Plan

chap. 2 Quantum Field Theory

chap 4 Global conformal Invariance

chap. 5 Conformal invariance in 2d.

chap 6 The operator formalism

chap 7. Minimal models I

and if time permits
superconformal extension.

Textbook:
Conformal Field Theory
(Di Francesco, P. Mathieu, D. Senechal)

Ref: Nucl. Phys. B241, 333
Belavin, Polyakov, Zamolodchikov.

Quantum Field Theory

§ Quantum Fields

Free Boson

Action $\quad S[\varphi] = \int dx\, dt\, \mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi)$

$$\mathcal{L} = \frac{1}{2}\left(\frac{1}{c^2}\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right)$$

Consider discretized version. in 1 spatial dim

$$L = \sum \frac{1}{2} a \left(\dot{\varphi}_n^2 - \frac{1}{a^2}(\varphi_{n+1} - \varphi_n)^2 - m^2\varphi_n^2\right)$$

Canonical Quantization

$$\Pi_n = \frac{\partial L}{\partial \dot{\varphi}_n} = a\dot{\varphi}_n$$

$$H = \frac{1}{2}\sum\left(\frac{\Pi_n^2}{a} + \frac{1}{a}(\varphi_{n+1} - \varphi_n)^2 + a m^2 \varphi_n^2\right)$$

and promote Π, φ to "operators"

$$[\varphi_n, \Pi_m] = i\delta_{nm}$$

$$[\Pi_n, \Pi_m] = [\varphi_n, \varphi_m] = 0$$

To momentum space representation.

$$\tilde{\varphi}_k = \frac{1}{\sqrt{N}}\sum e^{-2\pi i k n/N}\varphi_n$$

$$\tilde{\pi}_k = \frac{1}{\sqrt{N}}\sum e^{-2\pi i k n/N}\Pi_n$$

$$\left(\tilde{\varphi}_k^\dagger = \tilde{\varphi}_{-k}, \quad \tilde{\pi}_k^\dagger = \tilde{\pi}_{-k}\right)$$

Commutation relation in momentum space

$$[\tilde{\varphi}_k, \tilde{\pi}_q^\dagger] = \frac{1}{N}\sum_m \sum_n e^{-2\pi i (km - qn)/N} [\varphi_m, \pi_n]$$

$$= \frac{i}{N}\sum_n e^{-2\pi i n(k-q)/N}$$

$$= i\delta_{kq}$$

and $H = \frac{1}{2}\sum\left(\frac{1}{a}\tilde{\pi}_k\tilde{\pi}_k^\dagger + a\tilde{\varphi}_k\tilde{\varphi}_k^\dagger\left(m^2 + \frac{2}{a^2}\left(1-\cos\frac{2\pi k}{N}\right)\right)\right)$

$\because$ inverse F.T. $\varphi_n = \frac{1}{\sqrt{N}}\sum e^{2\pi i kn/N}\tilde{\varphi}_k$

$$\sum_n \varphi_n^2 = \sum_k \tilde{\varphi}_k \tilde{\varphi}_k^\dagger$$

$$\frac{1}{N}\sum e^{2\pi i k(n-l)/N} = \delta_{nl}$$

$$\sum \varphi_n\varphi_{n+1} = \frac{1}{N}\sum_n\sum_k\sum_q e^{2\pi i kn/N} e^{2\pi i q(n+1)/N}\tilde{\varphi}_k\tilde{\varphi}_q$$

$$= \sum_k\sum_q \delta_{k,-q} e^{2\pi i q/N}\tilde{\varphi}_k\tilde{\varphi}_q$$

$$= \sum_k \tilde{\varphi}_k\tilde{\varphi}_k^\dagger e^{-2\pi i k/N}$$

real part $\rightarrow$

$$= \sum_k \tilde{\varphi}_k\tilde{\varphi}_k^\dagger \cos\left(\frac{2\pi k}{N}\right)$$

An infinite (?) number of harmonic oscillators.

$$\omega_k^2 = m^2 + \frac{2}{a^2}\left(1 - \cos\frac{2\pi k}{N}\right)$$

As usual introduce ladder ops

$$a_k = \frac{1}{\sqrt{2a\omega_k}}\left(a\omega_k \tilde{\varphi}_k + i\tilde{\pi}_k\right)$$

$$a_k^\dagger = \frac{1}{\sqrt{2a\omega_k}}\left(a\omega_k \tilde{\varphi}_k^\dagger - i\tilde{\pi}_k\right)$$

$$[a_k, a_q^\dagger] = \delta_{kq}$$

$$\mathcal{H} = \frac{1}{2}\sum \left(a_k^\dagger a_k + \frac{1}{2}\right)\omega_k$$

vacuum $\quad a_k|0\rangle = 0$ for all k

excited states $\quad |k_1, k_2, \dots k_n\rangle = a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_n}^\dagger |0\rangle$

with

$$E = E_0 + \sum_i \omega_{k_i}$$

$$E_0 = \frac{1}{2}\sum_k \omega_k$$

Continuum limit $\quad a \to 0$

lattice spacing

$\sum \to \int$

$\vdots$

$$[a(p), a^\dagger(p')] = 2\pi\,\delta(p - p')$$

$$\omega = \sqrt{m^2 + p^2}$$

Free Fermion

wave functions antisymmetric under exchange

$$\{a(p),\ a^\dagger(q)\} = 2\pi\ \delta(p-q)$$
$$\{a(p),\ a(q)\} = \{a^\dagger(p),\ a^\dagger(q)\} = 0$$

"Anticommuting numbers"
= Grassmannian variables

$$\theta_i\theta_j + \theta_j\theta_i = 0$$

(Series expansion stops at some order)

$n=1$ $\quad f = c_0 + c_1\theta$

$n=2$ $\quad f = c_0 + c_1\theta_1 + c_2\theta_2 + c_{12}\theta_1\theta_2$.

$+, -, \times$.

Differentiation. for $f(\theta_1, \theta_2)$

$$\frac{\partial f}{\partial \theta_2} = c_2 - c_{12}\theta_1$$

$$\Rightarrow \quad \{\theta_i, \theta_j\} = 0$$
$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0$$
$$\left\{\theta_i, \frac{\partial}{\partial\theta_j}\right\} = \delta_{ij}$$

Integration = Differentiation

Gaussian Integration
(What physics is ALL about)

$$I = \int d\theta_1 \cdots d\theta_n \exp\left(-\tfrac{1}{2}\theta^T A \theta\right)$$
$$= \mathrm{Pf}(A) \qquad \text{Pfaffian} = (\det)^{1/2}.$$

~~That~~
for instance

$$\int d\theta_1 d\theta_2 \exp\left(-\tfrac{1}{2}(\theta_1\ \theta_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}\right)$$
$$= \int d\theta_1 d\theta_2 \left(1 - \tfrac{1}{2}\cdot 2a \cdot \theta_1\theta_2\right)$$
$$= (+a) \Leftarrow \text{square root of } a^2 = \underline{\underline{\det A}}$$

[But what about the sign error?
oh yes it does cause a lot of problems
in physical examples]

Compare with

$$\int dx\ e^{-\frac{1}{2}ax^2} \propto \frac{1}{\sqrt{a}}$$
$$\int \prod dx_i\ e^{-\frac{1}{2}x^T A x} \propto \frac{1}{\sqrt{\det A}}$$

Dirac Lagrangian

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi \qquad \underline{\bar{\psi} = \psi^\dagger\gamma^0}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

the $\not{\partial} \equiv \gamma^\mu\partial_\mu$

$$\not{\partial}^2 = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = \frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} = \partial^2$$

choose $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\{\gamma^0, \gamma^1\} = 0 \qquad (\gamma^0)^2 = 1, \quad (\gamma^1)^2 = -1$$

$$L = \frac{i}{2}\int dx \left(\psi_1(\partial_t + \partial_x)\psi_1 + \psi_2(\partial_t - \partial_x)\psi_2 \right)$$

Real ψ: Majorana condition since γ^0, γ^1 are all real.

Two non-interacting degrees of freedom

Consider ψ_1 only and discretize

$$L = \frac{i}{2}\sum \left(a\psi_n\dot{\psi}_n + a\cdot\frac{1}{a}\psi_n(\psi_{n+1} - \cancel{\psi_n}) \right)$$

$$\mathcal{H} = -\frac{i}{2}\sum \psi_n\psi_{n+1}, \qquad \{\psi_n, \psi_m\} = \frac{1}{a}\delta_{nm}$$

quantize nonzero

F.T. $\quad \psi_n = \frac{1}{\sqrt{aN}}\sum_k b_k e^{2\pi i k n/N}$

$$\{b_k, b_q^+\} = \delta_{kq}$$

$$\sum_n \psi_n\psi_{n+1} = \frac{1}{aN}\sum_n\sum_k\sum_q b_k b_q e^{2\pi i k n/N} e^{2\pi i q(n+1)/N}$$

$$= \sum b_q^+ b_q \cdot \frac{1}{a} \cdot i\left(\sin \frac{2\pi q}{N}\right)$$

Vacuum

$$b_k|0\rangle = 0 \qquad 0 < k \le N/2$$

excited states

$$b^+_{k_1} \cdots b^+_{k_n}|0\rangle \qquad k_1 \ldots k_n \text{ all distinct.}$$

$$\therefore \quad \sum_1^N \omega_k b_k^+ b_k|0\rangle = \sum_1^{2N/2} \omega_k b_k^+ b_k|0\rangle + \sum_{N/2}^N \omega_k \underbrace{b_k^+ b_k}_{= -b_k b_k^+ + 1}|0\rangle$$

$$= 0 + \sum_{N/2}^N \omega_k$$

$$= -\sum_1^{N/2} \omega_k$$

$$\therefore \quad H = E_0 + \sum^{N/2} \omega_k b_k^+ b_k$$

$$E_0 = -\frac{1}{2} \sum_1^{N/2} \omega_k \,.$$

(negative zero-pt energy)

Path integrals

$$\langle x | e^{-i(K+V)\delta t} | x' \rangle \qquad U = e^{-iHt}$$

time evolution

$$= \langle x | e^{-iK\delta t} e^{-iV\delta t} \mathcal{O}(\delta t^2) | x' \rangle \qquad e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \times e^{-[A,B]} \ldots$$

$$\approx \int \frac{dp}{2\pi} \langle x | e^{-iK\delta t} | p \rangle \langle p | e^{-iV\delta t} | x' \rangle$$

$$\langle p | x \rangle = e^{-ip\cdot x}$$

$$= \int \frac{dp}{2\pi} \exp \left\{ -i\delta t \left[\frac{p^2}{2m} - p \cdot \frac{x-x'}{\delta t} + V(x') \right] \right\}$$

$$= \sqrt{\frac{m}{2\pi i \delta t}} \exp \left\{ \underline{i\delta t \left[\frac{1}{2} m \frac{(x-x')^2}{\delta t^2} - V(x') \right]} \right\}$$

value of the acc

$$e^{\epsilon(A+B)} = e^{\epsilon A} e^{\epsilon B} e^{\mathcal{O}(\epsilon^2)}$$

$$\int \frac{dp}{2\pi} | p \rangle \langle p | \qquad \text{Completeness.}$$

$$; \quad \langle x | e^{-iH\delta t} | x' \rangle = \langle x | U(\delta t) | x' \rangle = \sqrt{\frac{m}{2\pi i \delta t}} \exp(i S)$$

$$\langle x_f | U(t) | x_i \rangle = \left(\frac{m}{2\pi i \delta t} \right)^{N/2} \int \prod dx_j \langle x_f | U\left(\frac{t}{N}\right) | x_{N-1} \rangle$$

$$\langle x_{N-1} | U\left(\frac{t}{N}\right) | x_{N-2} \rangle \ldots \qquad \langle x_1 | U\left(\frac{t}{N}\right) | x_i \rangle$$

$$= \lim_{N\to\infty} \left(\frac{mN}{2\pi i t} \right)^{N/2} \int \prod dx_j \exp iS[x]$$

functional integral measure. $\quad [dx] \equiv \lim_{N\to\infty} \prod \sqrt{\frac{mN}{2\pi i t}} dx_j$

The fundamental result

$$\langle x_f | U(t) | x_i \rangle = \int_{(x_i,0)}^{(x_f,t)} [dx] \exp iS[x]$$

Path integration for quantum fields

$$\langle \varphi_f(\vec{x}, t_f) | \varphi_i(\vec{x}, t_i) \rangle = \int [d\varphi(\vec{x}, t)] e^{iS[\varphi]}$$

$$\langle \psi_f(\vec{x}, t_f) | \psi_i(\vec{x}, t_i) \rangle = \int [d\bar{\psi} d\psi] e^{iS[\bar{\psi}, \psi]}$$

Path integral formulation ((treats ~~spatial~~ space and time in the same way. Better when generalizing to QFT

Correlation functions (or Green functions)

n-point correlation function

$$\langle x(t_1) x(t_2) \ldots x(t_n) \rangle \equiv \langle 0 | T(\hat{x}(t_1) \ldots \hat{x}(t_n)) | 0 \rangle$$

time-ordered product

$$T(\hat{x}(t_1) \ldots \hat{x}(t_n)) = x(t_1) \ldots x(t_n) \quad \text{if } t_1 > t_2 > \ldots > t_n$$

How to compute:

$$\langle x(t_1) x(t_2) \ldots x(t_n) \rangle = \lim_{\varepsilon \to 0} \frac{\int [dx]\, x(t_1) \ldots x(t_n) \exp iS_\varepsilon[x(t)]}{\int [dx] \exp iS_\varepsilon[x(t)]} \quad (*)$$

in $S_\varepsilon[x(t)]$, t is replace by $\underline{t - i\varepsilon}$

Proof: $\hat{x}(t) = e^{iHt} \hat{x} e^{-iHt}$

$\langle x(t_1)\, x(t_2) \cdots x(t_n)\rangle$

$$= \frac{\langle 0|\, \hat{x}\, e^{iH(t_2-t_1)}\, \hat{x}\, e^{iH(t_3-t_2)} \cdots \hat{x}\,|0\rangle}{\langle 0|\, e^{iH(t_n-t_1)}\,|0\rangle} \quad \cdots (*)$$

and
$$\frac{\langle 0|\mathcal{O}_1|0\rangle}{\langle 0|\mathcal{O}_2|0\rangle} = \lim_{\substack{T_i \to \infty \\ T_f \to -\infty}} \frac{\langle \overset{\text{arbitrary}}{\psi_f}|\, e^{-iT_f H(1-i\varepsilon)}\, \mathcal{O}_1\, e^{+iT_i H(1-i\varepsilon)}\,|\psi_i\rangle}{\langle \psi_f| \quad '' \quad \mathcal{O}_2 \quad '' \quad |\psi_i\rangle}$$

since
$$e^{-iT_i H(1-i\varepsilon)}|\psi_i\rangle = \sum e^{-iT_i H(1-i\varepsilon)}|n\rangle\langle n|\psi_i\rangle$$
$$\to e^{-iT_i E_0(1-i\varepsilon)}|0\rangle\langle 0|\psi_i\rangle$$

Now the r.h.s. of $(*)$ is written as

$$\lim_{\substack{T_i \to \infty \\ \epsilon \to 0 \\ T_f \to -\infty}} \frac{\langle \psi_f|\, e^{-iHT_f(1-i\varepsilon)}\, \hat{x}\, e^{-iH(t_1-t_2)(1-i\varepsilon)} \cdots \hat{x}\, e^{+iHT_i(1-i\varepsilon)}\,|\psi_i\rangle}{\langle \psi_f|\, e^{-iH(T_f+T_i+t_1-t_n)(1-i\varepsilon)}\,|\psi_i\rangle}$$

and this is

$$\int_{x_i, T_i}^{x_f, T_f} [dx]\, \left(x(t_1)\, x(t_2) \cdots x(t_n)\right) e^{iS} \qquad t_1 > t_2 \cdots > t_n$$

$$= \int dx_1\, dx_2 \cdots dx_n\; x_1 \cdots x_n \left(\int_{x_1, t_1}^{x_f, T_f} [dx]\, e^{iS}\right)\left(\int_{x_2, t_2}^{x_1, t_1} [dx]\, e^{iS}\right) \cdots$$
$$\cdots \left(\int_{x_i, T_i}^{x_n, t_n} [dx]\, e^{iS}\right)$$

$$= \int dx_1 \cdots dx_n\; x_1 \cdots x_n \;\langle x_f|\, e^{-iH(T_f-t_1)}\,|x_1\rangle \langle x_1|\, e^{-iH(t_1-t_2)}\,|x_2\rangle \cdots$$
$$\langle x_n|\, e^{-iH(t_n-T_i)}\,|x_i\rangle$$

$$= \langle x_f|\, e^{-iH(T_f-t_1)}\, \hat{x}\, e^{-iH(t_1-t_2)}\, \hat{x}\, e^{-iH(t_2-t_3)} \cdots \hat{x}\, e^{-iH(t_n-T_i)}\,|x$$

Euclidean formalism

$$t = -i\tau$$

$$\langle x(\tau_1)\, x(\tau_2) \cdots x(\tau_n)\rangle = \frac{\int [dx]\, x(\tau_1)\cdots x(\tau_n)\, e^{-S_E}}{\int [dx]\, e^{-S_E}}$$

$$i S_E = S[x(t \to -i\tau)] \qquad (iS \to -S_E)$$

$$L_E = -L(x(t\to -i\tau)) \qquad \left(\begin{array}{l} \int dt\, L = S \\ \int (-i d\tau) L_E = i S_E \end{array} \right)$$

for instance, a point particle action becomes identical to the Hamiltonian

$$S_E[x(\tau)] = \int d\tau \left\{ \frac{1}{2m}\dot{x}^2 + V(x) \right\}$$

Minkowski	$\eta_{\mu\nu}$	diag $(+1,-1,\ldots -1)$
Euclidean		diag $(+1,+1,\ldots +1)$

Generating Functional (for correlation functions)

$$Z[j] = \int [dx(t)] \exp - \left\{ S[x(t)] - \int dt\; j(t)\, x(t) \right\}$$

$$\frac{Z[j]}{Z[0]} = \left\langle \exp \int dt\; j(t)\, x(t) \right\rangle$$

or $\left\langle \exp \int d^dx\; j(x)\phi(x) \right\rangle$ for quantum fields

$$= \sum_{n=0}^{\infty} \int dt_1 \cdots dt_n\, \frac{1}{n!}\, j(t_1)\cdots j(t_n) \langle x(t_1) \cdots x(t_n)\rangle$$

$$\Rightarrow \langle x(t_1)\cdots x(t_n)\rangle = \frac{1}{Z[0]} \frac{\delta}{\delta j(t_1)} \cdots \frac{\delta}{\delta j(t_n)} Z[j] \Big|_{j=0}$$

Example: Free Boson

$$S = \frac{g}{2} \int d^2x \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right)$$

2-pt function (or propagator)

$$K(x,y) = \langle \phi(x) \phi(y) \rangle$$

$$S = \frac{1}{2} \int d^2x \, d^2y \; \phi(x) A(x,y) \phi(y)$$

and $A(x,y) = g\, \delta(x-y) \left(-\partial^2 + m^2 \right)$

Propagator = Inverse kernel

$$\boxed{g \left(-\partial_x^2 + m^2 \right) K(x,y) = \delta(x-y)}$$

WHY?

$$Z[j] = \int [d\phi] \, e^{\int d^2x d^2y \left(-\frac{1}{2} \phi(x) A(x,y) \phi(y) + j(x) \delta(x-y) \phi(y) \right)}$$

Gaussian Integration.

$$\int e^{-\frac{1}{2} a x^2 + b x} = \int e^{-\frac{1}{2} a \left(x - \frac{b}{a} \right)^2 + \frac{b^2}{2a}}$$

$$= (\text{independent of } b) \cdot e^{\frac{b^2}{2a}}.$$

$$\therefore \quad Z[j] = (\text{indep of } j) \cdot \exp \frac{1}{2} \int d^2x \, d^2y \; j(x) A^{-1}(x,y) j(y)$$

$\therefore$

$$\boxed{\frac{1}{Z[0]} \frac{\delta}{\delta j(x)} \frac{\delta}{\delta j(y)} Z[j] \Big|_{j=0} = A^{-1}(x,y)}$$

$\boxed{\text{2d case}}$ $K = K(\rho)$ $\boxed{1\text{-}13}$

$$\int_0^{2\pi} d\theta \int_0^r \rho\, d\rho \quad g\left(-\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho K') + m^2 K\right) = 1$$

$$= 2\pi g\left(-r K'(r) + m^2 \int_0^r d\rho\, \rho\, K(\rho)\right)$$

$\underline{\underline{m=0}}$ $K(r) = -\frac{1}{2\pi g} \ln r$

$$\langle \phi(\vec{x})\, \phi(\vec{y}) \rangle = -\frac{1}{4\pi g} \ln |\vec{x} - \vec{y}|^2$$

$m \neq 0$

$$-K' - rK'' + m^2 r K = 0$$

modified Bessel function

$$K(r) = \frac{1}{2\pi g} K_0(mr) \sim e^{-mr}$$

as $r \to \infty$

Wick's theorem

Contraction
(Set the notation)
$$:\phi_1 \phi_2 \phi_3 \phi_4: \; = \; :\phi_1 \phi_3: \langle \phi_2 \phi_4 \rangle$$

Time-ordered product	=	Normal-ordered product	+	contractions

$$T(\phi_1 \phi_2 \phi_3 \phi_4) = :\phi_1 \phi_2 \phi_3 \phi_4: + :\phi_1 \phi_2 \phi_3 \phi_4: + :\phi_1 \phi_2 \phi_3 \phi_4:$$

+ 4 others with single cont

$$+ :\phi_1 \phi_2 \phi_3 \phi_4: +$$ 2 others with double contraction

Simplest case
$$T(\phi_1 \phi_2) = :\phi_1 \phi_2: + \langle \phi_1 \phi_2 \rangle$$

Symmetries and Conservation Laws

Continuous Symmetry Transformations

$$S = \int d^dx \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\begin{cases} x \to x' \\ \phi(x) \to \phi'(x') \end{cases}$$ "active" transformation

$(\phi'(x') = F(\phi(x)))$

And now let's see how the action functional changes

$$S' = \int d^dx' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

$$= \int d^dx' \mathcal{L}(F(\phi(x)), \partial'_\mu F(\phi(x)))$$

$$= \int d^dx \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(F(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu F(\phi(x))\right)$$

Translation:

$$x' = x + a$$

$$\phi'(x+a) = \phi(x)$$

$\frac{\partial x}{\partial x'} = \mathbb{1}$ and F is trivial, $\boxed{S' = S}$

Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\phi'(\Lambda x) = L_\Lambda \phi(x)$$

Lorentz transf. $\boxed{\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}}$ $SO(d-1, 1)$

$$\cancel{\det} \left|\frac{\partial x'}{\partial x}\right| = \det\Lambda = 1$$

$$\therefore S' = \int d^dx\, \mathcal{L}\left(L_\Lambda \phi,\ \Lambda^{-1}\cdot\partial(L_\Lambda\phi)\right)$$

Scalar field $L_\Lambda = 1$ Lorentz inv ~~~~ if ∂_μ appear in a Lorentz-invariant way.
(In terms of Lorentz singlets)

$$\mathcal{L}(\phi,\ \partial_\mu\phi) = f(\phi) + g(\phi)\,\partial_\mu\phi\,\partial^\mu\phi$$

$\boxed{\text{Scale transformation}}$

$$x' = \lambda x$$
$$\phi'(\lambda x) = \lambda^{-\Delta}\phi(x)$$

$$S' = \int d^dx \underbrace{\left|\frac{\partial x'}{\partial x}\right|}_{\lambda^d} \mathcal{L}\left(\lambda^{-\Delta}\phi,\ \lambda^{-\Delta-1}\partial_\mu\phi\right)$$

$\boxed{\text{ex}}$ massless scalar field

$$S = \int d^dx\ \partial_\mu\phi\,\partial^\mu\phi$$

scale invariant if $\boxed{d = 2\Delta+2} \Rightarrow \Delta = \frac{d}{2}-1$

Noether's theorem

Consider infinitesimal transformation

$$x'^\mu = x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a}$$

$$\phi'(x') = \phi(x) + \omega_a\frac{\delta F}{\delta\omega_a}$$

"Generator" of a symmetry transf is defined as

$$\delta_\omega \phi \equiv \phi'(x) - \phi(x)$$

$$= -i\omega_a G_a \phi$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta F}{\delta \omega_a}$$

$$\shortparallel$$

$$\phi'\left(x + \omega_a \frac{\delta x}{\delta \omega_a}\right)$$

$$\shortparallel$$

$$\phi'(x) + \omega_a \frac{\delta x}{\delta \omega_a} \partial_\mu \phi'(x)$$

$$\therefore \quad i G_a \phi = \frac{\delta x}{\delta \omega_a} \partial_\mu \phi - \frac{\delta F}{\delta \omega_a}$$

Examples

$\boxed{\text{Translation}}$ $\Rightarrow P$ $\quad \frac{\delta x}{\delta \omega} = \delta^\mu_\nu, \; F = 1$

$$\boxed{P_\nu = -i\,\partial_\nu}$$

$\boxed{\text{Lorentz transformation}}$ $\quad x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$ ← infinitesimal

antisymmetric. $\quad \frac{1}{2}d(d-1)$ parameters

$$\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho\right)$$

$$F(\phi) = L_\Lambda \phi \qquad L_\Lambda \approx 1 - \frac{i}{2}\omega_{\rho\nu} \underline{\underline{S^{\rho\nu}}}$$

Hermian matrix

$$\frac{i}{2}\omega_{\rho\nu}L^{\rho\nu}\phi = \frac{1}{2}\omega_{\rho\nu}\left(x^\nu\partial^\rho - x^\rho\partial^\nu\right)\phi + \frac{i}{2}\omega_{\rho\nu}S^{\rho\nu}\phi$$

Generator of Lorentz transf

$$\therefore \quad L^{\rho\nu} = i\left(x^\rho\partial^\nu - x^\nu\partial^\rho\right) + S^{\rho\nu}$$

Noether's theorem (Classical)

Every continuous symmetry of the action is associated with a conserved current.

Proof: Consider a "local" transf.

$$\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$$

$$\det(1+E) \cong 1 + \mathrm{tr}\,E \qquad \text{for small } E$$

$$\therefore \text{ Jacobian} = \left|\frac{\partial x'}{\partial x}\right| \approx 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)$$

Inverse
$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$$

Now

$$S' = \int d^dx \left(1 + \partial_\mu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)\right)\mathcal{L}\Bigg(\phi + \omega_a\frac{\delta F}{\delta \omega_a},\ \underbrace{\delta^\nu_\mu - \partial_\mu\left(\omega_a\frac{\delta x^\nu}{\delta\omega_a}\right)}_{\frac{\partial x^\nu}{\partial x'^\mu}} \times \underbrace{\left(\partial_\nu\phi + \partial_\nu\left(\omega_a\frac{\delta F}{\delta \omega_a}\right)\right)}_{\partial_\nu\phi'}\Bigg)$$

$$= (\text{terms with } \omega_a) + (\text{terms with } \partial\omega_a)$$

vanishes if ω generates a symmetry

$$= -\int d^dx\, j^\mu_a\,\partial_\mu\omega_a$$

$$j^{\mu}_{a} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^{\mu}_{\nu}\mathcal{L}\right)\frac{\delta x^\nu}{\delta w_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta F}{\delta w_a}$$

If the fields satisfy the equation of motion S is stationary against ANY variation.

$$\therefore \boxed{\partial_\mu j^{\mu}_{a} = 0}$$

for a symmetry ~~of~~ of the action and sol to the eom

Conserved "charge" (quantity)

$$Q = \int d^{d-1}x\, j^{0}_{a}$$

since $\dot{Q}_a = \int d^{d-1}x\, \partial_0 j^0_a = -\int d^{d-1}x\, \partial_i j^i = -\int_\infty \vec{j}\cdot d\vec{S}$

$= 0$

Correlation functions

$$\langle \phi(x_1') \cdots \phi(x_n') \rangle = \frac{1}{Z}\int [d\phi]\, \phi(x_1') \cdots \phi(x_n') \exp -S[\phi]$$

$$= \frac{1}{Z}\int [d\phi']\, \phi'(x_1') \cdots \phi'(x_n') \exp -S[\phi']$$

$\downarrow$ transf $\qquad\qquad \downarrow$ inv.

$$= \frac{1}{Z}\int [d\phi]\, F(\phi(x_1)) \cdots F(\phi(x_n)) \exp -S[\phi]$$

$$= \langle F(\phi(x_1)) \cdots F(\phi(x_n)) \rangle$$

provide the "action" and the "functional ~~m~~ integr. measure" are invariant under the transf.

cf "Anomaly"

Ward identities (again assuming $[d\phi'] = [d\phi]$)

(Analog of Noether's theorem on QFT)

$$\langle X \rangle = \frac{1}{Z} \int [d\phi'] (X + \delta X) \exp - \left\{ S[\phi] + \int dx\, \partial_\mu j^\mu_a w_a(x) \right\}$$

$$\hookrightarrow \langle \delta X \rangle = \int dx\, \partial_\mu \langle j^\mu_a X \rangle w_a$$

and

$$\langle \delta X \rangle = -i \sum_{i=1}^{n} \left(\phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \right) w_a(x_i)$$

$$= -i \int d^d x\, w_a(x) \sum^{n} \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \delta(x - x_i)$$

the integrands should match since w is arbitrary

$$\boxed{\frac{\partial}{\partial x^\mu} \langle j^\mu_a \phi(x_1) \cdots \phi(x_n) \rangle = -i \sum \delta(x - x_i) \langle \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \rangle} \quad (*)$$

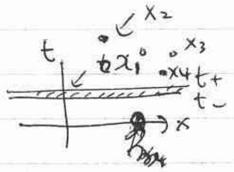
when integrated over the whole space,

$$\delta_w \langle \phi(x_1) \cdots \phi(x_n) \rangle = -i w_a \sum \langle \phi(x_1) \cdots G_a \phi(x_i) \cdots \phi(x_n) \rangle = 0$$

In operator formalism, the charge "Q" generates the transf

$$Q_a = \int d^{d-1}x\, j^0_a(x)$$

$$Y = \phi(x_2) \cdots \phi(x_n)$$



integration of (*)

$$\langle Q_a(t_+) \phi(x_1) Y \rangle - \langle Q_a(t_-) \phi(x_1) Y \rangle = -i \langle G_a \phi(x_1) Y \rangle$$

take the limit $t_- \to t_+$

$$\langle 0 | [Q_a, \phi(x_1)] Y \rangle = -i \langle 0 | G_a \phi(x_1) Y | 0 \rangle$$

$$\boxed{[Q_a, \phi] = -i G_a \phi}$$

Energy-momentum tensor

(Conserved current associated with translation)

$$x'^{\mu} \to x^{\mu} + \epsilon^{\mu}$$

$$\frac{\delta x'^{\mu}}{\delta \epsilon^{\nu}} = \delta^{\mu}_{\nu}, \quad \frac{\delta \phi}{\delta \epsilon^{\nu}} = 0$$

$$\therefore \quad T_c^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi$$

Conservation law $\quad \partial_\mu T_c^{\mu\nu} = 0$

Conserved Charge $\quad P^\nu = \int d^{d-1}x \, T_c^{0\nu}$

$\left(\text{energy: } P^0 = \int d^{d-1}x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right)\right)$

operator formalism $[P_\mu, \phi] = -\partial_\mu \phi$

Belinfante tensor

$T_c^{\mu\nu}$ is not symmetric in general

But

$$T_B{}^{\mu\nu} = T_c{}^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \qquad \boxed{B^{\rho\mu\nu} = -B^{\mu\rho\nu}}$$

Conservation law is not affected

and $T_B{}^{\mu\nu}$ can be made symmetric

provided the theory has rotation symmetry
(Lorentz invariant)

Alternate definition of the EM tensor

$$\delta S = \int d^dx\, T^{\mu\nu} \partial_\mu \epsilon_\nu$$
$$= \frac{1}{2}\int d^dx\, T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

but

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$
$$= (\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \epsilon^\beta)\, g_{\alpha\beta}$$
$$= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

(strictly speaking, $\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$)

$$\therefore \quad \delta S = -\frac{1}{2}\int d^dx\, T^{\mu\nu}\, \delta g_{\mu\nu}$$

ex $\quad S = \int d^dx \sqrt{g}\,\mathcal{L} = \frac{1}{2}\int d^dx \sqrt{g}\,(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2)$

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

$(\det A = e^{\mathrm{tr} \ln A},\ \delta g = g\,(g^{\mu\nu}\delta g_{\mu\nu}))$

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \partial^\mu\phi\,\partial^\nu\phi$$

$$\delta(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi) = \delta g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi = -\delta g_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi$$

* it is wrong to say

$$\delta(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi) = \delta(g_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi) = \delta g_{\mu\nu}\,\partial^\mu\phi\,\partial^\nu\phi.$$