1. The vacuum expectation value of the energy-momentum tensor, for free fields, is given as the sum of the zero-point energy. Considering general boundary conditions, the computation is reduced to the infinite sum

$$\sum_{n=1}^{\infty}(n-\theta)$$

where θ is determined by the particular boundary condition of interest, i.e. $\theta = 0$ for periodic boundary condition, $\theta = \frac{1}{2}$ antiperiodic boundary condition. The expression is apparently divergent, but regularization gives the finite part of the above expression

$$\frac{1}{24} - \frac{1}{8}(2\theta - 1)^2$$

Verify this.

Remark: This result can be found in Eq.(2.9.19) of Polchinski's String Theory Vol.1

2. (a) Find the conformal weight of scalar field in d-dimensions. You can start with the following covariant action

$$S = \int d^d x \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

and demand invariance under global scale transformation

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \qquad \phi \to \Omega^n \phi$$

(b) Now let us consider *local* scale transformation, i.e. we promote the parameter α to a coordinate dependent function $\alpha(x)$. First study how the present form of the equation of motion

$$\Box\phi \equiv \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = 0$$

transforms under the local conformal transformation.

(c) Consider general curved background metric and local scale transformation.

$$\overline{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$$

Let R and $\overline{R}$ as the scalar curvature evaluated using $g, \overline{g}$ respectively. It is possible to express $\overline{R}$ in terms of R and α. Show that

$$\overline{R} = \Omega^{-2} R + 2(d-1)\Omega^{-3}\Box\Omega + (d-1)(d-4)\Omega^{-4}(\nabla\Omega)^2$$

(d) Now verify that the following scalar field action coupled to the background curvature,

$$S = \int d^d x \sqrt{g} \Big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{d-2}{4(d-1)} R \phi^2 \Big)$$

gives rise to the equation of motion which is invariant under local conformal transformation.

3. Exercise 4.4 of Di Francesco et. al.

4. Exercise 4.8 of Di Francesco et. al.

5. Exercise 5.3 of Di Francesco et. al.

6. Exercise 5.5 of Di Francesco et. al.

7. Exercise 5.9 of Di Francesco et. al.

8. Exercise 6.7 of Di Francesco et. al.

9. Exercise 6.8 of Di Francesco et. al.

10. Exercise 6.12 of Di Francesco et. al.

11. Exercise 7.1 of Di Francesco et. al.

12. Exercise 7.2 of Di Francesco et. al.

13. Exercise 7.9 of Di Francesco et. al.

14. Let us consider the superconformal symmetry algebra in two dimensions, which contains fermionic (anticommuting) generators which mix bosonic and fermionic fields in addition to the usual generators of the Virasoro algebra. The minimal extension gives

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{1}{8}\hat{c}n(n^2-1)\delta_{n+m,0} \\
[L_n, G_m] &= \left(\frac{n}{2} - m\right) G_{n+m} \\
\{G_n, G_m\} &= 2L_{n+m} + \frac{1}{2}\hat{c}(n^2 - \frac{1}{4})\delta_{n+m,0}
\end{aligned}$$

In the Neveu-Schwarz (Ramond) sector the indices of G_n are half-integers (integers). The superconformal module is constructed from the ground state $|h\rangle$ which is annihilated by L_n, G_n for $n > 0$ and an eigenstate of L_0 with eigenvalue h.

Show that the first nontrivial null state of the NS algebra is at level $\frac{3}{2}$,

$$|\text{null}\rangle = \left(G_{-3/2} - \frac{2}{2h+1}L_{-1}G_{-1/2}\right)|h\rangle$$

Conformal Field Theory

Exercise 4.4~7.9

4.4

a) Show that the expression (4.62) for the four-point function is conformally covariant.

b) Show that there are only two independent cross-ratios of the form (4.23) that can be built out of four points, except in dimension two, where the two cross-ratios are related.

conformal invariants, the anharmonic ratios (4.23). The n-point function may have an arbitrary dependence (i.e., not fixed by conformal invariance) on these ratios. For instance, the four-point function may take the following form:

$$\langle \phi_1(x_1) \ldots \phi_4(x_4) \rangle = f\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}}\right) \prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \tag{4.62}$$

where we have defined $\Delta = \sum_{i=1}^{4} \Delta_i$.

It is therefore impossible to construct an invariant Γ with only 2 or 3 points. The simplest possibilities are the following functions of four points:

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|} \tag{4.23}$$

Such expressions are called *anharmonic ratios* or *cross-ratios*. With N distinct points, $N(N-3)/2$ independent anharmonic ratios may be constructed.

4.8 *Liouville field theory*

Consider the Liouville field theory in $d = 2$, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2 e^{\varphi}$$

Write down the canonical energy-momentum tensor and add a term that makes it traceless without affecting the conservation laws.

5.3 *Four-point function for the free boson*

Calculate the four-point function $\langle \partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$ for the free boson using Wick's theorem. Compare the result with the general expression (5.28). What is the function $f(\eta, \bar{\eta})$?

The four-point function may then depend on η and $\bar{\eta}$ in an arbitrary way—provided the result is real. The general expression (4.62) translates into

$$\langle \phi_1(x_1) \dots \phi_4(x_4) \rangle = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{h/3 - h_i - h_j} \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j} \tag{5.28}$$

where $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$. This form for the four-point function may

5.5 *Free complex fermion*

Given two real fermions ψ_1 and ψ_2, one may define a single *complex* fermion ψ and its Hermitian conjugate $\psi^\dagger$ this way (with holomorphic and antiholomorphic modes):

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad & \bar{\psi} &= \frac{1}{\sqrt{2}}(\bar{\psi}_1 + i\bar{\psi}_2) \\ \psi^\dagger &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \qquad & \bar{\psi}^\dagger &= \frac{1}{\sqrt{2}}(\bar{\psi}_1 - i\bar{\psi}_2) \end{aligned} \tag{5.194}$$

The real fermions ψ_1 and ψ_2 are governed by the action and energy-momentum tensor of Sect. 5.3.2.

a) Show that the OPE of the complex fermion with itself is

$$\psi^\dagger(z)\psi(w) \sim \frac{1}{z - w} \qquad \psi(z)\psi(w) \sim \psi(z)^\dagger\psi(w)^\dagger \sim 0 \tag{5.195}$$

b) Show that the energy-momentum tensor may be expressed as

$$T(z) = \frac{1}{2}(\partial\psi^\dagger\psi - \psi^\dagger\partial\psi) \tag{5.196}$$

and that the conformal dimension of ψ is $\frac{1}{2}$ and that the central charge is $c = 1$.

c) Show that the action describing the complex fermion system may be written as

$$S[\psi] = g \int d^2x \; \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \tag{5.197}$$

where $\Psi = (\psi, \bar{\psi}^\dagger)$ is a two-component field.

5.9 The Schwarzian derivative

a) Demonstrate explicitly the group property (5.130) of the Schwarzian derivative.

b) Show that the Schwarzian derivative of the $SL(2,\mathbb{C})$ transformation (5.12) vanishes.

$$f(z) = \frac{az+b}{cz+d} \quad \text{with} \quad ad-bc=1 \qquad (5.12)$$

$$\{u;z\} = \{w;z\} + \left(\frac{dw}{dz}\right)^2 \{u;w\} \qquad (5.130)$$

6.7 Contraction of two exponentials

Let A and B be two free fields whose contraction (with themselves and with each other) are c-numbers.

a) Show by recursion that

$$\overbrace{A(z) : B}^{n}(w) := n\,\overbrace{A(z)B(w)} : B^{n-1}(w) :$$

and therefore

$$\overbrace{A(z) : e}^{B(w)} = \overbrace{A(z)B(w)} : e^{B(w)} :$$

As usual, $: \cdots :$ denotes normal ordering for free fields.

b) By counting correctly multiple contractions, show that

$$\overbrace{e^{A(z)}e^{B}}(z) = \sum_{m,n,k} \frac{k!}{m!n!} \binom{m}{k} \binom{n}{k} [\overbrace{A(z)B(w)}]^k : A^{m-k}(w)B^{n-k}(w) :$$

$$= \exp\left\{\overbrace{A(z)B(w)}\right\} : e^{A(w)}e^{B(w)} :$$

And deduce from this the OPE (6.65) of two vertex operators.

$$\mathcal{V}_\alpha(z,\bar{z})\mathcal{V}_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta/4\pi g}\, \mathcal{V}_{\alpha+\beta}(w,\bar{w}) + \cdots \qquad (6.65)$$

6.8 Calculate $([T,(TT)])$, first using Eq. (6.222) and the OPE $T(z)(TT)(w)$ given in Eq. (6.213), and then directly in terms of modes, from the equality

$$[T,(TT)] = [T_{-2},(TT)_{-4}]$$

with $T_{-2} \equiv L_{-2}$ and

$$(TT)_{-4} = 2\sum_{l\geq 0} L_{-l-3}L_{l-1} + L_{-2}L_{-2}$$

which follows from Eq. (6.213).

$$([A,B]) = \sum_{n>0} \frac{(-1)^{n+1}}{n!} \partial^n \{AB\}_n(w) \qquad (6.222)$$

$$T(z)(TT)(w) \sim \frac{3c}{(z-w)^6} + \frac{(8+c)T(w)}{(z-w)^4} + \frac{3\partial T(w)}{(z-w)^3} + \frac{4(TT)(w)}{(z-w)^2} + \frac{\partial(TT)(w)}{(z-w)} \qquad (6.213)$$

6.12 *The classical limit of the Virasoro algebra*

The Poisson bracket form of the Virasoro algebra is obtained by replacing the commutator by a Poisson bracket times a factor i, that is,

$$i\{L_n, L_m\} = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Let $u(x,t)$ be the classical field defined on a cylinder ($u(x+2\pi, t) = u(x,t)$) whose Fourier modes are the L_n's:

$$u(x) = \frac{6}{c}\sum_{n\in\mathbb{Z}} L_n e^{-inx} - \frac{1}{4}$$

(the explicit time-dependence is omitted from now on). This is the classical form of the energy-momentum tensor. Show that its equal-time Poisson bracket is

$$\{u(x), u(y)\} = \frac{6\pi}{c}[-\partial_x^3 + 4u(x)\partial_x + 2(\partial_x u(x))]\delta(x-y) \qquad (6.247)$$

Use:

$$\delta(x-y) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-in(x-y)}$$

Recover the classical Korteweg–de Vries equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u$$

from the following canonical formulation:

$$\partial_t u = \{u, H\}, \qquad \text{with} \qquad H = \frac{c}{12\pi}\int dx\, u^2$$

The above Poisson bracket defines the so-called second Hamiltonian structure of the KdV equation. (The relative sign between the two terms on the r.h.s. of the KdV equation is not as in the classical form derived in Ex. 6.10 (cf. Eq. (6.242)); this is explained by a 'space Wick rotation', i.e., the space variables used in the two cases are related by a factor i.)

6.10 *The quantum Korteweg–de Vries equation*

Let us introduce an equation of evolution in time for the energy-momentum tensor through the canonical equation of motion

$$\partial_t T = -[H, T] \quad , \quad H = \frac{1}{2\pi i} \oint dw \, (TT)(w)$$

a) Using the OPE (6.213), check that the resulting evolution equation is

$$\partial_t T = \frac{1}{6}(1-c)\partial^3 T - 3\partial(TT) \qquad (6.241)$$

This is called the quantum Korteweg–de Vries (KdV) equation since in the classical limit $c \to -\infty$,[2] the substitution $T = cu(z,t)/6$ and a rescaling of the time variable transforms it into the standard KdV equation:

$$\partial_t u = \partial^3 u + 6u\partial u \qquad (6.242)$$

b) The quantum KdV equation (like its classical counterpart) is a completely integrable system in the sense that it has an infinite number of conserved integrals H_n

$$\partial_t H_n = 0$$

(whose densities are polynomial derivatives in T), all commuting with each other. Each of these conserved integrals has a definite spin. The spin of these charges is always odd, and there is one charge for each odd value of the spin. To illustrate this statement, check that there can be no nontrivial conserved integral of spin 2 and 4. A conserved integral is nontrivial if its density is not a total derivative.

7.1 *Inner product*
Show that the norm of the state $(L_{-1})^n|h\rangle$ is

$$2^n n! \prod_{i=1}^{n} \left(h - (i-1)/2\right)$$

7.2 *Gram matrix*
Show that the Gram matrix for level 3 is

$$M^{(3)} = \begin{pmatrix} 24h(1+h)(1+2h) & 12h(1+3h) & 24h \\ 12h(1+3h) & h(8+c+8h) & 10h \\ 24h & 10h & 2c+6h \end{pmatrix}$$

(the states are ordered as in Table 7.1).

Table 7.1. Lowest states of a Verma module.

l	$p(l)$	
0	1	$\|h\rangle$
1	1	$L_{-1}\|h\rangle$
2	2	$L_{-1}^2\|h\rangle$, $L_{-2}\|h\rangle$
3	3	$L_{-1}^3\|h\rangle$, $L_{-1}L_{-2}\|h\rangle$, $L_{-3}\|h\rangle$
4	5	$L_{-1}^4\|h\rangle$, $L_{-1}^2L_{-2}\|h\rangle$, $L_{-1}L_{-3}\|h\rangle$, $L_{-2}^2\|h\rangle$, $L_{-4}\|h\rangle$

7.9 Equation of motion for the Yang-Lee model

Consider the minimal model $\mathcal{M}(5,2)$ associated with the Yang-Lee edge singularity. The module of the identity operator $\mathbb{I} = \phi_{(1,1)} = \phi_{(1,4)}$ contains a null vector at level four:

$$|\chi\rangle = (L_{-2}^2 - \frac{3}{5}L_{-4})|0\rangle \tag{7.125}$$

a) Show that the field associated with the state $|\chi\rangle$ is

$$T_4(z) = (TT) - \frac{3}{10}T''$$

b) Compute the singular terms in the short-distance product of this field with any primary field Φ of the theory, with dimension h.

Result:

$$\begin{aligned} T_4(z)\Phi(0) &= z^{-4}h(h+\frac{1}{5})\Phi(0) \\ &+ z^{-3}2(h+\frac{1}{5})\partial\Phi(0) \\ &+ z^{-2}\left(\frac{5h+1}{2h+1}\partial^2\Phi(0) + 2h\Phi^{(2)}(0)\right) \\ &+ z^{-1}\left(\frac{5h+1}{(2h+1)(h+1)}\partial^3\Phi(0) + \frac{6h}{h+2}\Phi^{(2)}(0) + 2(h-1)\Phi^{(3)}(0)\right) \end{aligned}$$

where

$$\begin{aligned} \Phi^{(2)} &= (L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2)\Phi \\ \Phi^{(3)} &= (L_{-3} - \frac{2}{h+1}L_{-1}L_{-2} + \frac{1}{(h+1)(h+2)}L_{-1}^3)\Phi \end{aligned}$$

c) Deduce that the only possible primary fields of the theory have dimensions 0 or $-1/5$. Show, moreover, that when $h = -1/5$, we have $\Phi^{(2)} = \Phi^{(3)} = 0$.

The vanishing of $T_4(z)$ therefore implies most of the structure of the corresponding minimal model: It may be viewed as the equation of motion of the Yang-Lee model. This may be generalized to any minimal model (p,p'). In those cases, the identity has a nontrivial singular descendant at level $(p-1)(p'-1)$: It is a composite field $T_{(p-1)(p'-1)}$ of T and its derivatives. Its vanishing forms the equation of motion of the corresponding minimal model and completely determines the spectrum of the theory.