Full set of equations

- CDM (energy and momentum conservation equation)

$$\frac{\partial \delta_c^N}{\partial \tau} = -kV_c + 3\frac{\partial \Phi}{\partial \tau}$$

$$\frac{\partial V_c}{\partial \tau} = -aHV_c + k\Psi$$

- Baryons

$$\frac{\partial \delta_b^N}{\partial \tau} = -kV_b + 3\frac{\partial \Phi}{\partial \tau}$$

$$\frac{\partial V_b}{\partial \tau} = -aHV_b + c_s^2 k\delta_b^N + k\Psi + \frac{4\rho_\gamma}{3\rho_b} a n_e \sigma_T (V_\gamma - V_b)$$

- Neutrino (the Boltzmann hierarchy for the neutrino brightness function

$$\frac{\partial \delta_\nu^N}{\partial \tau} = -\frac{4}{3}kV_\nu$$ ← massless neutrino

$$\frac{\partial V_\nu}{\partial \tau} = -aH\cancel{V_\nu} + k\left(\frac{1}{4}\delta_\nu^N - \frac{1}{6}\Pi_\nu\right) + k\Psi$$

$$V_\nu \equiv 3\Theta_{\nu 1}$$

$$\Pi_\nu \equiv 12\Theta_{\nu 2}$$

$$\frac{\partial \Theta_{\nu\ell}}{\partial \tau} = \frac{k}{2\ell+1}\left[\ell\Theta_{\nu(\ell-1)} - (\ell+1)\Theta_{\nu(\ell+1)}\right] \quad (\ell \geq 2)$$

- Photon

$$\frac{\partial \delta_\gamma}{\partial \tau} = -\frac{4}{3}kV_\gamma$$

$$\frac{\partial V_\gamma}{\partial \tau} = -aHV_\gamma + k\left(\frac{1}{4}\delta_\gamma - \frac{1}{6}\Pi_\gamma\right) + k\Psi + a n_e \sigma_T (V_b - V_\gamma)$$

$$V_\gamma \equiv 3\Theta_{\gamma 1}$$

$$\Pi_\gamma \equiv 12\Theta_{\gamma 2}$$

$$A \equiv \Theta_2 - 12\hat{E}_2$$

$$\frac{\partial \Theta_2}{\partial \tau} = \frac{2}{15} k V_\gamma - \frac{3}{5} k \Theta_3 + a n_e \sigma_T \left(-\Theta_2 + \frac{1}{10} A\right)$$

$$\frac{\partial \Theta_\ell}{\partial \tau} = \frac{k}{2\ell+1}\left[\ell \Theta_{\ell-1} - (\ell+1)\Theta_{\ell+1}\right] - a n_e \sigma_T \Theta_\ell \quad (\ell \geq 3)$$

$$\frac{\partial \hat{E}_\ell}{\partial \tau} = \frac{k}{2\ell+1}\left[(\ell-2)\hat{E}_{\ell-1} - (\ell+3)\hat{E}_{\ell+1}\right] - a n_e \sigma_T \left[\hat{E}_\ell + \frac{1}{20}\delta_{\ell 2} A\right]$$

$\delta\rho = \sum_i \delta\rho_i$ ← density perturbation

$(\rho + P)V = \sum_i (\rho_i + P_i) V_i$ ← peculiar velocity

$\delta P = \sum_i \delta P_i$ ← pressure purturbation

$P\Pi = \sum_i P_i \Pi_i$ ← anisotropic stress

$\delta = \delta_N + 3\frac{aH}{k}(1+w)V_N$ ← density contrast.

- Metric

$$k^2 \Phi \equiv -4\pi G a^2 \rho \delta$$

$$k^2(\Psi - \Phi) \equiv -8\pi G a^2 P\Pi$$

- Background

Initial conditions

* during radiation domination
* well before horizon entry

$$\Psi_k = -\frac{2}{3}\frac{1}{1+\frac{4}{15}X_\nu}\mathcal{R}_{\vec{k}}$$

$$\Phi_{\vec{k}} = -\frac{2}{3}\frac{1+\frac{2}{5}X_\nu}{1+\frac{4}{15}X_\nu}\mathcal{R}_{\vec{k}}$$

$$V_{\vec{k}} = \frac{1}{2}\frac{k}{aH}\Psi_{\vec{k}}$$

$$\Pi_k = -\frac{2}{5}\left(\frac{k}{aH}\right)^2\Psi_k$$

$$\delta_{\vec{k}} = -\frac{2}{3}\left(\frac{k}{aH}\right)^2\Phi_{\vec{k}}$$

* photon contribution ~~to~~ $\Pi_{\vec{k}}$ is negligible because of the frequent collisions.

* All higher moments of the brightness functions are negligible for photons and neutrinos.

adiabatic condition

$$\delta_{\vec{k}} = \delta_{\vec{k}\gamma} = \delta_{\vec{k}\nu} = \frac{4}{3}\delta_{kb} = \frac{4}{3}\delta_{\vec{k}c}$$

$$X_\nu = \frac{\rho_\nu}{\rho_\gamma+\rho_\nu} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}N_\nu \simeq 0.68$$

$(N_\nu = 3)$

For each species

$$\begin{cases} \delta_i = \delta & \text{for radiation} \\ \delta_i = \frac{3}{4}\delta & \text{for matter} \end{cases}$$

$$V_i = V$$

Transfer functions

Set $\mathcal{R}_{\vec{k}} = 1$

CMB transfer function $\left.\begin{array}{l} T_\Theta \Rightarrow \Theta_\ell \\ T_E \Rightarrow E_\ell \end{array}\right\}$ at the present epoch

Matter-density transfer function $T(k) = \mathcal{R}^{(m)}_{\vec{k}}$

Brightness function

- CMBA measured at positions other than our own, and at earlier

$$\Theta(t, \vec{x}, \hat{n}) \equiv \frac{\delta T(t, \vec{x}, \hat{n})}{T(t)}$$

direction of observation
$\hat{e} = -\hat{n}$

$$\rightarrow \frac{\delta I}{I} = 4\Theta \quad \leftarrow \quad I \propto T^4$$

$$\Theta_{00}(t, \vec{x}) = \frac{1}{4}\delta_r \quad \text{monopole.}$$

- the observed anisotropy

$$\frac{\delta T}{T} = \Theta(t_{ls}, \vec{x}_{ls}, \hat{n}) + \left(\frac{\delta T}{T}\right)_{\text{journey}}$$

$\vec{x}_{ls} = -\chi_{ls}\hat{n}$

$\chi_{ls} = 2/H_0$

additional CMBA ~~[illegible]~~ acquired on the journey toward us

Boltzmann equation

Matter

Gas (Radiation)

$$\frac{df}{dt} = C[f] \quad \leftarrow \text{distribution function}$$

$$C[f_a] = \frac{1}{2E_a}\int Dp_b Dp_c Dp_d \, (2\pi)^4 \delta(p_a^\mu + p_b^\mu - p_c^\mu - p_d^\mu)\,|M|^2$$
$$\times [f_c f_d (1-f_a)(1+f_b) - f_a f_b (1-f_c)(1+f_d)]$$

$$Dp = \frac{d^3p}{(2\pi)^3\, 2E(p)}$$

Thermal equilibrium $\quad f = \dfrac{1}{e^{(E-\mu)/T} \pm 1}$

Gas dynamics in the perturbed universe.

$\vec{q} \equiv a\vec{p} \quad \leftarrow$ constant in unperturbed univer.

$\hat{n} \equiv \dfrac{\vec{q}}{q}, \quad \epsilon \equiv aE = (q^2 + m^2 a^2)^{1/2}$

$$\boxed{\frac{dq}{d\tau}} \longrightarrow \frac{1}{q}\frac{dq}{d\tau} = 2\frac{\partial \Phi}{\partial \tau} - \frac{d\Phi}{d\tau}$$

perturbed Boltzmann equation

$$f(\tau, \vec{x}, \hat{n}, q) = f(\tau, q) + \delta f(\tau, \vec{x}, \hat{n}, q)$$

$$f(\tau, q) = \frac{1}{e^{q/T_0} \pm 1} \quad \text{indep. of } \tau.$$

* Thomson scattering does not change the energy of the photon.

* Fourier transformed.

$$\frac{\partial \delta f}{\partial \tau} + ik\mu \frac{q}{\epsilon}\delta f + q\frac{df}{dq}\left(\frac{\partial \Phi}{\partial \tau} - ik\frac{\epsilon}{q}\mu \Psi\right) = a\left(\frac{d}{dt}\delta f\right)$$

collision terms

$* \; \mu = \hat{k}\cdot\hat{n}$

Boltzmann equation written in <u>Brightness function</u>.

$$\delta f(\tau, \vec{x}, \hat{n}, q) = -q \frac{df}{dq} \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\Theta(\tau, \vec{x}, \hat{n}, q)}$$

$$T_0 \to T_0(1+\Theta)$$

$$\frac{\partial \Theta}{\partial \tau} + ik\mu\Theta - \left(\frac{\partial \Phi}{\partial \tau} - ik\mu\Psi\right)\Theta = a\frac{d\Theta}{dt}$$

<u>Boltzmann hierarchy</u>

$$\Theta(\tau, \vec{k}, \hat{n}) = \sum_{l=0}^{\infty} (-i)^l (2l+1) \Theta_l(\vec{k}, \tau) \underline{P_l(\mu)}$$

Legendre expansion

$$\Theta_l(\vec{k}, \tau_0) \longrightarrow \text{transfer function } T_\Theta(k, l)$$

using $(l+1) P_{l+1}(\mu) = (2l+1)\mu P_l(\mu) - l P_{l-1}(\mu)$

$$\frac{\partial \Theta_0}{\partial \tau} = -k\Theta_1 + 4\frac{\partial \Phi}{\partial \tau} + a\frac{d\Theta_0}{dt}$$

collision terms

$$\frac{\partial \Theta_1}{\partial \tau} = \frac{k}{3}(\Theta_0 - 2\Theta_2) + \frac{k}{3}\Psi + a\frac{d\Theta_1}{dt}$$

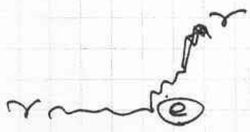
$$\frac{\partial \Theta_l}{\partial \tau} = \frac{k}{2l+1}\left[l\Theta_{l-1} - (l+1)\Theta_{l+1}\right] + a\frac{d\Theta_l}{dt} \quad (l \geq 2)$$

<u>Stress energy tensor</u>

$$\begin{cases} \delta^N = 4\Theta_0 \\ V = 3\Theta_1 \\ \Pi = 12\Theta_2 \end{cases}$$

collision terms

* Thomson scattering



(energy conserving
isotropic momentum.
polarization.

$$\sigma_T \equiv \frac{8\pi}{3}\alpha^2$$

$\frac{df}{dt} \longrightarrow$ perturbation, first order

$t \sim$

$f+\delta f$

$$\frac{df(\hat{n})}{dt} = \frac{n_e\sigma_T}{4\pi}\int d^2n' \left\{ f(\hat{n}')[1+f(\hat{n})) - f(\hat{n})(1+f(\hat{n}')) \right\}$$

electron rest frame

$$= \frac{n_e\sigma_T}{4\pi}\int d^2n' \left[\delta\tilde{f}(\hat{n}') - \delta\tilde{f}(\hat{n}) \right] = \frac{d}{dt}\delta\tilde{f}(n)$$

$$\frac{d\tilde{\Theta}}{dt} = \frac{n_e\sigma_T}{4\pi}\int d^2n' \left(\tilde{\Theta}(\hat{n}') - \tilde{\Theta}(\hat{n}) \right)$$

$$= n_e\sigma_T \left(-\tilde{\Theta}(\hat{n}) + \frac{1}{4\pi}\int d^2n' \tilde{\Theta}(\hat{n}') \right)$$

$$\rightarrow \quad \frac{d\tilde{\Theta}_0}{dt} = 0$$

$$\frac{d\tilde{\Theta}_{\ell\ell}}{dt} = -n_e\sigma_T\tilde{\Theta}_\ell \quad (\ell \geq 1)$$

back to locally orthomal frame.
(electron velocity is V_b)
$\rightarrow$ affect dipole Θ_1, $\sim V_\gamma$, $\underline{V_\gamma = \tilde{V}_\gamma + V_b}$

4.8 δ outside the horizon

- curvature perturbation — constant well outside the horizon
- adiabatic condition

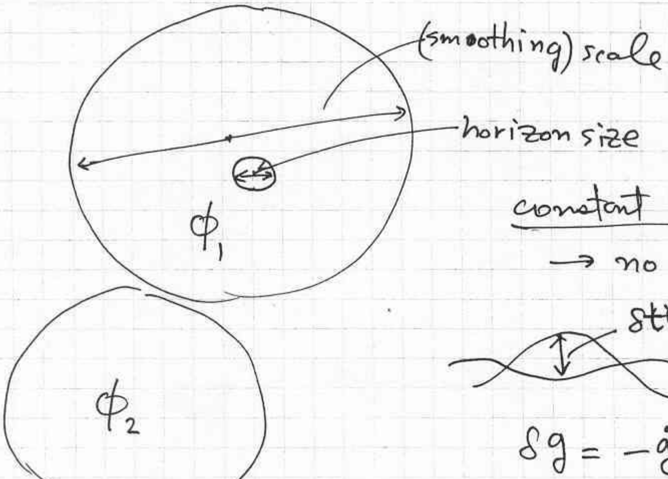
4.8.1 Generalized adiabatic condition

- Inflaton field has one component
- Vacuum fluctuation of non-inflaton fields has no significant effect after inflation.

On scales well outside the horizon, each region evolves like a separate FRW universe.

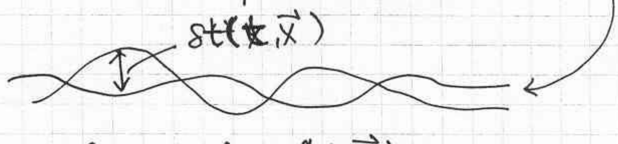
Inflaton field perturbation determining the difference between the locally measured proper times.

→ density perturbation after horizon entry.



constant ϕ - slicing

→ no perturbation



$$\delta g = -\dot{g}(t)\, \delta t(t, \vec{x})$$

for any quantity g.

$$\rightarrow \frac{\delta g}{\dot{g}} = \frac{\delta \rho}{\dot{\rho}} \quad \text{generalized adiabatic condition}$$

continuity equation $\dot{\rho}_i = -3H(\rho_i + P_i)$

$$\frac{\delta \rho_i}{3H(\rho_i + P_i)} = \frac{\delta \rho}{3H(\rho + P)} \rightarrow \frac{\delta_i}{1 + P_i/\rho_i} = \frac{\delta}{1 + P/\rho}$$

RD epoch : $P = \frac{1}{3}\rho$ $\quad \delta_i = \delta$: Radiation

$\delta_i = \frac{3}{4}\delta$ for matter.

$g \to$ total pressure.

$$\frac{\delta P}{\dot{P}} = \frac{\delta \rho}{\dot{\rho}} \longrightarrow$$ constancy of the curvature perturbation.

4.8.2 The curvature perturbation

$$H^2(\vec{x},t) \equiv \frac{8\pi G}{3}\rho(\vec{x},t) + \underbrace{\frac{2}{3}\nabla^2 R(\vec{x},t)}_{\delta K/a^2} \quad \text{cf. } R^{(3)} = 4\frac{k^2}{a^2}R$$

- δP is negligible on scales far outside the horizon.

$$R = -\Phi - V\frac{aH}{k} \qquad \boxed{R = -\frac{H}{\dot{\phi}}\delta\phi}$$

$$3\delta H = \frac{kV}{a} \qquad 3H\dot{\phi} \approx -V'$$

$$H^2 = V/3M_p^2$$

$$V\frac{aH}{k} = \frac{3\delta H a^2 H}{k^2} \qquad 2H\delta H = V'\delta\phi / 3M_p^2$$

$$= \frac{3}{2}\frac{a^2}{k^2}\cdot\frac{V'}{3M_p^2}\delta\phi \qquad 3$$

$$\dot{R} = -\dot{\Phi} - \dot{V}\frac{aH}{k} - V\left(\frac{aH}{k}\right)^{\cdot} \qquad -\Phi$$

$$= -H\frac{\delta P - \frac{2}{3}P\Pi}{\rho + P}$$

Just after the horizon exit.

$$R = -\frac{H}{\dot{\phi}}\delta\phi$$

During RD : $R = -\frac{3}{2}\Phi$

Multifluid evolution equation

- Baryon & CDM

$$\ddot{\delta}_{ki} + 2H\dot{\delta}_{ki} + \left(\frac{k}{a}\right)\frac{\delta P_{ki}}{\rho_i} = 4\pi G\rho\delta_{\vec{k}}$$

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - 4\pi G\rho\delta_{\vec{k}} = 0$$

$$\rightarrow \delta_k = f_{1\vec{k}} D_1 + f_{2\vec{k}} D_2$$

$$\mathrm{MD}: H = \frac{2}{3t} \rightarrow D_1 \propto t^{2/3} \propto a \quad \text{growing}$$

$$D_2 \propto t^{-1} \quad \text{decaying.}$$

$$\ddot{\delta}_k^b + 2H\dot{\delta}_k^b + c_s^2(t)\left(\frac{k}{a}\right)^2\delta_k^b = 4\pi G\rho\delta_k$$

$$\ddot{\delta}_k^c + 2H\dot{\delta}_k^c = 4\pi G\rho\delta_k$$

After decoupling $c_s^2 \rightarrow 0$.

$$S_{cb} = \delta_c - \delta_b \rightarrow \ddot{S}_{cb} + 2H\dot{S}_{cb} = 0$$

$$S_{cb} = \underline{A} + Bt^{-1/3}$$

$$\rightarrow \delta_b \simeq \delta_c$$

Jeans length

$$c_s^2\left(\frac{k}{a}\right)^2 = 4\pi G\rho \rightarrow \lambda_J = \frac{2\pi c_s}{(4\pi G\rho)^{1/2}} \quad \frac{a}{k_J}$$

Jeans mass - sphere of $\lambda_J/2$

$$M_J = \frac{\pi^{5/2}}{6}\frac{c_s^3}{G^{3/2}\rho^{1/2}} \sim 10^6 h^{-1}\left(\frac{T_b}{T_r}\right)^{3/2} M_\odot$$

$\lambda < \lambda_J$: Acoustic oscillation.

— Matter equation

∘ CDM : Non-interacting, $\omega = 0$.

$$\dot{\delta}_c^N = -kV_c + 3\dot{\Phi}$$

$$\dot{V}_c = -aHV_c + k\Psi$$

∘ Baryons

pressure, baryon (electron)-photon interaction

equation of state $\frac{dP_b}{d\rho_b} = c_s^2 \longrightarrow \delta P_b = c_s^2 \delta\rho_b$

$c_s^2 = ?$ $\quad \rho_b = nm$, $P_b = nT_b$, $n \propto a^{-3}$

$$d\rho_b = -3nm\frac{da}{a} = -3nm\, d\ln a$$

$$dP_b = -3nT_b\, d\ln a + nT_b\, d\ln T_b$$

$$c_s^2 = \frac{dP_b}{d\rho_b} = \frac{T_b}{m}\left(1 - \frac{1}{3}\frac{d\ln T_b}{d\ln a}\right)$$

baryon temperature evolution

$$\frac{dT_b}{d\tau} = -2aHT_b + \frac{8}{3}\frac{\mu}{m_e}\frac{\rho_r}{\rho_b}an_e\sigma_T(T_r - T_b)$$

$$\dot{\delta}_b^N = -kV_b + 3\dot{\Phi}$$

free electron density

$$\dot{V}_b = -aHV_b + k\Psi + \underbrace{kc_s^2\delta_b^N}_{\text{pressure}} + \underbrace{\frac{4\rho_r}{3\rho_b}an_e\sigma_T(V_r - V_b)}_{\text{Coupling to photons}}$$

Gas dynamics in flat spacetime

– Distribution function

$$dN = g_i f(t, \vec{r}, \vec{p})\, d^3\vec{r}\, \frac{d^3\vec{p}}{(2\pi)^3}$$

stress-energy tensor

$$T^{\mu\nu} = g_i \int f\, p^\mu p^\nu \frac{d^3\vec{p}}{(2\pi)^3 E}$$