— The cosmic microwave background.

Multipoles $\dfrac{\delta T(\hat{e})}{T} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{e})$

$$a_{\ell m} = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty T_\Theta(k,\ell)\, \mathcal{R}_{\ell m}(k)\, k\,dk$$

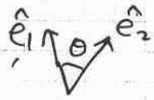
variance $C_\ell \equiv \langle |a_{\ell m}|^2 \rangle$ ← average over ~~scatt. surface~~ univ. ensemble.

(or $\langle a_{\ell m} a^*_{\ell' m'} \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'}$

CMBA spectrum C_ℓ

$$C_\ell = 4\pi \int_0^\infty T_\Theta^2(k,\ell)\, P_{\mathcal{R}}(k) \frac{dk}{k}$$

$$C(\theta) \equiv \left\langle \frac{\delta T(\hat{e}_1)}{T} \frac{\delta T(\hat{e}_2)}{T} \right\rangle$$



$$= \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\cos\theta)$$

scales explored by CMBA

last scattering surface : $2H_0^{-1} = 6000h^{-1}$ Mpc.

$$\frac{x}{100h^{-1}\,\mathrm{Mpc}} \simeq \frac{\theta}{1^\circ}$$

$$k^{-1} \simeq x \simeq \frac{2}{H_0 \ell} = \frac{6000h^{-1}\,\mathrm{Mpc}}{\ell}$$

Silk scale – thickness of last scattering surface.

COBE regime : Sachs–Wolfe effect

$$\frac{\delta T(\hat{e})}{T} = \frac{1}{3}\Phi(\vec{x}_{ls}) = -\frac{1}{5}\mathcal{R}(\vec{x}_{ls})$$

→ transfer function $T_\Theta(k,\ell) = -\frac{1}{5} j_\ell(2k/H_0)$

$$C_\ell = \pi \int_0^\infty \frac{dk}{k} j_\ell^2\left(\frac{2k}{H_0}\right) \delta_H^2(k)$$

$$\ell(\ell+1)C_\ell = \frac{\pi}{2}\underbrace{\left\{\frac{\sqrt{\pi}}{2}\,\ell(\ell+1)\frac{\Gamma[(3-n)/2]\,\Gamma[\ell+(n-1)/2]}{\Gamma((4-n)/2]\,\Gamma[\ell+(5-n)/2]}\right.}_{\simeq 1 \text{ for } n=1.}\delta_H^2(H_0/2)$$

$$\ell(\ell+1)C_\ell \simeq \frac{\pi}{2}\delta_H^2\left(\frac{\ell H_0}{2}\right)$$

Sachs-Wolfe effect.

Brightness function $\Theta(t,\vec{x},\hat{n}) = \dfrac{\delta T(t,\vec{x},\hat{n})}{T(t)}$

The observed anisotropy

$$\frac{\delta T}{T}(\hat{n}) = \Theta(t_{ls},\vec{x}_{ls},\hat{n}) + \left(\frac{\delta T}{T}\right)_{journey}$$

$\vec{x}_{ls} = -x_{ls}\hat{n}$, $x_{ls} = 2/H_0$

redshift perturbation

velocity gradient $u_{ij}(\vec{r},t) \equiv \dfrac{\partial u_i}{\partial r_j}$, $\vec{r} = a(t)\vec{x}$ ← physical coordinate

$$u_{ij} = H(t)\delta_{ij} + \frac{1}{a}\frac{\partial v_i}{\partial x_j}$$

$$-\frac{d\lambda}{\lambda} = \frac{da}{a} + n_i n_j \frac{\partial v_i}{\partial x_j}dx_j$$

using $\vec{V} = -t\vec{\nabla}\Phi$ (MD, $\Omega=1$)

$\left(V_k = \frac{2}{3}\frac{k}{aH}\Phi_k = -\frac{aH}{k}\delta_k\right)$ $\quad \hat{e}\cdot\vec{V}$

$$\left(\frac{\delta T}{T}\right)_{journey} = \int_0^{x_{ls}} \frac{t}{a}\frac{d^2\Phi}{dx^2}dx = \frac{t}{a}\frac{d\Phi}{dx}\Big|_0^{x_{ls}} - \int_0^{x_{ls}} \underbrace{\frac{d}{dx}\left(\frac{t}{a}\right)}_{=\frac{1}{3}}\frac{d\Phi}{dx}dx$$

$$x(t) = \int_t^{t_0}\frac{dt}{a} = 3\left(\frac{t_0}{a_0} - \frac{t}{a}\right)$$

$$\left(\frac{\delta T}{T}\right)_{jour} = \frac{1}{3}[\Phi(x_{ls}) - \Phi(0)] + \hat{e}\cdot[\vec{V}(0,t_0) - V(\vec{x}_{ls},t_{ls})$$

– From horizon entry to galaxy formation

Very large scale that enters the horizon after matter domination

$$\Phi_{\vec{k}} = -\frac{3}{5}\mathcal{R}_{\vec{k}} \longleftarrow \Phi_{\vec{k}} = -\frac{3+3\omega}{5+3\omega}\mathcal{R}_{\vec{k}} \quad \text{with } \omega = 0$$

$$\delta_{\vec{k}}(t) = -\frac{2}{3}\left(\frac{k}{aH}\right)^2 \Phi_{\vec{k}} = \frac{2}{5}\left(\frac{k}{aH}\right)^2 \underbrace{\mathcal{R}_{\vec{k}}}_{\text{constant}}$$

Transfer function

$$\mathcal{R}_k^{(m)\leftarrow \text{MD}} = \underbrace{T(k)}_{\text{transfer function}} \mathcal{R}_k^{\leftarrow \text{primodial value}} \quad (T(k\to 0) = 1)$$

$$\delta_{\vec{k}}(t) = \frac{2}{5}\left(\frac{k}{aH}\right)^2 T(k)\,\mathcal{R}_k \qquad \underline{\left(\frac{1}{aH}\right)^2 \propto a \text{ for MD} \text{ growth fact}}$$

$$P_\delta(k,t) = \frac{4}{25}\underbrace{\left(\frac{k}{aH}\right)^4}_{\text{time-dep.}} T^2(k)\,P_{\mathcal{R}}(k) = \left(\frac{k}{aH}\right)^4 T^2(k)\,\underbrace{\delta_H^2(k)}_{\equiv \frac{4}{25}P_{\mathcal{R}}(k)}$$

Normalization : $\delta_H(k) = 1.91\times 10^{-5}\left(\frac{k}{k_{pivot}}\right)^{\frac{(n-1)}{2}}$; $k_{pivot} \hookrightarrow 7.5 a_0 H_0$

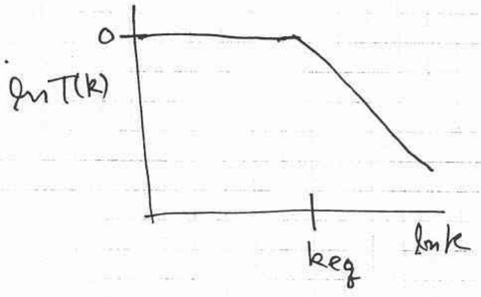
Calculating the transfer function

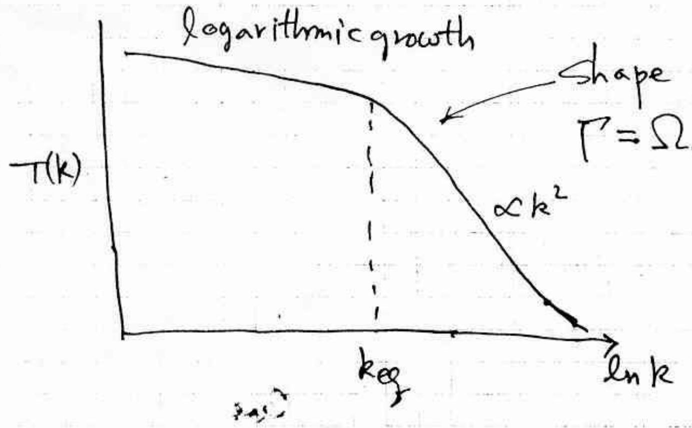
Rough estimate – no growth between horizon entry and equality

$k > k_{eq} = 14\,\Omega_0^{-1} h^{-1}\,\text{Mpc}^{-1}$

$T(k) = (k_{eq}/k)^2$

$k < k_{eq}$: $T(k) = 1$.



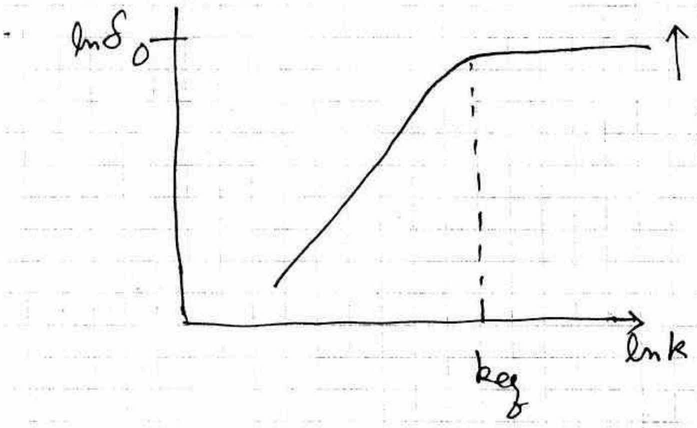


shape parameter

$\Gamma = \Omega_0 h \, e^{(-\Omega_b - \sqrt{2h}\,\Omega_b/\Omega_0)}$

$$\delta \sim T(k)\, a k^2$$

Bottom-up picture of structure formation.



$\uparrow$ as a grows.

small ~~scale~~ scale structure form first.