the extension of ordinary stable maps, it suffices to verify that targets \mathcal{W}_1 and \mathcal{W}_2 are isomorphic by an extension of φ . To do so, we use Tangent Line Map Condition.

By Lemma 3.2.3¹, we have the tangent line map

$$\mathbb{P}(Tf_i) : \mathcal{C}_i \rightarrow \mathbb{P}(TX)$$

and hence and its ordinary stabilization (with n markings)

$$(\mathbb{P}(Tf_i))' : \mathcal{C}'_i \rightarrow \mathbb{P}(TX)$$

where \mathcal{C}'_i is a suitable contraction of \mathcal{C}_i . Since the n -pointed stable maps $(\mathbb{P}Tf_1)'$ and $(\mathbb{P}Tf_2)'$ are equivalent over K , they are equivalent over R . Hence there is an isomorphism $\phi' : \mathcal{C}'_1 \rightarrow \mathcal{C}'_2$ satisfying that:

- the diagram

$$\begin{array}{ccc} \mathcal{C}_1|_K & \xrightarrow{\phi} & \mathcal{C}_2|_K \\ \downarrow & & \downarrow \\ \mathcal{C}'_1|_K & \xrightarrow{\phi'|_K} & \mathcal{C}'_2|_K \end{array}$$

commutes,

- $(\mathbb{P}Tf_1)' = (\mathbb{P}Tf_2)' \circ \phi'$, and
- the n markings are preserved under ϕ' .

¹This is the only place where Lemma 3.2.3 is used in this paper. However we can avoid the use of the Lemma, if we take the normalization of \mathcal{C}_i and then consider tangent line map on each component. It is well-defined, otherwise there will be a component of the closed fiber of \mathcal{C}_i which is mapped into a singular locus of the closed fiber of \mathcal{W}_i by Lemma 4.2.3.

At the closed fiber of \mathcal{C}'_1 , consider all components which have nonconstant images under $(\mathbb{P}Tf_1)'$. Now for each such component, there is the corresponding one in the closed fiber of \mathcal{C}_1 . Consider two smooth points on such each component and two sections passing through those points. These sections together with the n markings form M sections of $\mathcal{C}_1 \rightarrow \text{Spec}R$. By ϕ' , we obtain the corresponding sections of $\mathcal{C}_2 \rightarrow \text{Spec}R$. We may assume that the images of those M sections under f_i are disjoint. Here we used DPC. Now applying Corollary 2.6.2 to g_i with those M sections, we claim that the images of the naturally induced maps $g_{i,M} : \text{Spec}R \rightarrow X[n_i, M]$ are in fact in $X[n_i : M]$. Hence combined with Lemma 2.4.1, the claim shows that \mathcal{W}_1 and \mathcal{W}_2 are isomorphic under an extension of φ .

Proof of the claim: Restricted to the closed fiber, $g_{i,M}^* X[n_i, M]^+$ is still stable without n_i labels and also stable without M labels. The latter stability is clear. The first stability results from Lemma 3.1.2 as following. (So, here we use NC and SC.) On end screens of $g_{i,M}^* X[n_i, M]^+$, by the lemma, either there are two distinct images of marked points or there is a nonlinear image. Both obviously survive under $(\mathbb{P}Tf_i)'$. On a ruled screen, by the lemma, either there is a marking which maps into the screen, or there is a nonfiber image of f_i on the screen. Any nonfiber image survives as a noncontracted image under $(\mathbb{P}Tf_i)'$. \square

4.2. Properness. We want to show that the stack of μ -ramified stable maps is proper over \mathbb{C} . Since we already proved the separatedness over \mathbb{C} , we will prove the existence part of the valuative criterion for properness stated in [DM].

Proposition 4.2.1. *Given an n -pointed stable ramified map (f, \mathbf{p}) over the quotient field K of any DVR R , there is a stable ramified map extension $(\tilde{f}, \tilde{\mathbf{p}})$ of (f, \mathbf{p}) , namely, $(\tilde{f}, \tilde{\mathbf{p}})$ is a stable ramified map defined over a DVR R' which is the integral closure of R in a finite extension K' of K and $(\tilde{f}|_{K'}, \tilde{\mathbf{p}}) \cong (f, \mathbf{p})|_{K'}$.*

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{f} & \mathcal{W} \\
 \downarrow \left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\} p_i & & \downarrow \\
 \text{Spec}K & \xlongequal{\quad} & \text{Spec}K
 \end{array}
 \qquad
 \begin{array}{ccc}
 \tilde{\mathcal{C}} & \xrightarrow{\tilde{f}} & \tilde{\mathcal{W}} \\
 \downarrow \left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\} \tilde{p}_i & & \downarrow \\
 \text{Spec}R' & \xlongequal{\quad} & \text{Spec}R'
 \end{array}$$

It is enough to prove the above when $R = \mathbb{C}[[t]]$. Let $R = \mathbb{C}[[t]]$ from now on in this section.

Proof of Proposition 4.2.1 when $\dim X = 1$. Let \mathcal{W}/K be the pullback of the “universal family” $X[m]^+$ by a map $\text{Spec}K \rightarrow X[m]$, then using the image of the marked points under $\mathcal{C} \rightarrow \mathcal{W} \rightarrow X[m]^+$, there is the induced map $\text{Spec}K \rightarrow X[m : n]$. Thus, we may assume that \mathcal{W} is the pullback of $X[n]^+$ under $g : \text{Spec}K \rightarrow X[n]$ such that $f \circ p_i = \sigma_i \circ g$. Now take the usual stable map limit of this $f : \mathcal{C} \rightarrow X[n]^+$ *without markings*. The extension is a stable ramified map as shown in the proof of Theorem 4 in

[HM] using Abhyankar Lemma (which removes possible dimension 1 branch locus in the special fiber), the purity of branch locus [Za] (which removes possible dimension 0 branch locus in the special fiber), and the universal covering of A_1 -singularity. \square

Local Analysis. Let $(\pi : \mathcal{C} \rightarrow \text{Spec}R, \{p_1, \dots, p_n\}, f : \mathcal{C} \rightarrow X)$ be an *ordinary stable map* over $\text{Spec}R$ such that:

- Over $\mathcal{C}_K := \mathcal{C} \times_{\text{Spec}R} \text{Spec}K$, it is a μ -ramified stable map.
- Over $\mathcal{C}_0 :=$ the closed fiber, it is not anymore μ -ramified.

Here $\mathcal{W} = X \times S$. Assume that $\dim X = r \geq 2$ and \mathcal{C} is *irreducible*. Then only the following cases are possible unless some two markings have the same image at the closed point of $\text{Spec}R$.

- (1) (Jump of a Ramification Index) There exists a smooth point p on a component of \mathcal{C}_0 non-contracted under f such that the ramification index jumps at p .
- (2) (Creation of a Nodal Point) There exists a nodal point p of \mathcal{C}_0 such that any component of \mathcal{C}_0 containing p does not contract under f .
- (3) (Contraction of a Component) There exists a contracted component E of \mathcal{C}_0 and there exists no pair of ramification markings approaching to E .

Lemma 4.2.2. *In Case (1) or (2), the tangent line map*

$$\mathbb{P}Tf : \mathcal{C} \dashrightarrow \mathbb{P}TX$$

is not well-defined at p . In Case (3), $(\mathbb{P}Tf)(E)$ is not a point or there is a smooth point $q \in E$ such that $\mathbb{P}Tf$ is not well-defined at q .

Let $(\mathcal{C} \rightarrow \text{Spec}R, f : \mathcal{C} \rightarrow X)$ be an R -family of stable map in the usual sense with no markings. Assume that \mathcal{C} is irreducible and $\dim X = r \geq 2$. If the rational map

$$\mathbb{P}Tf : \mathcal{C} \dashrightarrow \mathbb{P}TX$$

is not defined at a closed point on the closed fiber, then we obtain this:

Lemma 4.2.3. *After a base change there is a suitable extension of $f|_K$ to an R -family of stable maps, $\mathcal{C}' \rightarrow X[2]^+$ such that:*

- *In the closed fiber of \mathcal{C}' , there is a component D whose image under the extension is a nonlinear image on a screen.*
- *There are two sections of \mathcal{C}' passing through D , whose images are distinct and away from the relative singular locus of $X[2]^+/X[2]$.*

Proof. By a suitable blowup $\tilde{\mathcal{C}}$ of \mathcal{C} supported at the closed point we may find an exceptional divisor E such that $\mathbb{P}(Tf)(E)$ is not a constant. The nonconstancy follows from that \mathcal{C} is Cohen-Macaulay (in fact normal) and hence Hartogs extension property holds for \mathcal{C} . Let (w, t) be a system of local parameters of a smooth point p of $\tilde{\mathcal{C}}$ such that $p \in E$, and let (z_1, \dots, z_r) be

a system of local parameters of $\mathcal{O}_{f(p),X}$. Now we obtain an expression: for some integer k ,

$$(f^*(z_1), \dots, f^*(z_r)) = t^k(a_1(w), \dots, a_r(w)) + o(t^k)$$

where $a_i(w)$ is nonzero for some i . Note that $k \geq 1$ since the component E collapses to the origin under f . Now create a screen using two markings of two *general* points of E . Then by Example 2.1.2, on the screen we obtain the limit as

$$\lim_{t \rightarrow 0} [f^*(z_1), \dots, f^*(z_r), at^k] = [a_1(w), \dots, a_r(w), a]$$

for some nonzero $a \in \mathbb{C}$. Since

$$\lim_{t \rightarrow 0} \mathbb{P}Tf(w, t) = \left[\frac{\partial a_1(w)}{\partial w}, \dots, \frac{\partial a_r(w)}{\partial w} \right]$$

which is nonconstant in w , the image of the closed fiber of $\tilde{\mathcal{C}}$ is nonlinear on the screen created. Hence the stable map extension $\mathcal{C}' \rightarrow X[2]^+$ of $f|_K : \mathcal{C}|_K \rightarrow X[2]^+$ has a nonlinear image on the screen. \square

Proof of Proposition 4.2.1 when $\dim X \geq 2$. Choose a fiber square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & X[M]^+ , \\ \downarrow & & \downarrow \\ \text{Spec}K & \longrightarrow & X[M] \end{array}$$

where M is the minimal one among M' such that \mathcal{W} can be expressed as a fiber square $\text{Spec}K \times_{X[M']} X[M']^+$.

Step 1. For each end component (of the geometric fiber) of \mathcal{W} we choose two sections of $\mathcal{C}/\text{Spec}K$, and for each ruled component of \mathcal{W} we choose one section of $\mathcal{C}/\text{Spec}K$. Label the chosen sections by a letter \mathbb{M} . We show that it is possible to satisfy the following conditions by suitable choices of those \mathbb{M} sections.

- (1) Their images under f are distinct images in their associated end or ruled component and the images are away from the relative singular locus of $\mathcal{W}/\text{Spec}K$.
- (2) If an end component contains the images of two markings among $p_1 \dots, p_n$, then the sections associated to the component coincide with at least two of them. If a ruled component contains the image of a marking of p_i 's then the associated section coincides with one of them.
- (3) With the \mathbb{M} sections, apply Lemma 2.6.2 to alter the map $\text{Spec}K \rightarrow X[\mathbb{M}]$. Then take the closure of this new map to get a map $\text{Spec}R \rightarrow X[\mathbb{M}]$. Denote by $\overline{\mathcal{W}}$ the pullback of $X[\mathbb{M}]^+$ under the map $\text{Spec}R \rightarrow X[\mathbb{M}]$. Let $(\overline{\mathcal{C}}, \overline{f})$ be the stable map extension of $f : \mathcal{C} \rightarrow \overline{\mathcal{W}}$. Now we are ready to state the third condition: The closed fiber of $\overline{\mathcal{W}}$

under \bar{f} has at least two p -markings or a nonlinear image in each end component, where the word of p -markings is a shorthand for the sections p_1, \dots, p_n .

- (4) The closed fiber of $\overline{\mathcal{W}}$ under \bar{f} has a p -marking or a nonfiber image in each ruled component.

Only Conditions 3 and 4 are not obvious how to be achieved.

Proof of Condition 3. Choose sections satisfying Conditions 1 and 2. Then an end component of the special fiber might have neither two markings nor a nonlinear image. In this case, the end component must contain a branch point or the image of a nodal point of the special fiber of $\bar{\mathcal{C}}$. (To see it, one can use an identification of the data (the end, the distinguished divisor, two associated sections) with $(\text{Spec}R \times \mathbb{P}^r, \text{Spec}R \times H, q_1, q_2)$, where H is a fixed hyperplane; and q_1, q_2 are suitable two sections in \mathbb{P}^r , away from H .) Either case, we may assume that the special point is away from the singular locus by a suitable change of the chosen two sections. (Because: It is enough to use the induced change of two sections from the action by an R -family of automorphisms of (\mathbb{P}^r, H) . Recall that the identification is unique once the data (H, q_1, q_2) is fixed.) Now apply Lemma 4.2.2 and Lemma 4.2.3, further to change the chosen two sections so that they satisfy Condition 3 for the replaced end component.

Proof of Condition 4. Choose sections satisfying Conditions 1 and 2. Then a ruled component of the special fiber might have a noncontracted image lying over the singular locus. If so, then we alter the associated section to pull out the noncontracted image so that the new ruled component contains a nonfiber image. (To see it, one can use an identification of the data (the ruled component, D_{\pm} , the associated section) with $(\text{Spec}R \times \text{Bl}_p\mathbb{P}^r, \text{Spec}R \times H_{\pm}, q)$, where H_{\pm} are some fixed divisor; p is a point of \mathbb{P}^r ; and q is a section whose image is away from $\text{Spec}R \times H_{\pm}$. Use the change of the section induced from an R -family of automorphisms of $(\text{Bl}_p\mathbb{P}^r, H_{\pm})$.)

Step 2. Now take the union of all p -markings and \mathbb{M} -sections over $\text{Spec}K$, to define a map h from $\text{Spec}K$ to $X[L]$, where L is the cardinality of the union. Note that canonically

$$\mathcal{W} \cong h^* X[L]^+$$

by Lemma 2.6.1. Then the usual stable map limit of $\varphi \circ f : \mathcal{C} \rightarrow X[L]^+$ satisfies NC, SC and DPC.

Step 3. So far, we have shown that there is an extension

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{W}' \\ \downarrow p'_i & & \\ \text{Spec}R' & & \end{array}$$

satisfying NC, SC and DPC by expanding the target. If there is a component E on the closed fiber of \mathcal{C}' , which map to singular locus, then for such a trouble component, add a general section passing through it. Then this result in creating a ruled component. Then still, the extension $f'' : \mathcal{C}'' \rightarrow \mathcal{W}''$ obviously satisfy NC and DPC. We claim that SC holds in the extension. The reason is as follows:

1) If the component E was mapped to a nonconstant component in the singular locus, then the new extension still has a nonfiber image which can be seen by considering nearby sections and the natural composite $\mathcal{C}'' \rightarrow \mathcal{W}'' \rightarrow \mathcal{W}'$. (See Remark 4.2.5 for an explanation.)

2) This is the case when a component E was mapped to a point in the singular locus of the target. Since the component E' in \mathcal{C}'' corresponding to E is a stable component without markings, there are three cases appear.

2a) The image of E' in the new ruled component is a point.

2b) The map $f''|_{E'}$ is totally ramifying a fiber of the ruled component whose branch points are on the marked divisors D_{\pm} , then there is a nodal point of \mathcal{C}'' on E' mapping to a point away from D_{\pm} .

2c) The image of E' is a fiber with a branch point away from D_{\pm} .

All of subcases makes the ruled component is stable.

This process of pulling out trouble components stops at finite steps: Consider the relative ample divisor L of \mathcal{W}' , then there are only finite “energy” $L \cdot f'_*[\mathcal{C}'_0]$ on the components of the inverse image of the singular locus. This shows there are only finite many nonfiber images on limits.

Step 4. Now AC is automatically satisfied by Proposition 2.2 of [J.Li]. By further expanding target, we can extend f so that nodal points maps to nodal points and PRIC is satisfied by applying Lemmas 4.2.2 and 4.2.3 again. This process also can be seen to stop at finite steps by considering a rational map $\mathbb{P}T f'$. □

Finally we have proved the main goal of the paper.

Theorem 4.2.4. *The stack of stable μ -ramified maps to X with fixed degree $\beta \in H_2(X, \mathbb{Z})$ is a proper Deligne-Mumford stack over \mathbb{C} .*

Remark 4.2.5. Let $f : \mathcal{C} \rightarrow X[n]^+$ be a stable ramified map over K such that its usual stable map extension has a closed fiber one of whose components, call it E , is mapped into a relative singular locus of $X[n]^+/X[n]$. Then consider a general section σ passing through the component E and use it to obtain a lift $f' : \mathcal{C} \rightarrow X[n+1]^+$ over K and take the usual stable map

extension $\overline{f'}$. Then after base change, there is a diagram

$$\begin{array}{ccc} \overline{\mathcal{C}'} & \xrightarrow{\overline{f'}} & X[n+1]^+ \\ \downarrow c & & \downarrow \\ \overline{\mathcal{C}} & \xrightarrow{\overline{f}} & X[n]^+ \end{array}$$

of stable maps over R , where c is a contraction. Then the lifted section $\overline{\sigma'}$ passes through the corresponding component E' in $\overline{\mathcal{C}'}$ of E . This can be easily seen by considering nearby sections.

Proof of Lemma 4.2.2. Case (1). Let m be the ramification index of $f|_{\mathcal{C}_0}$ at p , and let x, t be local parameters of $\mathcal{O}_{p, \mathcal{C}}$, where t is a local parameter of the DVR $R = \mathbb{C}[[t]]$. Let $f = (f_1, \dots, f_r)$ after introducing a local coordinate system of X at $f(p)$. WLOG assume that the tangent line direction of the image curve of x -axis under $f(x, 0)$ is $[1, 0, \dots, 0] \in \mathbb{P}^{r-1} = \mathbb{P}T_{f(p)}X$. Consider the equation $\frac{\partial f_1(x, t)}{\partial x} = 0$. It defines a curve Z not containing x -axis, but containing the origin $(x, t) = (0, 0)$. If the curve has at least two branches, then f is unramified along one of branches (when $t \neq 0$) by assumption of stable ramified maps. Hence along the branch, $\mathbb{P}Tf$ approaches to a point $\neq [1, 0, \dots, 0]$. If the curve Z has only one branch, then along the curve, $\mathbb{P}Tf$ approaches to a point $\neq [1, 0, \dots, 0]$ since the generic point of the curve is a multiple $m-1$ zero of $\frac{\partial f_1(x, t)}{\partial x}$, however the generic point cannot be a $m-1$ or higher multiple zero of the derivative $\frac{\partial f_i(x, t)}{\partial x}$ for some $i \neq 1$ (otherwise, along the curve, f has the ramification index m).

Case (2). The domain surface \mathcal{C} is étale locally defined by relation $xy = t^k$ at $(0, 0, 0)$ for some positive integer k and $\pi(x, y, t) = t$. Furthermore, $f(x, y, t) \in R[[x, y]]^{\oplus r}$, $f(x, 0, 0) \neq 0$, $f(0, y, 0) \neq 0$. For notational simplicity let $\dim X = 2$. Hence we may let

$$f(x, y, t) = (f_1(x, y, t), f_2(x, y, t)) \quad \text{and} \quad f(0, 0, t) = (0, 0).$$

By affine transformations of $f(x, y, t)$, we may impose the following two conditions:

- There is a positive integer m such that a monomial term of form x^m appears in $f_1(x, y, t)$ but not in $f_2(x, y, t)$.
- If $l < m$, then any monomial term of form x^l appears in neither $f_1(x, y, t)$ nor $f_2(x, y, t)$.

Now the argument splits up into two cases.

(a) This is the case when there is a nonzero term y^α in $f_2(x, y, t)$. Consider all the lattice points of exponents which appear in

$$f_1(x, t^k/x, t) = \sum_{a_{\mu, \nu} \neq 0} a_{\delta, \nu} x^\delta t^\nu$$

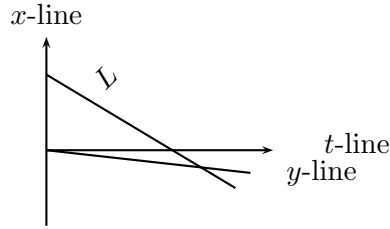


FIGURE 2. (ν, δ) -plane

or $f_2(x, t^k/x, t)$ on (ν, δ) -plane. Considering the intersection of the triangle with vertices $(0, m)$, $(0, 0)$, $(\alpha k, -\alpha)$ and the convex hull of all the lattice points, we see that there is a negative slope line L connecting two lattice points of exponents, one from f_1 and the other from f_2 , and all exponents lie on the upper half plane of L . This implies that there are positive rational numbers γ_1 and γ_2 such that along a curve of form $(x(t) = O(t^{\gamma_1}), y(t) = O(t^{\gamma_2}), t)$ on the domain surface of f , $\mathbb{P}Tf(x(t), y(t), t)$ approaches to a point, not equal to $[1, 0]$.

(b) This is the case when all terms of form y^α appear only in f_1 . Then by considering the lattice points of exponents, we see that there is a line L connecting two distinct exponents $t^a x^b$ and t^c/x^d in f_1 where $a \geq 0, b > 0, c > 0, d > 0$; and all exponents of f_1 and f_2 appear in the above half plane of L .

(i) Suppose that on L , an exponent of f_2 appears. Then as in Case (a), it is done.

(ii) Suppose that on L , no exponents of f_2 appear. Then put $x = zt^l$ and $y = t^{k-l}/z$ according to the line L (i.e., $l = \frac{c-a}{b+d}$ which is “deg x ”). Note that $k - l > 0$ (that is, “deg y ” > 0) since there is a lattice point of y^α lies on the upper half plane of L . Then we may set

$$f(zt^l, t^{k-l}/z, t) = t^{\gamma_1}(g_1(z, t), t^{\gamma_2}g_2(z, t)), \text{ for some } g_i(z, t) \in \mathbb{C}[[z, z^{-1}, t]]$$

with $g_1(z, 0) \neq 0$, and $\gamma_i \in \mathbb{Q}_{>0}$. Note that $\gamma_2 > 0$ since no exponents of f_2 appear on L . Now it is easy to see that $\frac{\partial g_1}{\partial z}(z_0, 0) = 0$ for some nonzero z_0 since two distinct exponents $t^a x^b$ and t^c/x^d in f_1 appear on L . Now we can go back to Case (1) with $g(z, t) = (g_1(z, t), g_2(z, t))$ with “new” $p = (z_0, 0)$ in (z, t) -coordinate. (Let us illustrate this procedure by example. Suppose $xy = t^2$, $f(x, y, t) = (x + y + x^2, x^2y)$ then $l = 1$ and $f(zt^l, t^{k-l}/z, t) = t(z + 1/z + tz^2, t^2z)$. Then set $z_0 = 1$ (which is a point where $\frac{\partial g_1}{\partial z}(z_0, 0) = 0$) and put $z = z_0 + u$ so that (u, t) becomes a new local coordinate system of the domain surface \mathcal{C} , away from $t = 0$. Now the expansion of f/t in terms of (u, t) becomes Case (1) since the ramification index jumps at $(u = 0, t = 0)$).

Case (3). This is a local problem at $f(E)$ in X so that we may assume that there is a projection to \mathbb{P}^1 from X such that the composite of f over $\text{Spec}K$ is a stable ramified map $f' : \mathcal{C} \rightarrow \mathbb{P}^1$. Since there is a component E of \mathcal{C}_0 mapping to a point in \mathbb{P}^1 , there are two ramification divisors of f' approaching to the component E . Now use the image of these two divisors (or their nearby ones if the images coincide) to obtain a new target space $X[2]^+$ and a map $\mathcal{C}' \rightarrow X[2]^+$. Then using the composite

$$\begin{array}{ccc} \text{Bl}_p(X \times \mathbb{C}) & \longrightarrow & \text{Bl}_p(\mathbb{P}^1 \times \mathbb{C}) \\ \downarrow & & \downarrow \\ X \times \mathbb{C} & \longrightarrow & \mathbb{P}^1 \times \mathbb{C} \end{array}$$

there is the induced map $\mathcal{C}' \rightarrow \mathbb{P}^1[2]^+$. Since \mathcal{C}'_0 has two branched points of the mapping $\mathcal{C}'_0 \rightarrow \mathbb{P}^1[2]^+$ on the screen, by choosing suitable projection $X \rightarrow \mathbb{P}^1$, we conclude that either $\mathcal{C}' \rightarrow X[2]^+$ has a nonlinear image on the screen or belongs to Case (1) on the screen. Since the stabilization of the composite $\mathcal{C}' \rightarrow X[2]^+ \rightarrow X$ is $\mathcal{C} \rightarrow X$, we conclude the proof. \square

5. RAMIFIED GROMOV-WITTEN INVARIANTS

Since the obstruction theory is determined by square free extension, the relative obstruction theory developed by J. Li [J.Li] works for the moduli stack of stable ramified maps. Only difference is to impose the condition of ramifications, which is straightforward. (See page 216 in [J.Li] and for the deformation theory for stable ramified maps, we add the condition that $\eta_i \in I \otimes \mathcal{O}_{U_\alpha}^\mu$ where $\mathcal{O}_{U_\alpha}^\mu$ is the A -sheaf of regular functions on U_α whose j -th derivatives are zero for all $j = 1, \dots, \mu_i - 1$ at p_i whenever i -th marking p_i is contained in U_α .) Therefore, there is a natural relative perfect obstruction theory $E^\bullet \rightarrow L_{\overline{\mathcal{M}}_{g,n} \times \mathcal{X}}$. (See [GV] for an expression of relative tangent/obstruction spaces (after the imposing the ramification condition).) Hence we can define the virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\mu}(X, \beta)]^{\text{vir}}$ by the relative perfect obstruction theory.

There is a natural morphism

$$Te_i : \overline{\mathcal{M}}_{g,n,\mu}(X, \beta) \rightarrow \mathbb{P}TX$$

and hence we define ramified Gromov-Witten Invariants by

$$\langle \tau_{a_1} \gamma_1, \dots, \tau_{a_n} \gamma_n \rangle_{g,n,\mu,\beta} = \int_{[\overline{\mathcal{M}}_{g,n,\mu}(X,\beta)]^{\text{vir}}} \prod \psi_i^{a_i} Te_i^*(\gamma_i)$$

where ψ_i are gravitational descendants associated to the i -th marking and γ_i are cohomology classes of $\mathbb{P}TX$.

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