5-1

Conformal map

analytic function ~~$w = f(z)$~~ $z = f(w)$

$\bar{z} = \bar{f}(\bar{w})$

$$dz\, d\bar{z} \to \underline{\left(\frac{df}{dw}\right)\left(\frac{d\bar{f}}{d\bar{w}}\right)} dw\, d\bar{w}$$

this takes the form of $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$.

in 2d: <u>any</u> analytic fn defines a <u>local</u> conformal transf.

Global conformal transf.

(bilinear transf
fractional linear

$$f(z) = \frac{az+b}{cz+d} \qquad \underline{ad-bc=1}$$

: invertible.
forms a group
$SL(2, \mathbb{C})$ (2×2 matrix unimodular complex entries)

$\approx SO(3,1)$

not surprising, since in d-dim $SO(d+1, 1)$

5-2

Generators of conformal transf

$$z' = z + \epsilon(z) \qquad \epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}$$

$$\phi'(z') = \phi(z) \Leftarrow \text{invariance..(?)}$$

$$= \phi(z' - \epsilon(z))$$

$$= \phi(z') - \epsilon(z)\partial'\phi(z')$$

$$\therefore \delta\phi = -\epsilon\partial'\phi = -c_n \underbrace{(z^{n+1}\partial_z)}_{\text{generator}} \phi(z)$$

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

$$[l_n, l_m] = [z^{n+1}\partial_z, z^{m+1}\partial_z]$$

$$= (m+1) z^{n+m+1}\partial_z - (n+1) z^{n+m+1}\partial_z$$

$$= -(n-m) z^{n+m+1}\partial_z$$

$$= (n-m) l_{n+m}$$

$$[\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}$$

$$[l, \bar{l}] = 0$$

$SL(2, \mathbb{C})$ is generated by l_{-1}, l_0, l_1

translation (l_{-1}), dilation (l_0), special conformal (l_1)

quasi-primary fields

$$\phi(w,\bar{w}) = \left(\frac{dw}{dz}\right)^{-h}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) \qquad - \circledast$$

primary fields (under infinitesimal transf)

$$\delta\phi = \phi'(z,\bar{z}) - \phi(z,\bar{z})$$

$$= (1+\partial\epsilon)^{-h}(1+\bar{\partial}\bar{\epsilon})^{-\bar{h}}\phi(z-\epsilon,\bar{z}-\bar{\epsilon}) - \phi(z,\bar{z})$$

$$= -(h\phi\,\partial_z\epsilon + \epsilon\,\partial_z\phi) - c.c.$$

EM tensor: quasi-primary but not primary.

Secondary : which are not primary
(e.g. derivative of primary fields).

Correlation functions

combining $\circledast$ and

$$\langle \phi(x')\cdots\phi(y')\rangle = \langle \mathcal{F}_i(\phi(x))\cdots\mathcal{F}_i(\phi(y))\rangle$$

$$x' \to w, \qquad \mathcal{F}_i(\phi(x)) \Rightarrow \left(\frac{dw}{dz}\right)^{-h}\phi(z)$$

$$\langle \phi_1(w_1,\bar{w}_1)\cdots\phi_n(w_n,\bar{w}_n)\rangle =$$

$$\prod_{i=1}^{n}\left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i}\langle \phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle$$

2, 3 - point fns again determined by $h, \bar{h}$.

4- point functions : extra condition for conformal theo

$$\eta = \frac{z_{12} z_{34}}{z_{13} z_{24}} \quad , \quad 1-\eta = \frac{z_{14} z_{23}}{z_{13} z_{24}}$$

$$\frac{\eta}{1-\eta} = \frac{z_{12} z_{34}}{z_{14} z_{23}}$$

$$\langle \phi_1(x_1) \cdots \phi_4(x_4) \rangle = f(\eta, \bar{\eta}) \prod z_{ij}^{h/3 - h_i - h_j} \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}$$

$$\left(\begin{matrix} h, & \cancel{h} = \sum h_i \\ \bar{h} & = \sum \bar{h}_i \end{matrix} \right)$$

Reduced to depend on one coord: since using global tr. it is possible to send $z_1 \to 0$ $z_2 \to 1$ $z_3 \to \infty$

WARD IDENTITIES

for translation $\partial_\mu \langle T^\mu{}_\nu X \rangle = - \sum_i \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle$

rotation $\varepsilon_{\mu\nu} \langle T^{\mu\nu} X \rangle = -i \sum s_i \delta(x - x_i) \langle X \rangle$

dilaton $\langle T^\mu{}_\mu X \rangle = - \sum_i \delta(x - x_i) \Delta_i \langle X \rangle$

When rewritten in complex coordinates,

$$\delta = \frac{1}{\pi} \partial_{\bar{z}} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar{z}}$$

$$2\pi \partial_z \langle T_{\bar{z}z} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{zz} X \rangle = - \sum \partial_{\bar{z}} \frac{1}{z - w_i} \partial_{w_i} \langle X \rangle$$

$$2\pi \partial_z \langle T_{\bar{z}\bar{z}} X \rangle + 2\pi \partial_{\bar{z}} \langle T_{z\bar{z}} X \rangle = - \sum \partial_z \frac{1}{\bar{z} - \bar{w}_i} \partial_{\bar{w}_i} \langle X \rangle$$

$$2 \langle T_{z\bar{z}} X \rangle + 2 \langle T_{\bar{z}z} X \rangle = - \sum \delta(x - x_i) \langle X \rangle \Delta_i$$

$$-2 \langle T_{z\bar{z}} X \rangle + 2 \langle T_{\bar{z}z} X \rangle = - \sum \delta(x - x_i) s_i \langle X \rangle$$

$$\varepsilon^{\cdot\cdot} = 2 \qquad \varepsilon_{\cdot\cdot} = \frac{1}{2}$$

From the last two eqs,

$$2\pi \langle T_{\bar{z}z} X \rangle = - \sum \partial_{\bar{z}} \frac{1}{z-w_i} h_i \langle X \rangle$$

$$2\pi \langle T_{z\bar{z}} X \rangle = - \sum \partial_z \frac{1}{\bar{z}-\bar{w}_i} \bar{h}_i \langle X \rangle$$

Plugging them into the first two eqs.

$$\partial_{\bar{z}} \left\{ \langle T X \rangle - \sum \left[\frac{1}{z-w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z-w_i)^2} \langle X \rangle \right] \right\} = 0$$

$$\partial_z \left\{ \langle \bar{T} X \rangle - \sum \left[\frac{1}{\bar{z}-\bar{w}_i} \partial_{\bar{w}_i} \langle X \rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2} \langle X \rangle \right] \right\} = 0$$

$$T = -2\pi T_{zz}, \quad \bar{T} = -2\pi T_{\bar{z}\bar{z}}$$

$$\therefore \langle T X \rangle = \sum \left\{ \frac{1}{z-w_i} \partial_{w_i} \langle X \rangle + \frac{h_i}{(z-w_i)^2} \langle X \rangle \right\} + \text{reg.} \quad \cdots (*)$$

<u>Conformal Ward identity</u>

$T^{\mu\nu}$: generators of conformal transf.

$$\delta_\epsilon \langle X \rangle = \int_M d^2x \, \partial_\mu \langle T^{\mu\nu}(x) \epsilon_\nu(x) X \rangle$$

$$= \frac{i}{2} \int_C \left(-dz \langle T^{\bar{z}\bar{z}} \epsilon_{\bar{z}} X \rangle + d\bar{z} \langle T^{zz} \epsilon_z X \rangle \right)$$

$$\left(\int d^2x \, \partial_\mu F^\mu = \frac{i}{2} \int_{\partial M} \left(-dz \, F^{\bar{z}} + d\bar{z} \, F^z \right) \right)$$

$$= - \frac{1}{2\pi i} \oint dz \, \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint d\bar{z} \, \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle$$

def: $\underline{\epsilon = \epsilon^z = 2\epsilon_{\bar{z}}, \quad \bar{\epsilon} = \epsilon^{\bar{z}} = 2\epsilon_z}$

the contour needs to include all positions $w_i, \bar{w}_i$

For primary fields, using $\circledast$ and performing the residue integral,

$$\delta_\epsilon \langle X \rangle = - \sum \left(\epsilon(w_i)\partial_{w_i} + \partial\epsilon(w_i)\, h_i \right) \langle X \rangle$$

$$\left(\Leftarrow \quad \delta_\epsilon \phi = -\epsilon\partial\phi - h\phi\,\partial\epsilon \right)$$

(discussion around $(\bar{z}, \bar{T})$ as exercise

Free Fields and the OPE

Representing a product of operators by a sum of terms, each a single operator multiplied by a c-number function of $z-w$

$$A(z)\,B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}$$

Example..
from the Ward identity of $\langle T X \rangle$...

$$T(z)\,\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w})$$

$\sim$: equaility up to regular terms as $w \to z$

Free Boson

Action $S = \frac{g}{2}\int d^2x\ \partial_\mu\varphi\,\partial^\mu\varphi$

2-pt function $\langle \varphi(x)\,\varphi(y)\rangle = -\frac{1}{4\pi g}\ln(x-y)^2 + \text{const}$

Holomorphic part:

$$\boxed{\partial\varphi(z)\,\partial\varphi(w) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2}} \qquad \text{"OPE"}$$

EM tensor $T_{\mu\nu} = g\left(\partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}(\partial\varphi)^2\right)$

In complex coordinates $T = -2\pi g :\partial\varphi\partial\varphi:$

Now let us compute the OPE $\underline{T\partial\varphi}$.

$$\begin{aligned}
T(z)\,\partial\varphi(w) &= -2\pi g\ :\partial\varphi(z)\partial\varphi(z):\ \partial\varphi(w) \\
&\sim -2\pi g\cdot 2\cdot\ :\partial\varphi\,\partial\varphi:\ \partial\varphi \\
&\sim \frac{\partial\varphi(z)}{(z-w)^2} \\
&\sim \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial^2\varphi(w)}{z-w}
\end{aligned}$$

$\therefore$ Implies "$\partial\varphi$" is a primary field with $\underline{h=1}$.

Let's tackle another example:

$$T(z)T(w) = 4\pi^2 g^2 :\partial\varphi(z)\partial\varphi(z): \; :\partial\varphi(w)\partial\varphi(w):$$

$$\sim \frac{1}{2}\frac{1}{(z-w)^4} - \frac{4\pi g :\partial\varphi(z)\partial\varphi(w):}{(z-w)^2}$$

$$\sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

T is NOT a primary field:

Free Fermion

$$S = \frac{g}{2}\int d^2x\, \psi^\dagger \gamma^0\gamma^\mu \partial_\mu \psi = g\int d^2x \left(\bar\psi\partial\bar\psi + \psi\bar\partial\psi\right)$$

$$\begin{cases} \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \\ \gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = 2\begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} \end{cases}$$

Classical eom: $\partial\bar\psi = 0$, $\bar\partial\psi = 0$

First we need to compute the "propagator"

$\approx$ Inverse of the kernel

$$S = \frac{1}{2}\int d^2x\, d^2y\, \psi_i(x)\, A_{ij}(x,y)\, \psi_j(y)$$

kernel $A_{ij} = g\,\delta(x-y)(\gamma^0\gamma^\mu)_{ij}\partial_\mu$

$$\int \left(g\,\delta(x-y)(\gamma^0\gamma^\mu)_{ik}\partial_\mu\right) K_{kj}^{(y-z)} = \delta(x-z)\,\delta_{ij}$$

$$\left((\text{Kernel})\cdot(\text{Propagator}) = \mathbb{1} \right)$$

In terms of complex coordinates

$$2g \begin{pmatrix} \bar\partial & 0 \\ 0 & \partial \end{pmatrix} \overset{\downarrow \text{ kernel}}{\begin{pmatrix} & \\ & \end{pmatrix}} = \frac{1}{\pi} \begin{pmatrix} \bar\partial \frac{1}{z-w} & 0 \\ 0 & \partial \frac{1}{\bar z - \bar w} \end{pmatrix}$$

$$\langle \psi(z)\, \psi(w) \rangle = \frac{1}{2\pi g} \frac{1}{z-w}$$

$$\langle \bar\psi \quad \bar\psi \rangle = \frac{1}{2\pi g} \frac{1}{\bar z - \bar w}$$

$$\langle \psi \quad \bar\psi \rangle = 0$$

OPE for fermions

$$\boxed{\psi(z)\, \psi(w) \sim \frac{1}{2\pi g} \cdot \frac{1}{z-w}}$$

OPE of $T\psi$?

$$T(z) \equiv -2\pi\, T_{zz} = -\frac{1}{2}\pi\, T^{\bar z \bar z}$$

$$= -\pi g : \psi(z)\, \partial\psi(z) :$$

$$T(z)\psi(w) = -\pi g : \psi(z)\, \partial\psi(z) : \psi(w)$$

$$\sim + \frac{1}{2} \frac{1}{z-w} \partial\psi(z) + \frac{1}{2} \frac{\psi(z)}{(z-w)^2}$$

$$\sim \frac{1}{2} \frac{\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}.$$

finally,

$$T(z)\, T(w) = \pi^2 g^2 : \psi(z)\partial\psi(z) :: \psi(w)\, \partial\psi(w) :$$

$$= \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$\boxed{\text{Ghost System}}$ $\Leftarrow$ string theory application

$$S = \frac{g}{2}\int d^2x \; b_{\mu\nu}\,\partial^\mu c^\nu$$: fermionic

b is symm. traceless

eom :

$$\partial^\alpha b_{\alpha\mu} = 0 \qquad \partial^\alpha c^\beta + \partial^\beta c^{\beta\alpha} = 0$$

$$c = c^z, \quad \bar{c} = c^{\bar{z}}, \quad b = b_{zz}, \quad \bar{b} = b_{\bar{z}\bar{z}}$$

$$\bar{\partial} b = 0 \qquad \bar{\partial} c = 0$$

$$\partial \bar{b} = 0 \qquad \partial \bar{c} = 0$$

$$\partial c = -\bar{\partial}\bar{c}$$

$$S = \frac{1}{2}\int d^2x\, d^2y \; b_{\mu\nu} \underbrace{A^{\mu\nu}_{\ \ \alpha}}_{\text{kernel}} c^\alpha$$

$$A^{\mu\nu}_{\ \ \alpha} = \frac{g}{2}\,\delta^\nu_\alpha\,(x-y)\,\partial^\mu$$

$$\left(\frac{1}{2} g\, \delta^\mu_{\ \alpha}\, \partial^\nu\right)\left(K_{\mu\nu}^{\ \ \beta}\right) = \left(\delta(x-y)\, \delta_\alpha^\beta\right)$$

$$\Rightarrow \quad g\, \partial_{\bar{z}} K^{\beta z}_{zz} = \frac{1}{\pi}\,\partial_{\bar{z}}\,\frac{1}{z-w}\; \delta_{\beta z}$$

$$\Rightarrow \quad \boxed{\langle b(z)\, c(w)\rangle = K^z_{zz} = \frac{1}{\pi g}\cdot\frac{1}{z-w}}$$

$$b(z)\, c(w) \sim \frac{1}{\pi g}\,\frac{1}{z-w}$$

$$\langle c(z)\, b(w)\rangle = \frac{1}{\pi g}\cdot\frac{1}{z-w}$$

$$\langle b(z)\, \partial c(w)\rangle = +\frac{1}{\pi g}\cdot\frac{1}{(z-w)^2}$$

$$\langle \partial b(z)\, c(w)\rangle = -\frac{1}{\pi g}\cdot\frac{1}{(z-w)^2}$$ (sign error in 5.109)

EM tensor.

$$T^{\mu\nu} = \frac{g}{2}\left(b^{\mu\alpha}\partial^{\nu} c_{\alpha} - \eta^{\mu\nu} b^{\alpha\beta}\partial_{\alpha} c_{\beta}\right)$$

Belinfante form?

$$B^{\rho\mu\nu} = -\frac{g}{2}\left(b^{\nu\rho} c^{\mu} - b^{\nu\mu} c^{\rho}\right)$$

and $\frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right) = \frac{g}{4}\left(b^{\mu\alpha}\partial^{\nu} c_{\alpha} - b^{\nu\alpha}\partial^{\mu} c_{\alpha}\right)$

and
$$\partial_{\rho} B^{\rho\mu\nu} = -\frac{g}{2}\Big(\partial_{\rho} b^{\nu\rho} c^{\mu} + b^{\nu\rho}\partial_{\rho} c^{\mu} - (\partial_{\rho} b^{\nu\mu}) c^{\rho} - b^{\nu\mu}\partial_{\rho} c^{\rho}\Big) \quad -b^{\nu\rho}\partial^{\mu} c_{\rho}$$

symmetric

$$\left(\partial_{\rho} B^{\rho\mu\nu}\right)_{\text{antisym}} = -\frac{g}{4}\left(-b^{\nu\rho}\partial^{\mu} c_{\rho} + b^{\mu\rho}\partial^{\nu} c_{\rho}\right)$$

$$\therefore\ T^{\mu\nu}_{B} = \frac{g}{2}\Big(b^{\mu\alpha}\partial^{\nu} c_{\alpha} + \underbrace{b^{\nu\alpha}\partial^{\mu} c_{\alpha} + \partial_{\alpha} b^{\mu\nu} c^{\alpha}}_{\text{Belinfante term}} - \eta^{\mu\nu} b^{\alpha\beta}\partial_{\alpha} c_{\beta}\Big)$$

$$T = \pi g : 2\partial c b + c\partial b :$$

$$T(z)\, c(w) = \pi g : 2\partial c b + c\partial b : c$$

$$\sim +2\frac{\partial_z c(z)}{(z-w)} - \frac{c(z)}{(z-w)^2}$$

$$\sim -\frac{c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w} \qquad \text{primary field of } h = -1$$

$$T(z)\, b(w) = \pi g : 2\partial c b + c\partial b : b$$

$$\sim +2\frac{b(z)}{(z-w)^2} - \frac{\partial_z b(z)}{z-w}$$

$$\sim 2\frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w} \qquad \text{Primary of } h = +2$$

Now very important!

$$T(z)T(w) = \pi g^2 :2\partial c(z)\, b(z) + c(z)\partial b(z):$$
$$:2\partial c(w)\, b(w) + c(w)\partial b(w):$$
$$\sim -\frac{13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

Central charge

$$T(z)\,T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$$\delta_\epsilon T(w) = -\frac{1}{2\pi i}\oint dz\, \epsilon(z) T(z) T(w)$$
$$= -\frac{1}{12} c\, \partial^3_w \epsilon(w) - 2T(w)\,\partial_w \epsilon(w) - \epsilon(w)\partial_w T$$

For finite transformations

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w;z\}\right] \quad -(*)$$

$$\{w;z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

Check if $(*)$ reduces to $\delta_\epsilon T$ correctly

$$T'(z+\epsilon) = \cancel{T'(z) + \epsilon(z)\partial T(z)}$$
$$(1-2\epsilon)\left(T(z) - \frac{c}{12}\partial^3\epsilon(z)\right)$$
$$= T(z) - 2\epsilon T(z) - \frac{c}{12}\partial^3\epsilon(z)$$

$$\delta_\epsilon T(z) \equiv T'(z) - T(z) = -\frac{1}{12}c\,\partial_z^3\epsilon(z) - 2\partial_z\epsilon(z)T(z) - \epsilon(z)\,\partial_z T(z)$$

Group property of — (*) (chain rule)

$z \to w \to u$

$$T''(u) = \left(\frac{du}{dw}\right)^{-2}\left[T'(w) - \frac{c}{12}\{u; w\}\right]$$

$$= \left(\frac{du}{dw}\right)^{-2}\left[\left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w; z\}\right] - \frac{c}{12}\{u; w\}\right]$$

$$= \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{u; z\}\right]$$

since $\{u; z\} = \{w; z\} + \left(\frac{dw}{dz}\right)^2 \{u; w\}$

exercise.

physical meaning of c (central charge)

* conformal anomaly
* Casimir energy $\langle T \rangle$.
* Trace anomaly $\langle T^\mu_\mu \rangle = \frac{c}{24\pi} R$.

$z \to w = \frac{L}{2\pi} \ln z$ (plane $\to$ cylinder)

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 z^2 \left(T_{pl}(z) - \frac{c}{12}\left(\frac{1}{2}\frac{1}{z^2}\right)\right)$$

$$= \left(\frac{2\pi}{L}\right)^2 \left(T_{pl}(z) \cdot z^2 - \frac{c}{24}\right)$$