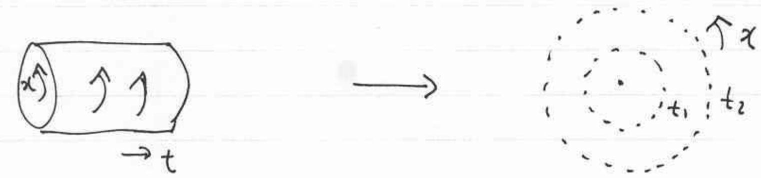
Operator Formalism

§ Radial Quantization

$$\xi = t + ix$$

and consider $z = e^{2\pi\xi/L}$



$$\phi_{in} \propto \lim_{t \to -\infty} \phi(x, t)$$

in radial quantization

$$|\phi_{in}\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle$$

Inner product, Conjugate operator

$$z \to \frac{1}{z^*} \quad (r \to r^{-1},\ \theta \to \theta)$$

$$[\phi(z, \bar{z})]^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z)$$

let's compute, with $\langle \phi_{out}| = |\phi_{in}\rangle^\dagger$

$$\langle \phi_{out} | \phi_{in} \rangle = \lim_{z, \bar{z}, w, \bar{w} \to 0} \langle 0 | \phi(z, \bar{z})^\dagger \phi(w, \bar{w}) | 0 \rangle$$

$$= \lim_{z, \bar{z}, w, \bar{w} \to 0} \bar{z}^{-2h} z^{-2h} \langle 0 | \phi(1/\bar{z}, 1/z)\, \phi(w, \bar{w}) | 0 \rangle$$

$$= \lim_{\xi, \bar{\xi} \to \infty} \bar{\xi}^{2h} \xi^{2\bar{h}} \langle 0 | \phi(\xi, \bar{\xi})\, \phi(0, 0) | 0 \rangle$$

(the limit is well-behaved)

Mode Expansions

$$\phi(z,\bar z) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} z^{-m-h}\,\bar z^{-n-\bar h}\,\phi_{m,n}$$

Inverting,

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\, z^{m+h+1}\,\frac{1}{2\pi i}\oint d\bar z\, \bar z^{n+\bar h-1}\,\phi(z,\bar z)$$

and $\phi_{m,n}^{\dagger} = \phi_{-m,-n}$ since

$$\phi(z,\bar z)^{\dagger} = \sum_{m,n} \bar z^{-m-h} z^{-n-\bar h}\,\phi_{m,n}^{\dagger}$$

$$\|$$

$$\bar z^{-2h} z^{-2\bar h}\,\phi\left(\frac{1}{\bar z},\frac{1}{z}\right) = \bar z^{-2h} z^{-2\bar h}\sum \phi_{m,n}\,\bar z^{m+h} z^{n+\bar h}$$

$$= \sum \phi_{-m,-n}\,\bar z^{-m-h}\,z^{-n-h}$$

For the "in" and "out" states to be well-defined.

$$\phi_{m,n}|0\rangle = 0 \quad \text{for} \quad m > -h,\ n > -\bar h$$

from now on, we'll write only the holomorphic parts since the formulae for anti-holomorphic are identical

$$\boxed{\begin{aligned}\phi(z) &= \sum z^{-m-h}\,\phi_m \\ \phi_m &= \frac{1}{2\pi i}\oint dz\, z^{m+h-1}\,\phi(z)\end{aligned}}$$

Radial ordering and OPE

(time ordering)

$$R\,\phi_1(z)\,\phi_2(w) = \begin{cases}\phi_1(z)\,\phi_2(w) & \text{if } |z|>|w| \\ \phi_2(w)\,\phi_1(z) & \text{if } |z|<|w|\end{cases}$$

radial ordering will be always implicit

$$\oint_w dz\, a(z)\, b(w) = \oint_{0w} dz\, a(z)\, b(w) - \oint_0 dz\, b(w)\, a(z)$$

$$= [A,\ b(w)]$$

inside correlation function

$$A = \oint a(z)\, dz$$

or commutator

$$\boxed{[A, B] = \oint_0 dw \oint_w dz\; a(z)\, b(w)}$$

$$A = \oint a(z)\, dz$$
$$B = \oint b(z)\, dz$$

§ Virasoro algebra

Conformal generators

$$Q = \frac{1}{2\pi i} \oint dz\, \epsilon(z)\, T(z)$$

Conformal Ward identity

$$\delta_\epsilon (\cdot) = -\frac{1}{2\pi i} \oint dz\, \epsilon(z)\, T(z)(\cdot)$$

$$\delta_\epsilon \phi = -[Q_\epsilon, \phi]$$

Laurent expansion of T

$$T(z) = \sum z^{-n-2} L_n, \qquad L_n = \frac{1}{2\pi i} \oint dz\; z^{n+1}\, T(z)$$

$$\Rightarrow \quad Q_\epsilon = \sum \epsilon_n L_n \quad \text{for} \quad \epsilon(z) = \sum z^{n+1} \epsilon_n$$

$$[L_n, L_m] = \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{m+1} \oint_w dz\, z^{n+1}\, T(z)\, T(w)$$

$$= \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{m+1} \oint_w dz\, z^{n+1} \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{t.reg} \right\}$$

$$= \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{m+1} \left\{ \frac{c}{12}(n+1)n(n-1)\, w^{n-2} + (n+1)\, w^{n}\, 2T(w) + w^{n+1}\, \partial T(w) \right\}$$

$-(k+2) \sum w^{-k-3}$

$$= \frac{c}{12}(n^3 - n) \cdot \delta_{m+n,0} + \underbrace{2(n+1) L_{m+n} + (-m-n-2) L_{m+n}}_{(n-m) L_{m+n}}$$

Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}$$
$$[L_n, \bar{L}_m] = 0$$
$$[\bar{L}_n, \bar{L}_m] = (n-m) \bar{L}_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}$$

Hilbert space

vacuum $|0\rangle$: inv under "global" conformal tr.

annihilated by L_{-1}, L_0, L_1 and all "lowering ops"

$$L_n|0\rangle = 0 \qquad n \geq -1$$
$$\bar{L}_n|0\rangle = 0$$

primary fields $\approx$ eigenstates of the Hamiltonian (asymptotic states)

$$[L_n, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint_w dz\, z^{n+1}\, T(z)\phi(w,\bar{w})$$

primary

$$= \frac{1}{2\pi i}\oint_w dz \cdot z^{n+1}\left[\frac{h\,\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{z-w} + \text{reg.}\right]$$

$$= h(n+1)\, w^n \phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \quad (n \geq -1)$$

ee

Now consider ~~the~~ an asymptotic state

$$|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle$$ ($\Leftarrow$ a nontrivial ground state.)

then $L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle$, $\bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle$

and $L_n|h,\bar{h}\rangle = 0$ for $n > 0$

$\bar{L}_n|h,\bar{h}\rangle = 0$

($\because$ $w^n, w^{n+1} \to 0$ as $w \to 0$)

Excited states are obtained by acting on ϕ_m ~~m>0~~ $m<0$
or L_m $(m<0)$

$$[L_n, \phi_m] = [n(h-1) - m]\,\phi_{n+m}$$

from $[L_n, \phi(w)]$ and $\phi(w) = \sum z^{-m-h}\phi_m$

Descendants

$$L_{-k_1} L_{-k_2} \cdots L_{-k_n} |h\rangle$$

with L_0 eigenvalue

$$h' = h + k_1 + \cdots k_n = h + N$$

N is called the <u>level</u>

\# the generating fn for #(N) = # of partitions

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)\, q^n \equiv \frac{1}{\varphi(q)}$$

Euler fn

$$\| \quad (1+q+q^2+\ldots)(1+q^2+q^4+\ldots)(1+q^3+q^6+\ldots)$$

$|h\rangle$ and its descendants form an irreducible rep of the Virasoro algebra:

called "Verma module"

Free Boson

$$\phi(x,t) = \sum e^{2\pi i n x/L} \phi_n(t)$$

(on the cylinder)

$$\phi_n(t) = \frac{1}{L}\int dx\, e^{-2\pi i n x/L}\phi(x,t)$$

$$L = \frac{g}{2}\int dx\left(\dot\phi^2 - \phi'^2\right) = \frac{gL}{2}\sum_n\left(\dot\phi_n\dot\phi_{-n} - \left(\frac{2\pi n}{L}\right)^2\phi_n\phi_{-n}\right)$$

Lagrangian

$$\Pi_n = gL\dot\phi_{-n} \quad \Rightarrow \quad \overset{\text{Quantization}}{[\phi_n, \Pi_m] = i\delta_{nm}}$$

$$H = \frac{1}{2gL}\sum_n\left(\Pi_n\Pi_{-n} + (2\pi n g)^2\phi_n\phi_{-n}\right)$$

Sum of (decoupled) harmonic oscillators with different frequencies

$$w_n = \frac{2\pi|n|}{L}$$

except for $n=0$ case (zero mode)

ladder op $\quad \tilde a_n = \dfrac{1}{\sqrt{4\pi g|n|}}\left(2\pi g|n|\,\phi_n + i\Pi_{-n}\right)$

$[\tilde a_n, \tilde a_m] = 0, \quad [\tilde a_n, \tilde a_m^\dagger] = \delta_{nm}$ except for the zero mod

$$a_n = \begin{cases} -i\sqrt{n}\,\tilde a_n & n>0 \\ i\sqrt{-n}\,a_{-n}^\dagger & n<0\end{cases} \qquad \bar a_n = \begin{cases} -i\sqrt{n}\,\tilde a_{-n} & n>0 \\ i\sqrt{-n}\,\tilde a_n^\dagger & n<0\end{cases}$$

$$[a_n, a_m] = n\delta_{n+m}$$

$$[a_n, \bar a_m] = 0$$

$$[\bar a_n, \bar a_m] = n\delta_{n+m}$$

$\overset{n>0}{\Longleftarrow}$ $\quad a_n = -i\sqrt{n}\dfrac{1}{\sqrt{4\pi g n}}\left(2\pi g n\phi_n + i\Pi_{-n}\right)$

$\bar a_{-n} = i\sqrt{n}\dfrac{1}{\sqrt{4\pi g n}}\left(2\pi g n\phi_n - i\Pi_{-n}\right)$

$$H = \frac{1}{2gL}\pi_0^{\,2} + \frac{2\pi}{L}\sum_{n\neq 0}\left(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n\right)$$

and $\quad [H, a_{-m}] = \frac{2\pi}{L} m\, a_{-m} \Rightarrow f(H)\, a_{-m} = a_{-m} f\left(H + \frac{2\pi m}{L}\right)$

Back to the Fourier modes!

$$\phi_n = \frac{i}{n\sqrt{4\pi g}}\left(a_n - \bar{a}_{-n}\right)$$

$$\therefore \phi(x) = \phi_0 + \frac{i}{\sqrt{4\pi g}}\sum \frac{1}{n}\left(a_n - \bar{a}_{-n}\right)$$

in the Heisenberg picture,

$$\phi_0(t) = \phi_0(0) + \frac{1}{gL}\pi_0 t \qquad [H, \phi_0] = \frac{-i}{gL}\pi_0$$

$$a_n(t) = a_n(0)\, e^{-2\pi i n t/L}$$

$$\bar{a}_n(t) = \bar{a}_n(0)\, e^{-2\pi i n t/L}$$

Thus we can write

$$\phi(x,t) = \underset{\approx}{\phi_0} + \frac{1}{gL}\underset{\approx}{\pi_0} t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(\underset{\approx}{a_n} e^{2\pi i n(x-t)/L} - \underset{\approx}{\bar{a}_{-n}} e^{2\pi i n(x+t)/L}\right)$$

In the Euclidean space and using complex coordinates

$$z = e^{2\pi(t - ix)/L}$$

$$\phi(z,\bar{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \underline{\bar{a}_n}\,\bar{z}^{-n}\right)$$

Primary field

$$i\partial\varphi = \frac{1}{\sqrt{4\pi g}}\sum a_n z^{-n-1} \qquad \left(a_0 = \frac{\pi_0}{\sqrt{4\pi g}}\right)$$

(right-moving and left-moving part
π_0 : cm momentum

Vertex operators

$$V_\alpha(z,\bar z) = :e^{i\alpha\phi(z,\bar z)}:$$

or

$$V_\alpha = \exp\left\{ i\alpha\phi_0 + \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n}\left(a_{-n} z^n + \bar a_{-n} \bar z^n\right)\right\}$$
$$\cdot \exp\left\{ \frac{\alpha}{4\pi g}\pi_0 - \frac{\alpha}{\sqrt{4\pi g}} \sum_{n>0} \frac{1}{n}\left(a_n z^{-n} + \bar a_n \bar z^{-n}\right)\right\}$$

primary, with $h = \bar h = \dfrac{\alpha^2}{8\pi g}$

Consider the OPE

$$\partial\phi(z) V_\alpha(w,\bar w) = \sum \frac{(i\alpha)^n}{n!} \partial\phi(z) :\phi(w,\bar w)^n:$$
$$\sim -\frac{1}{4\pi g}\frac{1}{z-w} \sum \frac{(i\alpha)^n}{(n-1)!} :\phi(w,\bar w)^{n-1}:$$
$$\sim -\frac{i\alpha}{4\pi g}\frac{V_\alpha(w,\bar w)}{z-w}$$

$$T(z)V_\alpha(w,\bar w) = -2\pi g \sum \frac{(i\alpha)^n}{n!} :\partial\phi(z)\partial\phi(z)::\phi(w,w)^n:$$
$$\sim -\frac{1}{8\pi g}\frac{1}{(z-w)^2}\sum \frac{(i\alpha)^n}{(n-2)!} :\phi(w,\bar w)^{n-2}:$$
$$+ \frac{1}{z-w}\sum \frac{(i\alpha)^n}{n!} n :\partial\phi(z)\phi(w,\bar w)^{n-1}:$$
$$\sim \frac{\alpha^2}{8\pi g}\frac{V_\alpha}{(z-w)^2} + \frac{\partial_w V_\alpha}{(z-w)}$$

For the computation of V's we need

$$:e^A::e^B: = :e^{A+B}: e^{\langle AB\rangle}$$

(can be derived from the Hausdorff formula

$$e^A e^B = e^B e^A e^{[A,B]} \quad \text{for } [A,B] = \text{const}$$

$$V_\alpha(z,\bar{z})\, V_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta/4\pi g}\, V_{\alpha+\beta}(w,\bar{w}) + \cdots$$

$$\hookrightarrow \quad V_\alpha(z,\bar{z})\, V_{-\alpha}(w,\bar{w}) \sim |z-w|^{-2\alpha^2} + \cdots$$

$$\left(\text{for } g = \frac{1}{4\pi}\right)$$

<u>The Fock Space</u>

Define the ground state

$$a_n|\alpha\rangle = \bar{a}_n|\alpha\rangle = 0 \qquad n>0$$

$$a_0|\alpha\rangle = \bar{a}_0|\alpha\rangle = \alpha|\alpha\rangle \qquad \left(a_0 = \pi_0/\sqrt{4\pi g}\right)$$

$$T(z) = -2\pi g \,:\partial\phi(z)\,\partial\phi(z): \\ = +\frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-n-m-2} :a_n a_m:$$

$$\Rightarrow \left\{ \begin{array}{ll} L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m} a_m & n\neq 0 \\ L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2 & \end{array} \right.$$

and $$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0\right)$$

Excited states $$a_{-1}^{n_1} a_{-2}^{n_2}\cdots\, \bar{a}_{-1}^{m_1}\, \bar{a}_{-2}^{m_2}\cdots |\alpha\rangle.$$

and $|\alpha\rangle$ from $|0\rangle$

$$\boxed{|\alpha\rangle = V_\alpha(0)|0\rangle}$$

$z=0$

$$e^{-A} B e^{A} = B - [A,B] + \cdots \qquad \text{when } [A,B] = \text{const}$$

$$\hookrightarrow \quad [B, e^A] = e^A[B,A].$$

① $B = \pi_0 .\quad A = i\alpha\phi \Rightarrow [\pi_0, V_\alpha] = \alpha V_\alpha$

② $B = a_n \quad A = i\alpha\phi \Rightarrow [a_n, V_\alpha] = -\alpha z^n V_\alpha \qquad n>0$

Twisted Boundary Conditions

The fact that $n \in \mathbb{Z}$ comes from the periodicity

e.g. for antiperiodic b.c. n is half-integer.

$$\langle \phi(z) \partial\phi(w) \rangle = \sum_{m,n\neq 0} \frac{1}{n} \underbrace{\langle a_n a_m \rangle}_{n\delta_{n+m} \text{ for } n>0,\ 0 \text{ otherwise}} z^{-n} w^{-m-1}$$

$$= \frac{1}{w} \sum_{n>0} \left(\frac{w}{z}\right)^n$$

periodic case $\quad \langle \phi(z) \partial\phi(w) \rangle = \frac{1}{w} \frac{w/z}{1-w/z} = \frac{1}{z-w}$

anti-periodic case $\quad \langle \phi(z) \partial\phi(w) \rangle = \frac{1}{w} \sqrt{\frac{w}{z}} \frac{1}{1-w/z} = \sqrt{\frac{z}{w}} \frac{1}{z-w}$

Now consider $\langle T(z) \rangle$

point-splitting regularization

$\langle T(z) \rangle = -\frac{1}{2} \langle \partial\phi(z+\varepsilon) \partial\phi(z) \rangle$ and keep the finite part

periodic $\quad 0$

antiperiodic $\quad +\frac{1}{16z^2}$

$$+\frac{1}{4} \frac{(1+\frac{\varepsilon}{z})^{\frac{1}{2}} + (1+\frac{\varepsilon}{z})^{-\frac{1}{2}}}{\varepsilon^2} = +\frac{1}{4}\left(\frac{1}{\varepsilon^2} + \frac{1}{4}\frac{\varepsilon^2}{z^2}\right)$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 \cdots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 \cdots$$

On the cylinder the shift is $-\frac{1}{24}$.

$$\therefore \quad -\frac{1}{24}\left(\frac{2\pi}{L}\right)^2 \quad \text{periodic}$$

$$+\frac{1}{48}\left(\frac{2\pi}{L}\right)^2 \quad \text{antiperiodic.}$$

$\boxed{\text{Compactified Boson}}$

$$\phi(x+L, t) \equiv \phi(x, t) + 2\pi m R$$

periodic ~~equal~~ mod $2\pi m R$

$$\phi(x,t) = \phi_0 + \frac{n}{gRL} t + \frac{2\pi R m}{L} x + \frac{i}{\sqrt{\ }} \Sigma (\cdots)$$

m : winding number

Free Fermion

Canonical Quantization on a cylinder

$$S = \frac{g}{2} \int d^2x \, \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi$$

$$\psi(z)\psi(w) \sim \frac{1}{z-w}$$

$$T(z) = -\frac{1}{2} :\psi(z)\partial\psi(z):$$

$$c = \frac{1}{2} \quad \left(h = \frac{1}{2} \text{ for } \psi\right)$$

Mode expansion

$$\psi(x) = \sqrt{\frac{2\pi}{L}} \Sigma b_k e^{2\pi i k x/L}$$

$$\{b_k, b_q\} = \delta_{k+q,0}$$

$$\psi(x,t) = \sqrt{\frac{2\pi}{L}} \Sigma b_k e^{-2\pi k w/L} \qquad w = \cancel{t}\, i x$$

Boundary conditions

$$\psi(x+2\pi L) = \psi(x) \qquad \text{Ramond}$$
$$\psi(x+2\pi L) = -\psi(x) \qquad \text{Neveu-Schwarz}$$

R : mode index k is integral
NS : half-integral

Mapping onto the Plane

$\tau + ix$
cylinder

plane

$$z = e^{2\pi w/L}$$

$$\psi_{cyl}(w) \to \psi_{cyl.}(w) = \left(\frac{dz}{dw}\right)^{1/2} \psi_{pl.}(z)$$
$$= \sqrt{\frac{2\pi z}{L}}\, \psi_{pl}(z)$$

$$\therefore \quad \psi(z) = \sum b_k z^{-k-1/2}$$

$$R : \quad \psi(e^{2\pi i} z) = -\psi(z) \qquad \text{(because of the } z^{-1/2} \text{ factor)}$$
$$NS : \quad \psi(e^{2\pi i} z) = \psi(z)$$

NS sector

$$\langle \psi(z)\psi(w)\rangle = \sum_{k,q\in \mathbb{Z}+1/2} z^{-k-1/2} w^{-q-1/2} \langle b_k b_q\rangle$$
$$= \sum_{\substack{k\in \mathbb{Z}+1/2,\\ k>0}} z^{-k-1/2} w^{k-1/2}$$
$$= \frac{1}{z}\sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n = \frac{1}{z-w}$$

R sector

$$\langle \psi(z)\psi(w)\rangle = \sum_{k,q\in \mathbb{Z}} z^{-k-1/2} w^{-q-1/2} \langle b_k b_q\rangle$$

$b_0^2 = \frac{1}{2}$

$$= \frac{1}{2\sqrt{zw}} + \sum_{1}^{\infty} z^{-k-1/2} w^{+k-1/2}$$
$$= \frac{1}{2}\,\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}$$

Again employing the point-splitting reg. $\boxed{6\text{-}13}$

$$\langle T(z)\rangle = \frac{1}{2}\lim\left(-\langle\psi(z+\varepsilon)\partial\psi(z)\rangle + \frac{1}{\varepsilon^2}\right)$$

NS	0
R	$\frac{1}{16z^2}$

<u>Vacuum energies</u>

	<u>Plane</u>	<u>cylinder</u> $\left(-\frac{c}{24}\right)$ shift
NS	0	$-\frac{1}{48}$
R	$\frac{1}{16}$	$+\frac{1}{24}$

<u>Remark</u> ζ- function regularization

$\langle T_{cylinder}\rangle \sim \Sigma n \sim \zeta(-1)$