Lyth, Cosmological inflation and large-scale structure.

Ch.7. Scalar fields and the vacuum fluctuation.

7.4 Vacuum fluctuation of the inflaton field

7.4.1. Scalar field equation in an expanding universe

unperturbed equation

$$\ddot{\phi} + 3H\dot{\phi} + \nabla^2\phi + \frac{dV}{d\phi} = 0.$$

possibly contain quantum effects in itself.

$\nabla^2 \to a^{-2}\sum_i \frac{\partial^2}{\partial x_i^2}$

possibly $-\xi R\phi^2$, $R = -(\rho - 3p)/M_{pl}^2 = -12H^2$ during inflation

$|\xi| \ll 1$ is needed.

Perturbation

$$\phi(\vec{x},t) = \phi(t) + \delta\phi(\vec{x},t)$$

$$(\ddot{\phi} + \ddot{\delta\phi}) + 3H(\dot{\phi} + \dot{\delta\phi}) + \nabla^2(\cancel{\phi} + \delta\phi) + V'(\phi + \delta\phi) = 0$$

$V'(\phi + \delta\phi) = V'(\phi) + V''(\phi)\delta\phi + \cdots$

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \nabla^2\delta\phi + m^2\delta\phi = 0.$$

Fourier component.

$$\ddot{\delta\phi}_k + 3H\dot{\delta\phi}_k + \left(\frac{k}{a}\right)^2\delta\phi_k + \cancel{\tfrac{1}{2}}m^2\delta\phi_k = 0.$$

* compare two terms

Horizon exit : $k = aH$ (or $H^{-1} = \frac{a}{k}$) — horizon size — proper wavelength

$\left(\frac{k}{a}\right)^2 \sim H^2 \gg m^2$

(slow-roll condition $\eta = M_p^2 \frac{V''}{V} \ll 1$)

Before horizon exit : m^2 is negligible compared to $\left(\frac{k}{a}\right)^2$

After horizon exit : $a \propto e^{Ht}$. After a few Hubble times m^2 term dominates.

As long as the inflaton mass is negligible

$$(\ddot{\delta\phi}_k) + 3H(\dot{\delta\phi}_k) + \left(\frac{k}{a}\right)^2\delta\phi_k = 0.$$

Well after horizon exit : $\ddot{\delta\phi}_k + 3H\dot{\delta\phi}_k = 0$ $\quad \left(\frac{k}{a}\right)^2$ term neglected

$\delta\phi_k \to$ const.

Well before horizon exit $\ddot{\delta\phi}_k + \left(\frac{k}{a}\right)^2\delta\phi_k = 0$

7.4.4. Vacuum fluctuation after horizon exit.

– Well before horizon exit

$$\delta\phi_{\vec{k}}(t) = \omega_k(t)\, a_{\vec{k}} + \omega_k^*(t)\, a_{-\vec{k}}^\dagger \qquad [a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta_{\vec{k}\vec{k}'} \quad [a_{\vec{k}}, a_{\vec{k}'}] = 0.$$

$$\ddot{\omega}_k + 3H\dot{\omega}_k + \left(\frac{k}{a}\right)^2 \omega_k = 0.$$

→ Solution to this equation matching $\omega_k \equiv V^{-1/2}\sqrt{\frac{1}{2E_k}}\, e^{-iE_k t}$ well before horizon exit. (up to phase factor that varies slowly on the Hubble time scale)

Ignore the variation of H (around the horizon exit)

$$\boxed{\omega_k(t) = L^{-3/2}\frac{H}{(2k^3)^{1/2}}\left(i + \frac{k}{aH}\right)e^{\frac{ik}{aH}}}$$

*check

$$\dot{\omega}_k = \cancel{L^{-3/2}\frac{H}{(2k^3)^{1/2}}}\ \left[-\frac{\dot{a}k}{a^2H}e^{\frac{ik}{aH}} + \left(i+\frac{k}{aH}\right)e^{\frac{ik}{aH}}\left(-\frac{i\dot{a}k}{a^2H}\right)\right]$$

$$= e^{\frac{ik}{aH}}\left[-\cancel{\frac{k}{a}} + \cancel{\frac{k}{a}} - \frac{ik^2}{a^2H}\right]$$

$$\ddot{\omega}_k = e^{\frac{ik}{aH}}\left[+\frac{2i\dot{a}k^2}{a^3H} - \frac{ik^2}{a^2H}\cdot\frac{-i\dot{a}k}{a^2H}\right]$$

$$= e^{\frac{ik}{aH}}\left[2i\frac{k^2}{a^2} - \frac{k^3}{a^3H}\right]$$

$$2i\frac{k^2}{a^2} - \frac{k^3}{a^3H} + 3H\left(-\frac{ik^2}{a^2H}\right) + \left(\frac{k^2}{a^2}\right)\left(i + \frac{k}{aH}\right) = 0.$$

* well before horizon exit

$$\frac{k}{aH} = \frac{k}{aH}\bigg|_{t=T} + \underbrace{\left(-\frac{\dot{a}k}{a^2H}\right)\bigg|_{t=T}}_{-\frac{k}{a}}(t-T) + \cdots$$

$$\omega_k = L^{-3/2}\frac{H}{(2k^3)^{1/2}}\left(\cancel{i} + \underbrace{\frac{k}{aH}}_{\text{negligible}}\right)e^{i\frac{k}{aH}}$$

$\sqrt{\left(\frac{k}{a}\right)^2+m^2}$ $\quad$ $\sqrt{\left(\frac{k}{a}\right)^2+m^2}$ $\quad$ E_k

$$\frac{k}{aH}\bigg|_{t=T} - \frac{k}{a}(t-T) \qquad e^{i\frac{k}{aH}\big|_{t=T}}\, e^{-i\frac{k}{a}(t-T)}$$

slowly varying phase factor.

$$\left(\frac{1}{aL}\right)^{-3/2}\frac{H}{\left(2\left(\frac{k}{a}\right)^3\right)^{1/2}}$$

$$\left[\left(\frac{k}{a}\right)^2+m^2\right]^{3/2} = \left(\frac{1}{aL}\right)^{3/2}\left(\frac{a}{2k}\right)^{1/2}e^{-i\left(\frac{k}{a}\right)t}$$

↑ physical momentum.

The mean square vacuum fluctuation

$$\langle|\delta\phi_k|^2\rangle = |w_k|^2 = \frac{H^2}{2k^3L^3}\left|i+\frac{k}{aH}\right|^2 \qquad k^3\left[1+\frac{m^2}{H^2}\right]^{3/2} = \left[k^2+a^2m^2\right]^{3/2}$$

Before

$$\frac{H^2}{2k^3L^3}\cdot\frac{k^2}{a^2H^2} = \frac{1}{a^3L^3}\,\frac{a}{2k} \longrightarrow \text{After } \frac{H^2}{2L^3k^3}$$

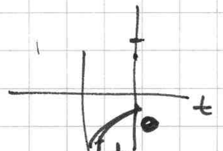
neglect the variation of H

Spectrum

$$P_\phi(k) = V\cdot\frac{k^3}{2\pi^2}|w_k|^2 = L^3\frac{k^3}{2\pi^2}\cdot\frac{H^2}{2L^3k^3} = \left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH}$$

7.4.5 Conformal-time formalism.

Conformal time : $d\tau \equiv \frac{dt}{a}$



(inflation) $\tau = -(aH)^{-1} = -H^{-1}e^{-Ht}$ $\quad \int d\tau = \int\frac{dt}{e^{Ht}} = H^{-1}e^{-Ht}\Big|_{t_0}^{t}$

(RD) $\tau = (aH)^{-1}$ $\quad \left(t^{1/2}\cdot\frac{1}{2t}\right)^{-1}$ $\quad \int\frac{dt}{t^{1/2}} = 2t^{1/2}$, $H=\frac{1}{2t}$

(MD) $\tau = 2(aH)^{-1}$ $\quad 2\left(t^{2/3}\cdot\frac{2}{3t}\right)^{-1}$ $\quad \int\frac{dt}{t^{2/3}} = 3t^{1/3}$, $H=\frac{2}{3t}$

perturbation $u \equiv a\delta\phi$

$$u(\vec{x},\tau) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, u(\vec{k},\tau)\,e^{i\vec{k}\cdot\vec{r}}$$

$$u(\vec{k},\tau) = w(k,\tau)\,a(\vec{k}) + w^*(k,\tau)\,a^\dagger(-\vec{k})$$

Equation of motion

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(\frac{k}{a}\right)^2\delta\phi = 0$$

$$\frac{\partial u}{\partial\tau} = a\frac{\partial}{\partial t}(a\delta\phi) = a^2\dot{\delta\phi} + a\dot{a}\delta\phi$$

$$\frac{\partial^2 u}{\partial\tau^2} = a\frac{\partial}{\partial t}\left(a^2\dot{\delta\phi}+a\dot{a}\delta\phi\right) \qquad \tau = -\frac{1}{\dot{a}}$$

$$= \underbrace{a^3\ddot{\delta\phi} + 3a^2\dot{a}\dot{\delta\phi}}_{-a^3\left(\frac{k}{a}\right)^2\delta\phi = -k^2u} + \underbrace{(a\dot{a}^2 + a^2\ddot{a})\delta\phi}_{(\dot{a}^2+a\ddot{a})u}$$

$a = e^{Ht}$, $\tau = -H^{-1}e^{-Ht}$

$H^2e^{Ht} = -\frac{1}{\tau^2}$

$$\underline{\frac{\partial^2 u(\vec{k},\tau)}{\partial\tau^2} + \left(k^2 - \frac{2}{\tau^2}\right)u(\vec{k},\tau) = 0}$$

7.5 Spectrum of the primordial curvature perturbation

7.5.1 spectrum

"primordial" curvature perturbation $\mathcal{R}_{\vec{k}} = -\frac{H}{\dot{\phi}} \delta\phi_{\vec{k}} \Big|_{t=t_*}$ — inflaton perturbation; $t=t_*$ horizon exit

$$P_\phi(k, t_*) = \left(\frac{H}{2\pi}\right)^2 \Big|_{k=aH}$$

$$P_{\mathcal{R}}(\vec{k}, t) = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Big|_{k=aH}$$

Using the slow-roll formula $\begin{cases} 3H\dot{\phi} = -V' \\ 3H^2 M_p^2 = V \end{cases}$

$$= \left(\frac{3H^2}{V'}\right)^2 \left(\frac{H}{2\pi}\right)^2 = \left(\frac{V/M_p^2}{V'}\right)^2 \frac{V/3M_p^2}{4\pi^2}$$

$$= \frac{V^3}{12\pi^2 M_p^6 V'^2} = \frac{V}{24\pi^2 M_p^4 \epsilon} \qquad \epsilon \equiv \frac{M_p^2 V'^2}{2V^2}$$

Density perturbation

$$\delta_H^2(k) \equiv \frac{4}{25} P_{\mathcal{R}}(k) \simeq \frac{V^3}{75\pi^2 M_p^6 V'^2} = \frac{V}{150\pi^2 M_p^4 \epsilon}$$

value at the horizon entry

COBE normalization

$k_{pivot} \equiv 7.5\, a_0 H_0$

$\delta_H(k_{pivot}) = 1.91 \times 10^{-5}$

adiabatic density perturbation only : $\begin{cases} \dfrac{V^{3/2}}{M_p^3 V'} = 5.2 \times 10^{-4} \\ \dfrac{V^{1/4}}{\epsilon^{1/4}} = 0.027 M_p = 6.6 \times 10^{16}\ \text{GeV} \end{cases}$

→ crucial constraint on ~~the~~ models of inflation (overall normalization of the potential)

7.5.2 spectral index

scale dependence of the spectrum

- "effective spectral index" $n(k)-1 \equiv \frac{d\ln P_R(k)}{d\ln k}$

e.g. $P_R(k) \propto k^{\alpha}$ $\qquad \frac{\ln P_R(k)}{d\ln k} = \frac{d}{d\ln k}\alpha \ln k = \alpha = n-1$

$\therefore n = 1+\alpha$

$$P_R(k) = \frac{V^3}{12\pi^2 M_p^6 V'^2}\Big|_{k=aH}$$

$$d\ln k = \frac{dk}{k} = \frac{daH}{aH} = \frac{\dot{a}dt}{a} = H dt$$

slow-roll $\quad 3H\dot{\phi} + V' \simeq 0, \quad 3M_p^2 H^2 = V$

$$\to dt = -\frac{3H}{V'}d\phi$$

$$d\ln k = H dt = -\frac{3H^2}{V'}d\phi = \frac{-V}{M_p^2 V'}d\phi$$

$$\epsilon = \frac{M_p^2 V'^2}{2V^2}$$

$$\eta = \frac{M_p^2 V''}{V}$$

$$\frac{d}{d\ln k} = -M_p^2 \frac{V'}{V}\frac{d}{d\phi}$$

Derivatives of the slow-roll parameters

$$\frac{d\epsilon}{d\ln k} = -M_p^2\frac{V'}{V}\frac{d}{d\phi}\left(\frac{M_p^2 V'^2}{2V^2}\right) = -M_p^2\frac{V'}{V}\left(\frac{M_p^2 V' V''}{V^2} - \frac{M_p^2 V'^3}{V^3}\right)$$

$$= -2\,\frac{M_p^2 V'^2}{2V^2}\cdot\frac{M_p^2 V''}{V} + 4\left(\frac{M_p^2 V'^2}{2V^2}\right)^2$$

$$= -2\epsilon\eta + 4\epsilon^2$$

$$\frac{d\eta}{d\ln k} = -M_p^2\frac{V'}{V}\frac{d}{d\phi}\left(\frac{M_p^2 V''}{V}\right) = -M_p^2\frac{V'}{V}\left(\frac{M_p^2 V'''}{V} - \frac{M_p^2 V' V''}{V^2}\right)$$

$$= 2\cdot\frac{M_p^2 V'^2}{2V^2}\cdot\frac{M_p^2 V''}{V} - \frac{M_p^4 V' V'''}{V^2} \equiv \xi^2$$

Makes the order in slow-roll parameter manifest

$$= 2\epsilon\eta - \xi^2$$

$$\frac{d\xi^2}{d\ln k} = -M_p^2\frac{V'}{V}\frac{d}{d\phi}\left(\frac{M_p^4 V' V'''}{V^2}\right) = -M_p^2\frac{V'}{V}\left(-\frac{M_p^4 V'^2 V'''}{V^3} + \frac{M_p^4 V'' V'''}{V^2} + M_p^4 V' V''''\right)$$

$$= 2\cdot\frac{M_p^2 V'^2}{2V^2}\cdot\frac{M_p^4 V' V'''}{V^2} - \frac{M_p^2 V''}{V}\frac{M_p^4 V' V'''}{V^2} - \frac{M_p^6 V' V''''}{V^3} \equiv \sigma^3$$

$$= 2\epsilon\xi^2 - \eta\xi^2 - \sigma^2$$

*check $V' \propto \phi^p \quad (p \neq 1,2)$, $V = \dfrac{\phi^{p+1}}{p+1}$

$$\epsilon = \frac{M_p^2 V'^2}{2V^2} = \frac{(p+1)^2}{2}\left(\frac{M_p}{\phi}\right)^2$$

$$\eta = \frac{M_p^2 V''}{V} = (p+1)p\left(\frac{M_p}{\phi}\right)^2 \qquad \underline{\epsilon \sim \eta \sim \xi^2 \sim \sigma}$$

$$\xi^2 = \frac{M_p^4 V' V'''}{V^2} = (p+1)p^2(p-1)\left(\frac{M_p}{\phi}\right)^4$$

$$\sigma^3 = \frac{M_p^6 V' V''''}{V^3} = (p+1)^3 p^2 (p-1)(p-2)\left(\frac{M_p}{\phi}\right)^6$$

$$\frac{d \ln P_R}{d\ln k} = \frac{d\ln V}{d\ln k} - \frac{d\ln\epsilon}{d\ln k} = \frac{1}{V}\cdot - \frac{M_p^2 V'}{\cancel{M_p} V}\frac{d}{d\phi}V - \frac{1}{\epsilon}\frac{d\epsilon}{d\ln k}$$

$$= -2\frac{M_p^2 V'^2}{2V^2} - \frac{1}{\epsilon}(-2\epsilon\eta + 4\epsilon^2)$$

$$= -2\epsilon + 2\eta - 4\epsilon = n-1$$

$$\boxed{n-1 = -6\epsilon + 2\eta}\Big|_{k=aH}$$

negligible for most model of inflaton

Slow-roll : $\epsilon \ll 1$, $|\eta| \ll 1$.

→ Inflation predicts that the variation of the spectrum is small in an interval $\Delta \ln k \sim 1$.

observation: $|n-1| < \underline{0,3}$ ← check for new value.

- The rate of change of n

$$\frac{dn}{d\ln k} = -6\frac{d\epsilon}{d\ln k} + 2\frac{d\eta}{d\ln k} = -6(-2\epsilon\eta + 4\epsilon^2) + 2(2\epsilon\eta - \xi^2)$$

$$= +12\epsilon\eta + 24\epsilon^2 - 2\xi^2$$

- observable scale : $\Delta \ln k \sim 10$

Planck $\left(\dfrac{\Delta \ln k \sim 3}{n \text{ up to } 0{,}01}\right)$, $\dfrac{dn}{d\ln k} \gtrsim 10^{-3}$ observable.

- n has negligible variation : $P_R \propto k^{n-1}$

$n \neq 1$: tilted spectrum

$n > 1$: blue spectrum.

7.7 Gravitational waves

7.7.1 Gravitational-wave spectrum

gravitational wave $h_{+,\times}$ — same as massless scalars (up to numerical factor)

$$P_{grav}(k) = \frac{2}{M_{P.}^2}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH} \longrightarrow \frac{3V/M_P^2}{4\pi^2}$$

spectral index

$$n_{grav} - 1 = \frac{d \ln P_{grav}}{d \ln k} = -M_P^2 \frac{V'}{V}\frac{d}{d\phi}\ln V = -M_P^2 \frac{V'^2}{V^2} = -2\epsilon$$

7.7.2. Relative amplitude of gravitational waves

Contribution to CMB anisotropies

(adiabatic) $$\ell(\ell+1)C_\ell = \frac{\pi}{2}\left\{\frac{\sqrt{\pi}}{2}\ell(\ell+1)\frac{\Gamma\left[\frac{3-n}{2}\right]\Gamma\left(\ell+\frac{n-1}{2}\right)}{\Gamma\left(\frac{4-n}{2}\right)\Gamma\left(\ell+\frac{5-n}{2}\right)}\right\}\delta_H^2\left(\frac{H_0}{2}\right)$$

(grav.)

$$\ell(\ell+1)C_\ell = \frac{\pi}{9}\left(1+\frac{48\pi^2}{385}\right)P_{grav}\, c_\ell \leftarrow \begin{cases} c_2 = 1.118 \\ c_3 = 0.878 \\ \vdots \\ c_\infty = 1 \end{cases}$$

$$r \equiv \frac{C_\ell(\text{grav})}{C_\ell(\text{ad})} \simeq 12.4\epsilon \leftarrow \frac{\frac{V}{2\pi^2 M_P^4} \leftarrow P_{grav.}}{\frac{1}{150\pi^2 M_P^4}\cdot\frac{V}{\epsilon} \leftarrow \delta_H^2}$$

— approximations used.

- ℓ-dependence (scale invariance assumed.)
- Sachs-Wolfe effect only

— the ratio at the quadrapole : $\tilde{r} = 13.8\epsilon$

— Amplitude on much shorter scale :

: too small to be detected by LIGO, VIRGO, …

①

7.7.3 Consistency relation

$$\left.\begin{array}{l} n_{grav} = -2\epsilon \\ r = 12.4\epsilon \end{array}\right\} \Rightarrow \underline{r = -6.2(n_{gra} - 1)}$$

indep. of inflation model.
(for single field)

7.7.4 Why r is negligible in most models of inflation

Improving experimental status of r

Planck with polarized detector : $\Delta r \approx 0.05$

$\underline{r \gtrsim 0.1}$ to be detected (95% confidence level)

$r = 12.4\epsilon \gtrsim 0.1$

$V^{1/4} = 6.6\times10^{16}\,\mathrm{GeV}\,\epsilon^{1/4} \gtrsim (2-6)\times10^{16}\,\mathrm{GeV}$

gravitational wave contribute up to $\ell \sim 100$

$\rightarrow \Delta \ln k \simeq 4$

$N = \frac{1}{M_p^2}\int \frac{V}{V'}d\phi$

$dN = \frac{1}{M_p^2}\frac{V}{V'}d\phi$

$\Downarrow$

$dN = -d\ln k$

$$\frac{1}{M_p}\left|\frac{d\phi}{dN}\right| = M_p\left|\frac{V'}{V}\right| = (2\epsilon)^{1/2} = \left(\frac{r}{6.2}\right)^{1/2}$$

$\Delta N \simeq 4.$ for $2 \lesssim \ell \lesssim 100$

the change of ϕ during $\Delta N \simeq 4 \Rightarrow \Delta\phi$

$$\frac{\Delta\phi}{M_p} \simeq 4\left(\frac{r}{6.2}\right)^{1/2} = 0.5\left(\frac{r}{0.1}\right)^{1/2}$$

$$\underbrace{r \gtrsim 0.1}_{\text{to be detected}} \Rightarrow \underbrace{\Delta\phi \gtrsim 0.5M_p}_{\text{variation in }\phi}$$

must be larger or comparable to M_p $\Rightarrow$ $\underline{\phi \gtrsim M_p \text{ region}}$
uncontrolable.

$\rightarrow$ Almost all of the particle physics-motivated models of inflation give negligible gravitational wave.