0

graviton

Vacuum fluctuation of inflaton $\delta\phi$ $\qquad$ δh

$\Downarrow$ $\qquad\qquad$ $\downarrow$

classical perturbation $\underline{\delta\phi}$ $\qquad$ δh

setting the initial condition $\rightarrow$ density perturbation $\delta\rho$

(curvature perturbation)

(scalar) (vector) (tensor)

Reheating

primodial values.

RD

Evolution of perturbation

(R-M Equality)

MD

Recombination

$\frac{\delta T}{T}$ CMBA $\qquad$ structure formation. $\frac{\delta\rho}{\rho}$

$\underbrace{\qquad\qquad\qquad}_{\text{observation.}}$

Cosmological perturbation theory.

— The unperturbed universe

$$ds^2 = a^2(\tau)\,(\underset{\substack{\uparrow \\ \text{conformal time}}}{-d\tau^2} + \underbrace{dx^2+dy^2+dz^2}_{\text{flat}})$$

$$T^{\mu\nu} = (\rho+p)u^\mu u^\nu + p g^{\mu\nu} \quad , \quad \underbrace{u^\mu = (-1,0,0,0)}_{\text{rest in the comoving frame}}$$

— Purturbed metric

- No uniquely preferred coordinate system
- Requirement : reduced to the unperturbed metric in the limit of zero perturbation

Most general first-order perturbation

$$\boxed{ds^2 = a^2(\tau)\{-(1+2\underline{\underline{A}})d\tau^2 - \underline{\underline{B_i}}\,d\tau dx^i + [(1+2D)\delta_{ij} + 2E_{ij}]dx^i dx^j\}}$$

lapse function shift function traceless

A, B_i, D, E_{ij} : 10

— perturbed stress energy tensor

Global coordinate system $x^\mu = (\tau, x^i)$

At each spacetime point, a locally orthonormal coordinate system (t, r_i) with the following property

$$dt = a(1+A)d\tau$$
$$dr_i = a\,dx^i$$

Let $v^i = \frac{dr^i}{dt}$: the fluid velocity in the locally orthonormal frame.

Fluid velocity in the global coordinate

$$u^\mu = \frac{dx^\mu}{dt} \qquad u^0 =$$
$$= \frac{dx^\mu}{a\,d\tau} \qquad u^i =$$

density perturbation

$$\boxed{\begin{aligned} T^0{}_0 &= -(\rho+\delta\rho) \\ T^0{}_i &= (\rho+p)(v_i - B_i) \\ T^i{}_0 &= -(\rho+p)v^i \\ T^i{}_j &= (p+\delta p)\delta^i_j + \Sigma^i{}_j \end{aligned}}$$

fluid velocity in locally orthonormal frame.

anisotropic stress (traceless)

pressure perturbation

$* \ \Pi_{ij} \equiv \frac{\Sigma_{ij}}{p}$

$\delta\rho,\ \delta p,\ v^i,\ \Sigma^i_j$: 10

– Scalar – vector – tensor decomposition

$$\delta G_{\mu\nu} = \kappa^2 \delta T_{\mu\nu}$$

metric perturbation stress-energy perturbation

$\Rightarrow$ { evolution equation
constraint equation

- v_i decomposition

$v_i = v_i^S + v_i^V$ (scalar + vector)

$\partial_i S$ (derivative of some scalar function)

In Fourier space

$v_i^S = -\frac{ik_i}{k} V$, (scalar) $k_i v_i^V = 0$

parallel to $\vec{k}$ perpendicular to $\vec{k}$

- Π_{ij} decomposition

$$\Pi_{ij} = \underbrace{\Pi_{ij}^S}_{\partial_i\partial_j S} + \underbrace{\Pi_{ij}^V}_{\partial_i V_j} + \Pi_{ij}^T$$

traceless

$\Pi_{ij}^S = \left(-\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\Pi$ scalar – ①

$\Pi_{ij}^V = \frac{-i}{2k}\left(k_i \Pi_j + k_j \Pi_i\right)$ with $k_i \Pi_i = 0$. vector – ②

$k_i \Pi_{ij}^T = 0$. ②

$\rightarrow 5$

$\vec{k} = (0,0,k)$: $\Pi_{ij}^S = \begin{pmatrix} \Pi & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & -2\Pi \end{pmatrix}$, $\Pi_{ij}^V = -\frac{i}{2}\begin{pmatrix} 0 & 0 & \Pi_1 \\ 0 & 0 & \Pi_2 \\ \Pi_1 & \Pi_2 & 0 \end{pmatrix}$, $\Pi_{ij}^T = \begin{pmatrix} \Pi^\times & \Pi^+ & 0 \\ \Pi^+ & -\Pi^\times & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- B_i Decomposition

$$B_i = B_i^S + B_i^V \ : \ B_i^S = -\frac{ik_i}{k} B, \quad k_i B_i^V = 0$$

- E_{ij} decomposition

$$E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T \ ; \quad E_{ij}^S = \left(-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right) E$$

$$E_{ij}^V = -\frac{i}{2k}(k_i E_j + k_j E_i)$$

$$k_i E_{ij}^T = 0.$$

* Equations break down into three independent sets.

	Metric	Stress-energy
Scalar	A, B, D, E	$\delta\rho, \delta p, V, \Pi$
Vector	B_i^V, E_i	v_i^V, Π_i
tensor	$E_{ij}^T \equiv a^{-2} h_{ij}^T$	Π_{ij}^T

* Physical significance

scalar perturbation — generated by the vacuum fluctuation of the inflaton field
"Density perturbation" included.

vector perturbation — relativistic generalization of purely rotational fluid flow.

tensor perturbation — gravitational waves (generated during inflation)

— Gauge transformations

$$\tilde{x}^0 = x^0 + \delta\tau(x^\mu), \quad \tilde{x}^i = x^i + \delta x^i(x^\mu)$$

tensor : gauge invariant
vector : not interested
scalar :

Metric	stress-energy
$\tilde{A} = A - \dot{\delta\tau} - aH\delta\tau$	$\tilde{V} = V + (\dot{\delta x})$
$\tilde{B} = B + (\dot{\delta x}) + k\delta\tau$	$\tilde{\delta} = \delta + 3(1+w)aH\delta\tau$
$\tilde{D} = D - \frac{k}{3}\delta x - aH\delta\tau$	$\widetilde{\delta p} = \delta p - \dot{p}\delta\tau$
$\tilde{E} = E + k\delta x$	$\tilde{\Pi} = \Pi$

— spatial curvature for scalar perturbations

spatial curvature of the slices of fixed τ

$$R^{(3)} = 4\frac{k^2}{a^2}\left(D + \frac{1}{3}E\right)$$

under the gauge transformation

$$(\tilde{D} + \tfrac{1}{3}\tilde{E}) = (D + \tfrac{1}{3}E) - aH\delta\tau \quad \text{— indep. of } \delta x$$

curvature perturbation

$$\mathcal{R} = D + \tfrac{1}{3}E \quad \text{(in comoving slice)}$$

$$R^{(3)} = 4\frac{k^2}{a^2}\mathcal{R}$$

— Evolution of the perturbations

∘ Tensor perturbations

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + \underset{\sim\sim}{2E^T_{ij}})dx^i dx^j\right]$$

$$\ddot{E}^T_{ij} + 2aH\dot{E}^T_{ij} + k^2 E^T_{ij} = 8\pi G a^2 p \Pi^T_{ij}$$

After horizon entry ($k \gtrsim aH$) : oscillates → gravitational wave

$$h_{ij} \equiv a E^T_{ij}$$

$$\frac{\partial^2 h_{ij}}{\partial t^2} + 3H\frac{\partial h_{ij}}{\partial t} + \left(\frac{k}{a}\right)^2 h_{ij} = 0 \quad \text{if } \underline{\Pi^T_{ij} = 0}$$

same as massless scalar field.

∘ Scalar perturbation

Choice of gauge $\xleftarrow{\mu=0} D_\mu T^{\mu\nu} = 0 \xrightarrow{\mu=i}$

the use of <u>conservation eq</u> & <u>Euler equation</u>

Conformal Newtonian gauge

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + (1-2\Psi)\delta_{ij}dx_i dx_j\right]$$

Evolution equations

$$\dot{\delta}_N = -(1+\omega)(kV_N - 3\dot{\Phi}) + 3aH\omega\delta_N - 3aH\frac{\delta P_N}{\rho}$$

$$\dot{V}_N = -aH(1-3\omega)V_N - \frac{\dot{\omega}}{1+\omega}V_N + k\frac{\delta P_N}{\rho+P} - \frac{2}{3}k\frac{\omega}{1+\omega}\Pi + k\Psi$$

Constraint equation

$$k^2\Phi = -4\pi G a^2\rho\left[\delta_N + 3\frac{aH}{k}(1+\omega)V_N\right]$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 P\Pi$$

Total matter gauge : make a gauge transformation ~~of~~ from conformal Newtonian gauge

$$\delta x = 0, \quad \delta\tau = \frac{V}{k}$$

stress-energy tensor perturbation

$$V = V_N$$

$$\delta = \delta_N + 3\frac{aH}{k}(1+\omega)V_N$$

$$\delta p = \delta P_N - \frac{\dot{P}V_N}{k}$$

Rewrite the equation (Φ, Ψ in conformal Newtonian gauge; $V, \delta, \delta p$ in total matter gauge)

$$\dot{\delta} - 3\omega aH\delta = -(1+\omega)kV - 2aH\omega\Pi$$

$$\dot{V} + aHV = k\frac{\delta p}{\rho+p} - \frac{2}{3}\frac{\omega}{1+\omega}k\Pi + k\Psi$$

$$k^2\Phi = -4\pi G a^2\rho\delta$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 p\Pi$$

— Four fluid : CDM, B, γ, ν

– Matter equation

∘ CDM : Non-interacting, $\omega = 0$.

$$\dot{\delta}_c^N = -kV_c + 3\dot{\Phi}$$

$$\dot{V}_c = -aHV_c + k\Psi$$

∘ Baryons

pressure, baryon (electron)-photon interaction

equation of state $\dfrac{dP_b}{d\rho_b} = c_s^2 \longrightarrow \delta P_b = c_s^2 \delta\rho_b$

$$c_s^2 = ? \quad \rho_b = nm, \quad P_b = nT_b, \quad n \propto a^{-3}$$

$$d\rho_b = -3nm\frac{da}{a} = -3nm\, d\ln a$$

$$dP_b = -3nT_b\, d\ln a + nT_b\, d\ln T_b$$

$$c_s^2 = \frac{dP_b}{d\rho_b} = \frac{T_b}{m}\left(1 - \frac{1}{3}\frac{d\ln T_b}{d\ln a}\right)$$

baryon temperature evolution

$$\frac{dT_b}{d\tau} = -2aHT_b + \frac{8}{3}\frac{\mu}{m_e}\frac{\rho_\gamma}{\rho_b} a n_e \sigma_T (T_\gamma - T_b)$$

free electron density

$$\dot{\delta}_b^N = -kV_b + 3\dot{\Phi}$$

$$\dot{V}_b = -aHV_b + k\Psi + \underbrace{kc_s^2\delta_b^N}_{\text{pressure}} + \underbrace{\frac{4\rho_\gamma}{3\rho_b} a n_e \sigma_T (V_\gamma - V_b)}_{\text{Coupling to photons}}$$

Gas dynamics in flat spacetime

- Distribution function

$$dN = g_i f(t, \vec{r}, \vec{p})\, d^3\vec{r}\, \frac{d^3\vec{p}}{(2\pi)^3}$$

stress-energy tensor

$$T^{\mu\nu} = g_i \int f\, p^\mu p^\nu \frac{d^3\vec{p}}{(2\pi)^3 E}$$