Introduction to Cosmology

Basic units

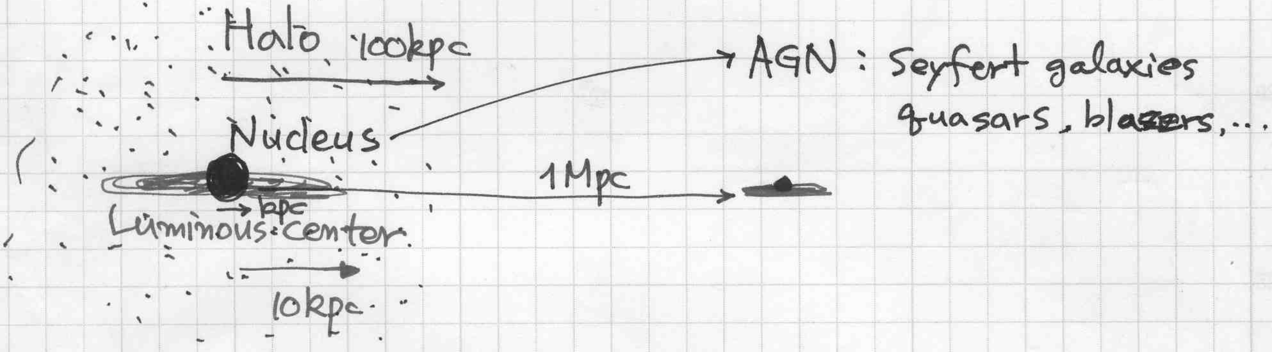
$\hbar = c = k_B = 1$

$1\,pc = 3.26$ light years $= 3.09 \times 10^{16}\,m$

$M_\odot = 1.99 \times 10^{33}\,g$

Building blocks

- Stars : $1 - 10\,M_\odot$
- Galaxies : $10^6\,M_\odot - 10^{12}\,M_\odot$ basic building blocks of the universe.



AGN : Seyfert galaxies
quasars, blazers, ...

- Clusters (Group) of galaxies : 2 – 1000 galaxies
- Superclusters : ~ 100 Mpc
  sheets, filaments & voids
- Above 100 Mpc : Homogeneous distribution of matter
- Observable universe : ~ 10^4 Mpc
- Beyond the observable universe

The universe observed

Observational basis for the standard cosmology

- The expansion of the universe SNIa
  - Hubble constant H_0
  - Deceleration parameter q_0 → matter content.
- The age of the universe t_0
- The matter content of the universe
  - Ω_i , $i = M(B, DM), \Lambda, R, \cdots$
- CBR : CMB, IR, UV, RW, X, γ, ν, CR, $\cdots$ CMBA
- The abundances of light elements : D, ^{3}He, ^{4}He, ^{7}Li
- The distribution of galaxies (SDSS, 2dF 자료참고)

① The expansion

- luminosity distance – red shift
- angular diameter – "
- galaxy count – "

Luminosity distance $d_L = \left(\frac{\mathcal{L}}{4\pi \mathcal{F}}\right)^{1/2}$

"known" absolute luminosity

"measured" flux

Red shift $z = \frac{\lambda_m}{\lambda_0} - 1$

measured wavelength

original "

$$H_0 d_L = z + \frac{1}{2}(1 - q_0) z^2 + \cdots$$

↑ Hubble constant

↑ Deceleration parameter

$H_0 = 100\, h$ Km/s/Mpc

→ Matter content

$$q_0 = \sum_i \Omega_i \frac{1 + 3w_i}{2}$$

H_0^{-1} → cosmic scale

Hubble time $9.78 h^{-1} \times 10^9$ yr

Hubble distance $3000 h^{-1}$ Mpc

② Large scale isotropy & homogeneity

- Uniformity of CMB temperature

$$T_0 = 2.726 \pm 0.01 \, K$$

$$\Delta T_{dipole} = 3.365 \pm 0.027 \, mK$$

$$\Delta T / T \approx 10^{-5}$$

$\Rightarrow$ At last scattering of CMB, the universe was highly isotropic & homogeneous.

- Distribution of galaxies
- Isotropy of X-ray background
- Peculia velocity field

③ The age of the universe

- The expansion rate of the universe
- Dating the oldest stars in globular clusters
- Dating the radio active elements
- The cooling of white dwarf stars
- The cooling of hot gas in clusters

$$10 \, Gyr \lesssim t_0 \lesssim 20 \, Gyr$$

$$t_0 = H_0^{-1} f(\Omega_i)$$

$H_0^{-1} = 9.78 h^{-1} Gyr$

Model	$f(\Omega_i)$
$\Omega_T = 0$	1
$\Omega_T = \Omega_M = 1$	$\frac{2}{3}$
$\Omega_M = 0.2, \Omega_\Lambda = 0.8$	

④ CMB

- The last scattering surface

  $z \sim 1100, \quad t \sim 180{,}000\,(\Omega_0 h^2)^{-1/2}\ \mathrm{yr}$

- Spectrum : Blackbody, with $T = 2.726 \pm 0.01\,\mathrm{K}$
- Anisotropy : Variation in the CMB temperature in different directions.
  - the motion of our local reference frame (the earth) w.r.t the cosmic rest frame.
  - rotation of the universe
  - anisotropic universe
  - the presence of density inhomgeneity

$\Delta T_{\mathrm{dipole}} = 3.365 \pm 0.027\ \mathrm{mK}$

$\rightarrow v_{\mathrm{local}} = 627 \pm 22\ \mathrm{km/s}$ to $\left(\begin{array}{l} RA = 166^\circ \pm 3^\circ \\ DEC = -27.1 \pm 3^\circ \end{array}\right.$

$\Delta T_Q = 11 \pm 3\,\mu\mathrm{K}$

$(\Delta T/T)_{\mathrm{rms}} = 10^{-5}$

→ Powerful test of structure formation theories.

⑤ Light element abundances

- Nucleosynthesis

  $t \simeq 0.01 - 100$ sec, $\quad T \simeq 10\,\mathrm{MeV} - 0.1\,\mathrm{MeV}$

  D : $D/H = \text{few} \times 10^{-5}$ ← No known astrophysical processe can account for these values.

  ^3He : $^3He/H \simeq \text{few} \times 10^{-5}$

  ^4He : $Y = {}^4He/N \simeq 0.25$ ⇒ only in "Nucleosynthesis".

  ^7Li : $^7Li/H \simeq 1\text{-}2 \times 10^{-10}$

- → Determination of baryon density

  $\eta = (4-7)\times 10^{-10}, \quad 0.015 \lesssim \Omega_B h^2 \lesssim 0.026$

- Constraint on the light species.

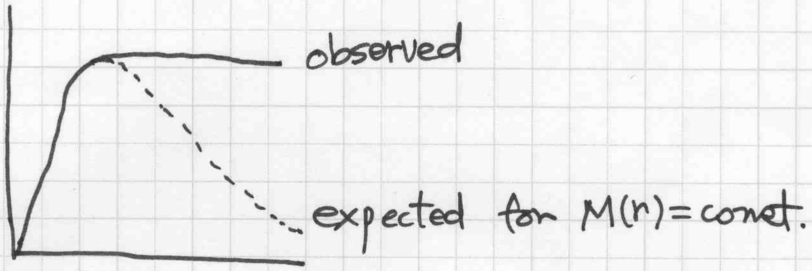
⑥ The matter density

$\Omega_{Lum} \lesssim 0.01$

$\Omega_{Halo} \gtrsim 0.1$ ← rotation curve

- rotation curve

$$\frac{GM(r)m}{r^2} = \frac{mv^2}{r}, \qquad v^2 = \frac{GM(r)}{r}$$



- Ω problem

$\Omega_{LUM} < \Omega_B < \Omega_M$

⑦ Large scale structure

- Hierarchy : stars – nebula – Lyα absorber
  galaxies – clusters – superclusters, voids, walls,
- Galaxy catalogs
- structure formation

One page review of General Relativity

Newtonian gravity : Gravitational potential = Matter (mass)

General relativity : $\underset{\substack{| \\ \text{metric} \\ g_{\mu\nu}}}{\text{Geometry}} = \underset{\substack{| \\ \text{EM tensor} \\ T_{\mu\nu}}}{\text{Matter}}$ (energy)

→ Einstein equation $G_{\mu\nu} = \underset{\substack{\uparrow \\ 8\pi G}}{\kappa^2} T_{\mu\nu}$

Einstein T $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

Ricci T $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$

Riemann T (curvature) $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$

RC symbols (Connection) $\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$

Einstein equation can be derived from the EH action

$$S = \int d^4x \sqrt{-g} \left(\kappa^{-2} R + \mathcal{L}_M \right)$$

Prob. 1. Derive Einstein equation from the EH action.

- Geodesic equation : $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0$

  Path of freely falling particle.
- Isometry and Killing vector

  Symmetry of manifold (spacetime) $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$.

  maximally symmetric space : $\frac{n(n+1)}{2}$ Killing vectors

  2D example : E^2, S^2, H^2

FRW universe

Copernican principle : the universe is pretty much the same "everywhere".

Observational facts

- The distribution of matter and radiation in the "observable universe" is homogeneous & isotropic.
- The universe is "not" static : distant galaxies are receeding from us.

→ Our local Hubble volume during Hubble time

~ spacetime with homogeneous and isotropic spatial sections.

$$M = \mathbb{R} \times \Sigma \quad \leftarrow \text{Maximally symmetric, 3D}$$

RW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d^2\theta + \sin^2\theta d\phi^2)\right]$$

$k = -1$ open H^3

$k = 0$ flat E^3

$k = +1$ closed S^3

Note: The assumption of local homogeneity and isotropy only implies that the spatial metric is locally S^3, H^3 or R^3 and can have ~~global~~ different global properties.

Kinematics of RW metric

- Particle horizon

Consider the light propagation in RW metric, $ds^2=0$.

$$\int_0^t \frac{dt}{a(t)} = \int_0^{r_H} \frac{dr}{\sqrt{1-kr^2}}$$

The proper distance to the horizon

$$d_H(t) = \int_0^{r_H} \sqrt{g_{rr}}\, dr = a(t)\int_0^t \frac{dt'}{a(t')}$$

$(e^{Ht}-1)$

$1-e^{-Ht}$

$e^{-Ht'}dt'$ $H^{-1}(e^{-Ht}-1)$

$H^{-1}(1-e^{Ht})$

- Freely falling particle

$u^\mu \equiv \frac{dx^\mu}{ds}$ "Peculiar velocity"
4-velocity w.r.t. the comoving frame

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0$$

$$\mu=0:\ \frac{du^0}{ds} + \Gamma^0_{\mu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \frac{\dot a}{a}|\vec{u}|^2 = 0.$$

> Prob. 2. For RW metric, verify
>
> $$\Gamma^0_{ij} = \frac{\dot a}{a} g_{ij},\quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot a}{a}\delta^i_j,\quad \Gamma^i_{jk} = ??$$

$u^2 = (u^0)^2 - |\vec{u}|^2 = 1,\quad u^0 du^0 = |\vec{u}|\, d|\vec{u}|$ $\quad (u^0)^{-1}|\vec{u}|\frac{d|\vec{u}|}{ds} + \frac{\dot a}{a}|\vec{u}|^2 = 0$

$\uparrow$

$\frac{dt}{ds}$ $\frac{d|\vec{u}|}{dt}$

$$\frac{\dot{|\vec{u}|}}{|\vec{u}|} = -\frac{\dot a}{a} \longrightarrow |\vec{u}| \propto a^{-1}$$

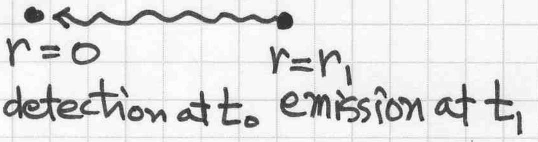
The magnitude of the 3-momentum of a freely falling particle red shifts as a^{-1}.

This is also true for massless particles.

$$\boxed{\frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)} \equiv 1+z \leftarrow \text{red shift parameter}}$$

- Hubble's law

A source (e.g. galaxy) with absolute luminosity $\mathcal{L}$



Luminosity distance

$$d_L^2 \equiv \frac{\mathcal{L}}{4\pi\mathcal{F}_1}$$ ← measured flux

$$\mathcal{F}_1 = \frac{\mathcal{L}}{4\pi\,(a(t_0)r_1)^2\underbrace{(1+z)^2}}$$

- red shift of photon
- increased time interval between detection

$$d_L^2 = [a(t_0)r_1]^2(1+z)^2$$

With the knowledge of $a(t)$, we can express d_L i.t.o. z. Taylor expansion gives

$$H_0 d_L = z + \frac{1}{2}(1-q_0)z^2 + \cdots$$

$$H_0 = \frac{\dot{a}_0}{a_0}, \quad q_0 = -\frac{a_0\ddot{a}_0}{\dot{a}_0^2} = \Omega_0\sum_i\left(\frac{1+3\omega_i}{2}\right)$$

- Galax count – red shift relationship
- Angular diameter – red shift relationship.

Dynamics of RW metric

Einstein equation

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

LHS : RW metric

Connection $\Gamma^0_{ij} = a\dot{a}\tilde{g}_{ij}$, $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$, $\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}$

Ricci tensor $R_{00} = -3\frac{\ddot{a}}{a}$, $R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij}$

Ricci scalar $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$

RHS : Matter content of the universe.

To be consistent with the symmetries of metric

$T_{\mu\nu}$ must be diagonal and $T_{11} = T_{22} = T_{33}$

→ Energy-momentum tensor of perfect fluid.

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p\, g_{\mu\nu}$$

p, ρ : functions of t only

$U_\mu = (1,0,0,0)$ fluid at rest in comoving frame.

$$(00) \quad 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = \kappa^2 \rho$$

$$(ii) \quad 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\kappa^2 p$$

"Friedmann equations"

Prob. 3 Verify Ricci tensor and Friedmann equations.

Prob 4. Derive the conservation equation $T_{\mu\nu}{}^{;\nu} = 0$ and discuss its physical meaning.

$$T_{\mu\nu;}{}^{\nu}=0 \longrightarrow \dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0.$$

We have two independent equations for three unknowns a, ρ & p. So we need one more equation.

- Matter dynamics
- The equation of state enters here. $p=p(\rho)$

Basic equations for dynamical cosmology

① Friedmann eq. $\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}=\frac{\kappa^2}{3}\rho$

② EM conservation eq. $\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0.$

③ Equation of state $p=p(\rho)$

Most kinds of matter content of our interest can be described by the simple equation of state $p=\omega\rho$.

EM conservation eq. $\longrightarrow \rho \propto a^{-3(1+\omega)}$

$\omega=\frac{1}{3}$	$p=\frac{1}{3}\rho$	Radiation	$\rho\propto a^{-4}$
$\omega=0$	$p=0$	Matter	$\rho\propto a^{-3}$
$\omega=-1$	$p=-\rho$	Vacuum energy (Cosmological constant)	$\rho=$ const.

Note 1. Particle physics enters cosmology through matter dynamics, the matter content of the universe.

Note 2. The existence of Big Bang.

$$\frac{\ddot{a}}{a}=-\frac{\kappa^2}{6}(\rho+3p)$$

$\underline{\rho+3p>0} \Longrightarrow a=0$ must have reached at some finite time in the past.

(Weak energy condition)

The expansion rate of the universe

Hubble parameter $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$

Note: The Hubble constant is the present value of Hubble parameter.

Note: H^{-1}, Hubble time and length
time scale for the expansion.

The critical density $\rho_c \equiv \frac{3H^2}{\kappa^2}$

The ratio of density to the critical density.

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \quad \text{Density parameter}$$

Correspondence between the signs of k and $\Omega - 1$

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\kappa^2}{3}\rho \rightarrow \frac{k}{H^2 a^2} = \frac{\rho}{\rho_c} - 1 = \Omega - 1.$$

$\Omega > 1 \longleftrightarrow k = +1$ closed
$\Omega = 1 \longleftrightarrow k = 0$ flat
$\Omega < 1 \longleftrightarrow k = -1$ open

Note: At early times, the curvature term is negligible.

$$\frac{k}{a^2} \propto a^{-2} \ll \frac{\kappa^2}{3}\rho \propto a^{-3} \text{ or } a^{-4} \text{ as } a \rightarrow 0.$$

So, the Friedmann equation at early time is

$$H^2 = \frac{\kappa^2}{3}\rho$$

Note: Behavior of $\Omega - 1$

$$\Omega - 1 = \frac{k}{H^2 a^2} \propto \frac{1}{\rho a^2} \propto \begin{cases} a & \text{for MD} \\ a^2 & \text{for RD} \end{cases}$$

$$|\Omega - 1| \simeq \begin{cases} (1+z)^{-1} & \text{MD} \\ 10^4 (1+z)^{-2} & \text{RD} \end{cases}$$

spatial curvature. $${}^3R^{(3)} = \frac{6k}{a^2} = 6H^2(\Omega-1)$$

radius of curvature $$R^{(3)}_{cur} \equiv a|k|^{-1/2} = \frac{H^{-1}}{|\Omega-1|^{1/2}}$$

- At earlier epoch, the universe was nearly critical.
- The universe close to critical density is very flat.

Integration of Friedmann equation.

$$\left(\frac{\dot a}{a}\right)^2 + \underset{H_0^2(\Omega_0-1)}{\boxed{\frac{k}{a^2}}} = \frac{\kappa^2}{3}\sum_i \rho_i = \frac{\kappa^2}{3}\sum_i \rho_{i0}\left(\frac{a}{a_0}\right)^{-3(1+w_i)}$$

$$p_i = w_i\rho_i \rightarrow \left(\frac{\rho_i}{\rho_{i0}}\right) = \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

$$\left(\frac{\dot a}{a_0}\right)^2 = H_0^2\left[1-\Omega+\sum_i\Omega_i\left(\frac{a}{a_0}\right)^{-(1+3w_i)}\right]$$

$$H_0 t = \int_0^{a/a_0}\left[1-\Omega+\sum_i\Omega_i x^{-1-3w_i}\right]^{-1/2}dx.$$

The age of the universe

$$H_0 t_0 = \int_0^1\left[1-\Omega+\sum_i\Omega_i x^{-1-3w_i}\right]^{-1/2}dx \equiv f(\Omega_i)$$

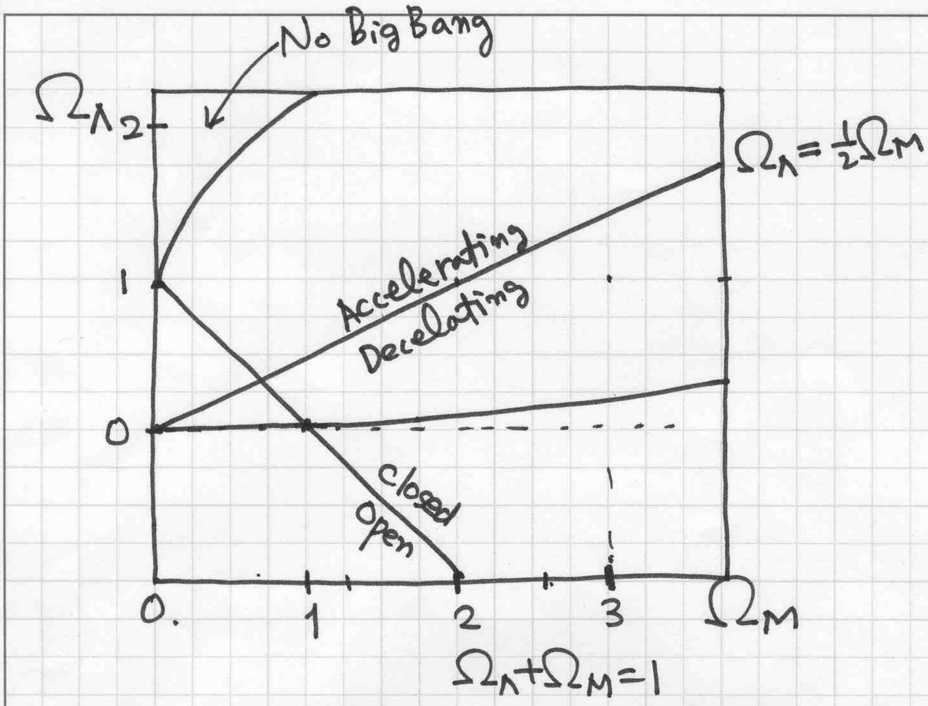
Scale factor as a function of time can be obtained by solving H_0t for a/a_0. $a \propto t^{2/3(1+w)}$ $(w\neq-1)$, e^{H_0t} $(w=-1)$

$t^{2/3}$ (MD), $t^{1/2}$ (RD)

Currently favored values

$\Omega = 1$, $\Omega_\Lambda = 0.7-0.8$, $\Omega_M = 0.2-0.3$

$(\Omega_R \simeq 10^{-4})$



Prob. 5. Identify the lines in the figure and draw the contour lines for constant t_0.

Note. The age of the present universe provides a very powerful constraint on Ω.

Prob. 6. Determine the relationship between the luminosity distance d_L and redshift z, as a function of the cosmological parameters Ω_M, Ω_Λ. To what order in z should we go to determine independently those parameters

prob 7. Calculate the horizon distance $d_H(t)$ as a function of the cosmological parameters.

Equilibrium thermodynamics

- Direct evidence for a hot early universe – CMB
  isotropic, accurate blackbody spectrum with $T \approx 3K$.

$\longrightarrow$ The early universe is filled with
Hot. ideal gas in thermal equilibrium

- Distribution function in thermal equilibrium

$$f(\vec{p}) = \frac{1}{e^{(E-\underset{\substack{\uparrow \\ \text{chemical} \\ \text{potential}}}{\mu})/\underset{\substack{\uparrow \\ \text{temperature}}}{T}} \mp 1} \qquad \begin{matrix} +\text{FD} \\ -\text{BE} \end{matrix}$$

Number density

$$n = \int \frac{g\, d^3\vec{p}}{(2\pi)^3} f(\vec{p}) \quad = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2-m^2)^{1/2} E}{e^{(E-\mu)/T} \pm 1} dE$$

Energy density

$$\rho = \int \frac{g\, d^3\vec{p}}{(2\pi)^3} E(\vec{p}) f(\vec{p}) = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2-m^2)^{1/2} E^2}{e^{(E-\mu)/T} \pm 1} dE$$

Pressure

$$P = \int \frac{g\, d^3\vec{p}}{(2\pi)^3} \frac{|\vec{p}|^2}{3E} f(\vec{p}) = \frac{g}{6\pi^2} \int_m^\infty \frac{(E^2-m^2)^{3/2}}{e^{(E-\mu)/T} \pm 1} dE$$

Relativistic, Non-degenerate $\quad T \gg m,\ T \gg \mu$

$$n = \left[\frac{3}{4}\right] \frac{\zeta(3)}{\pi^2} g T^3, \quad \rho = \left[\frac{7}{8}\right] \frac{\pi^2}{30} g T^4, \quad P = \frac{1}{3}\rho$$

Non-relativistic, $\quad m \gg T$

$$n = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}$$

$$\rho = mn + \frac{3}{2}P$$

$$P = nT \ll \rho$$

The ~~total~~ energy density : The energy density of nonrelativistic species is exponentially smaller than that of relativistic species

$$\rho_R = \frac{\pi^2}{30} g_* T^4$$

$$p_R = \frac{1}{3}\rho_R$$

$$g_* = \sum_{\text{bosons}} g_b + \frac{7}{8}\sum_{\text{fermions}} g_f$$

$\left(\frac{T_i}{T}\right)^4$ — $\left(\frac{T_i}{T}\right)^4$ if $T_i \neq T$.

- Entropy

The entropy in a comoving volume is conserved in thermal equilibrium.

→ Entropy is a useful fiducial quantity during expansion.

$$s = \frac{\rho + p}{T}$$

Entropy in the early universe is dominated by relativistic ptls.

$$s = \frac{2\pi^2}{45} g_* T^3 \qquad \left(\text{cf. relation to } n_\gamma ;\ s = \frac{\pi^4}{45\zeta(3)} g_* n_\gamma\right)$$

Uses of entropy conservation

① Number density $n \propto a^{-3}$, $s \propto a^{-3}$

$Y \equiv \frac{n}{s}$ is convenient to represent the abundances for decoupled particle.

② The evolution of temperature of the universe

$$S \propto g_* T^3 a^3 = \text{const.}$$

$$\underline{T \propto g_*^{-1/3} a^{-1}}$$

Thermal history of the universe

- Thermal equilibrium : The universe has for much of its history been very nearly in thermal equilibrium.
- Departure from thermal equilibrium Makes fossil records of the universe.

- Rule of thumb for thermal equilibrium.

Interaction rate Expansion rate

$$\Gamma_{int} \quad > \quad H$$

$$\Gamma_{int} \equiv n\langle\sigma|v|\rangle = \Gamma_{int}(T)$$

Note: For $\Gamma_{int} = aT^n \ (n>2)$

$$N_{int} = \int_t^\infty \Gamma_{int}(t')dt' = \frac{(\Gamma/H)|_t}{n-2}$$

A particle interacts less than once after $\Gamma = H$

- Rough understanding of decoupling

① interaction mediated by a massless gauge boson

$$\sigma \sim \frac{\alpha^2}{s}, \quad \alpha = \frac{g^2}{4\pi}$$

$$\Gamma \sim n\langle\sigma|v|\rangle \sim T^3 \cdot \frac{\alpha}{T^2} = \alpha^2 T$$

$$H \sim T^2/M_P$$

$$\Gamma/H \sim \alpha^2 M_P/T$$

$$T > \alpha^2 M_P \sim 10^{16}\,\text{GeV}, \quad \Gamma/H \gtrsim 1$$

$$T < \alpha^2 M_P, \quad \text{frozen}$$

② interaction mediated by a massive gauge boson

$$\sigma \sim G_X^2 S \ , \quad G_X \sim \frac{\alpha}{m_X}$$

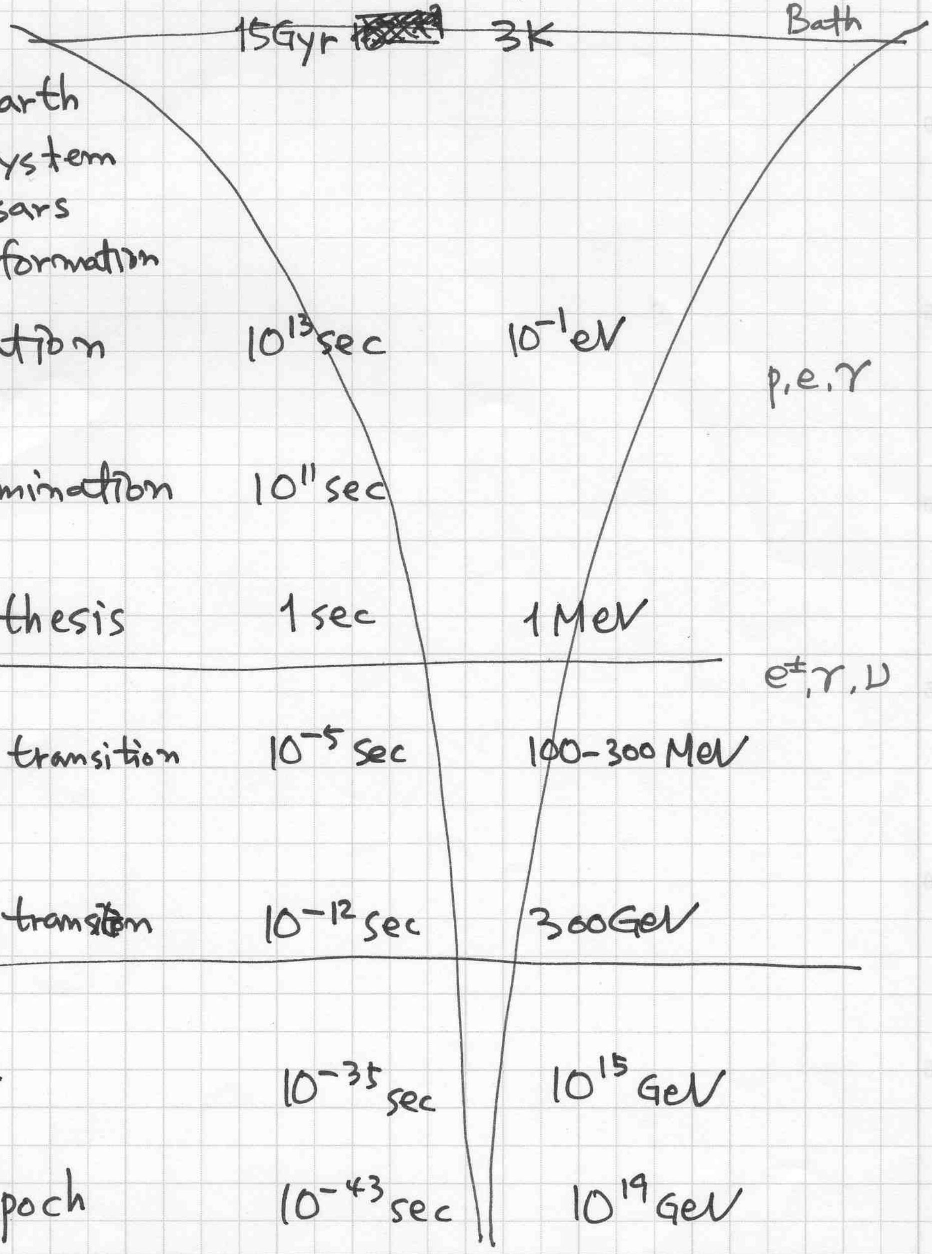
$$\Gamma \sim n\langle\sigma|v|\rangle \sim T^3 \cdot G_X^2 T^2 = G_X^2 T^5$$

$$\Gamma/H \sim G_X^2 M_p T^3$$

$$m_X \gtrsim T \gtrsim G_X^{-2/3} M_p^{-1/3} \sim \left(\frac{m_X}{100\,\mathrm{GeV}}\right)^{4/3} \mathrm{MeV} \rightarrow \Gamma/H \gtrsim 1$$

$$T \lesssim \quad " \quad \text{freez out.}$$

- Brief thermal history.



	t	T	Thermal Bath
Today	15Gyr	3K	
Life on earth			
Solar system			
Quasars			
Galaxy formation			
Recombination	10^{13} sec	10^{-1} eV	p, e, γ
Matter domination	10^{11} sec		
Nucleosynthesis	1 sec	1 MeV	$e^{\pm}$, γ, ν
QCD phase transition χSB,	10^{-5} sec	100-300 MeV	
EW phase transition	10^{-12} sec	300 GeV	
⋮			
GUT PT	10^{-35} sec	10^{15} GeV	
Planck epoch	10^{-43} sec	10^{19} GeV	

- The electroweak phase transition

SU(2)×U(1) symmetry is restored above $T_c \sim 100\,\text{GeV}$.

Around T_c, phase transition occurs $\langle\phi\rangle = 0 \rightarrow \langle\phi\rangle \neq 0$.

Consquences of PT.

1st order — bubble creation
2nd order — smooth transition
Topological defects.

SM : 2nd order
MSSM : ?
Electroweak baryogenesis

- The quark-hadron transition

$T \sim \Lambda_{QCD} \sim 100\,\text{MeV}$

thought to be 2nd order

- Neutrino decoupling

interaction $e^+e^- \leftrightarrow \nu\bar{\nu}$

$$\sigma \simeq G_F^2 s$$

$$\Gamma_{int} = n\langle\sigma|v|\rangle \simeq G_F^2 T^5$$

$$\frac{\Gamma_{int}}{H} \simeq \left(\frac{G_F^2 T^5}{T^2/M_p}\right) \simeq \left(\frac{T}{1\,\text{MeV}}\right)^3$$

- Neutrino temperature, $e^\pm$ annihilation

$$S = g_* T^3 = \text{const}$$

before $e^\pm$ annihilation after $e^\pm$ annihilation

$g_* = \frac{11}{2}$ $g_* = 2$

$$\frac{11}{2} \underset{\substack{\downarrow \\ T_\nu}}{T_b^3} = 2 \underset{\substack{\downarrow \\ T_\gamma}}{T_f^3} \qquad \left(\frac{T_\gamma}{T_\nu}\right) = \left(\frac{11}{4}\right)^{1/3} \simeq 1.40.$$

interaction $n \longleftrightarrow p e \bar{\nu}_e$

$$\frac{n}{p} = e^{-\Delta m/T} \quad , \quad \Delta m = m_n - m_p = 1.3\,\mathrm{MeV}$$

$\simeq 1/6$ at decoupling

$\rightarrow 1/7$ at $T \simeq 0.1$ MeV due to neutron decay

- Nucleosynthesis

Equilibrium number density of nuclear species

A : mass number
Z : charge

Kinetic equilibrium

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} e^{\overset{\uparrow}{(\mu_A - m_A)}/T}$$

chemical potential.

chemical equilibrium

$$\mu_A = Z \underbrace{\mu_p}_{\text{proton chemical potential}} + (A-Z) \underbrace{\mu_n}_{\text{neutron chemical potential}}$$

$\vdots$

$$n_A^{EQ} = \frac{g_A A^{3/2}}{2^A} \left(\frac{2\pi}{m_N T}\right)^{3(A-1)/2} n_p^Z n_n^{(A-Z)} e^{B_A/T}$$

mass fraction

$$X_A^{EQ} = \frac{g_A \zeta(3)^{A-1} 2^{(3A-5)/2}}{\pi^{(A-1)/2}} A^{5/2} \eta^{A-1} \left(\frac{T}{m_N}\right)^{3(A-1)/2} X_p^Z X_n^{A-Z} e^{B/T}$$

$$\eta = \frac{n_N}{n_\gamma} = 2.68 \times 10^{-8} (\Omega_B h^2)$$

$$\sim 10^{-10} - 10^{-9}$$

To determin whether a given set of nuclei will actually be in thermal equilibrium at a given epoch in the early universe, one has to take the rates of the relevant nuclear reactions from accelerator experiments.

$T \gtrsim 0{,}3\,\text{MeV}$: the lightest few nuclei in thermal equilibrium.

$T \sim 0{,}1\,\text{MeV}$: neutrons bind into ^4He

$$X = \frac{2n}{n+p} \approx 22\% \quad \text{for } n/p = 1/7$$

accurate calculations ⟶ computer.

- Photon decoupling

$e^-\gamma \leftrightarrow e^-\gamma$ (Thomson scattering)

$$\Gamma_{int} = \underset{\uparrow}{n_e}\,\sigma_T \leftarrow \text{Thomson cross section}$$

$\sigma_T = 6.65\times10^{-25}\,\text{cm}^2$

electron number density

→ Equilibrium abundance of free electron

$pe^- \leftrightarrow H\gamma$ ← Hydrogen atom

n_H, n_p, n_e, n_γ : # densities of H-atom, proton, free el. γ

charge neutrality : $n_p = n_e$

Baryon conservation : $n_B = n_p + n_H$ (we neglect neutrons in ^4He)

Equilibrium # density : $n_i = g_i\left(\frac{m_i T}{2\pi}\right)^{3/2} e^{(\mu_i - m_i)/T}$

Due to $pe^- \leftrightarrow H\gamma$, $\mu_p + \mu_e = \mu_H$

$$n_H = g_H\left(\frac{m_H T}{2\pi}\right)^{3/2} e^{(\mu_H - m_H)/T}$$
$$n_p = g_p\left(\frac{m_p T}{2\pi}\right)^{3/2} e^{(\mu_p - m_p)/T}$$
$$n_e = g_e\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{(\mu_e - m_e)/T}$$

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{2\pi}{T}\frac{m_H}{m_p m_e}\right)^{3/2} e^{[\underbrace{(\mu_H-\mu_p-\mu_e)}_{=0} - \underbrace{(m_H-m_p-m_e)}_{=-B=13.6eV}]/T}$$

$$n_H = \frac{g_H}{g_p g_e} n_p n_e \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{B/T}$$

$$g_p = g_e = 2,\ g_H = 4,\ n_B = \eta n_r$$

Fractional ionization $X_e \equiv \dfrac{n_p}{n_B}$

$$n_H = n_B - n_p,\ n_e = n_p$$

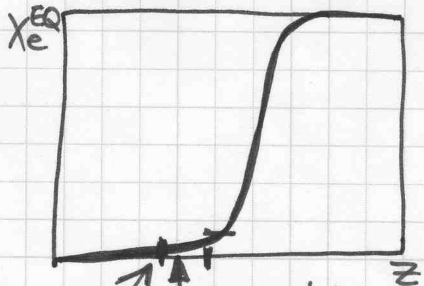
$$\frac{1-n_p/n_B}{n_p^2/n_B^2} = n_B\left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B/T} \qquad \frac{2\zeta(3)}{\pi^2}T^3$$

$$= \frac{1-X_e^{EQ}}{(X_e^{EQ})^2} = \eta n_r \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B/T}$$

$$\frac{1-X_e^{EQ}}{(X_e^{EQ})^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}}\eta\left(\frac{T}{m_e}\right)^{3/2} e^{B/T}$$

$$\eta = (\Omega_B h^2) \times (2.68 \times 10^{-8})$$

$$T = (1+z) \times (2.7K)$$



recombination, $X_e \approx 0.1$

$T_{rec} \approx 0.3eV \ll B \simeq 13.6eV$

$\Gamma_{int}(ep \leftrightarrow H\gamma) \gtrsim H$
for $(1+z) > 1100$ ⟶ freeze-in of free electron

Residual ionization fraction $X_\infty \simeq 3\times10^{-5}\ \Omega_0/\Omega_B h$

Decoupling of photon

$\Gamma_{int}(e\gamma \leftrightarrow e\gamma) = X_e \eta n_r \sigma_T \simeq H$ for $z \simeq 1100$

universe: $\boxed{\text{opaque}} \xrightarrow{\text{recombination}} \boxed{\text{transparent}}$

Ch. 3. Inflation

3.1 Motivation for inflation

Inflation $\Longrightarrow$ initial conditions required for Hot Big Bang.
- homogeneous & isotropic universe
- flat universe
- inhomogeneity of the universe.

3.1.1 Flatness problem

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{\rho}{3M_p^2} \quad , \quad \rho_c = 3M_p^2H^2 \; , \; \Omega = \frac{\rho}{\rho_c}$$

$$\rightarrow \Omega - 1 = \frac{K}{a^2H^2}$$

- Initially flat ($\Omega=1$) $\longrightarrow$ remains flat for all times.
- aH is a decreasing function of time

$$\text{MD}: \; a \propto t^{2/3}, \quad aH \propto t^{2/3} \cdot \frac{\frac{2}{3}t^{-1/3}}{t^{2/3}} = \frac{2}{3}t^{-1/3}$$

$$|\Omega - 1| \propto t^{2/3}$$

$$\text{RD}: \; a \propto t^{1/2}, \quad aH \propto t^{1/2} \cdot \frac{\frac{1}{2}t^{-1/2}}{t^{1/2}} = \frac{1}{2}t^{-1/2}$$

$$|\Omega - 1| \propto t$$

$$\underline{\Omega_0 \sim 1 \Longrightarrow |\Omega(t_{nuc}) - 1| \lesssim 10^{-16}}$$

- Finely tuned initial conditions are needed.-
- At Planck epoch

$\Omega \gtrsim 1 \longrightarrow$ recollapse almost $\overset{\text{a few}\times 10^{-43}\text{s}}{\underline{\text{immediately}}}$.

$\Omega \lesssim 1 \longrightarrow$ cools below 3K $\underset{\sim 10^{-11}\text{s}}{\underline{\text{within the first second}}}$.

- Flatness problem ~ Age problem

How did our universe get to be so old?

3.1.2 Horizon problem

Scale factor $a \propto t^{2/3}$ (MD), $t^{1/2}$ (RD)

particle horizon $d_H \propto t$

CMB : Why the temperature seen in different regions of tho sky is so accurately the same?

$\rightarrow$ The homogeneity must form part of the initial condition

BBN :

3.1.3 Unwanted relics

- gravitino , $\left.\begin{array}{l} m_{\tilde{G}} \sim 100\,\text{GeV} \\ \tau \sim 10^6\,\text{s} \end{array}\right\} \Rightarrow$ Spoil BBN $\Rightarrow T_R \lesssim 10^9\,\text{GeV}$
- Moduli
- Topological defect.

3.1.4. Homogeneity and isotropy

Wether we can develop a theory of the origin of the inhomogeneity? or Is it also one of initial condition.

3.2 Inflation in the abstract

Inflation — an epoch during which the scale factor of the univers
is accelerating.

$$\underline{\ddot{a} > 0} \quad \text{or} \quad \frac{d}{dt}\boxed{\frac{H^{-1}}{a}} < 0$$

comoving Hubble length
(comoving coordinate of Hubble lengt

* In comoving coordinates (coordinates fixed with expansio
the observable universe becomes smaller during inflation

How inflation solves the problems ?

- Flatness problem

$$\Omega - 1 = \boxed{\frac{k}{a^2H^2}}$$ decreases during inflation.

- Unwanted relics

Diluted by large expansion

- Horizon problem.

Huge reduction in comoving Hubble length
→ Our present observable universe originated from a tiny region that was well inside the Hubble radius.

Material driving inflation

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M_p^2} \Longrightarrow \underline{\rho + 3P < 0}$$

ρ is always ~~nega~~positive, "negative pressure is needed.

3.3 Scalar fields in cosmology

"material with the unusual property of a negative pressure"

- scalar field
  - * No direct observation of fundamental scalar
  - * crucial role in SSB in particle physics.
  - * vital role in cosmology ?

- Inflaton – the scalar field responsible for inflation

- Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \underline{V(\phi)}$ "potential"

homogeneous scalar field $\phi = \phi(t)$

energy density and pressure $\begin{cases} \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{cases}$

- The equations of motion

$$H^2 = \frac{1}{3M_p^2}\left[\underline{V(\phi)} + \frac{1}{2}\dot{\phi}^2\right]$$

V(φ) is more important than $\frac{1}{2}\dot{\phi}^2$ for inflation.

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \longleftarrow \underline{\Box^2\phi + V' = 0}$$

- The condition for inflation

$$\rho + 3p = \left(\frac{1}{2}\dot{\phi}^2 + V\right) + 3\left(\frac{1}{2}\dot{\phi}^2 - V\right) = 2(\dot{\phi}^2 - V) < 0$$

$$\underline{\dot{\phi}^2 < V(\phi)}$$

- The curvature term – neglected.

3.4 Slow-roll inflation

The slow roll approximation

$$\boxed{H^2 \simeq \frac{V(\phi)}{3M_p^2} \quad -(1)} \quad \boxed{3H\dot{\phi} \simeq -V'(\phi) \quad -(2)}$$

$$\left(\text{cf} \quad H^2 = \frac{V(\phi) + \boxed{\frac{1}{2}\dot{\phi}^2}}{3M_p^2} \qquad \boxed{\ddot{\phi}} + 3H\dot{\phi} = -V'(\phi) \right)$$

The slow-roll parameters

$$\epsilon(\phi) = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta(\phi) = M_p^2 \frac{V''}{V}$$

Necessary condition (<u>but not sufficient!</u>) for slow-roll approx.

$$\epsilon(\phi) \ll 1, \quad |\eta(\phi)| \ll 1$$

These two conditions involves only $V(\phi)$. We may choose $\dot{\phi}$ so that SRA is violated.

$$* \ \frac{1}{2}\dot{\phi}^2 = \frac{1}{2}\left(\frac{V'}{3H}\right)^2 \ll V \qquad 3M_p^2H^2 = V$$

use (2)

$$\frac{M_p^2}{6}\frac{V'^2}{V^2} \ll 1$$

$$* \ 3H\ddot{\phi} + 3\dot{H}\dot{\phi} = -V''\dot{\phi}, \quad \ddot{\phi} = -\frac{3\dot{H} + V''}{3H}\cancel{\dot{\phi}} \ll 3H\cancel{\dot{\phi}}$$

$$2H\dot{H} = \frac{V'}{3M_p^2}\dot{\phi} \qquad \dot{H} = \frac{2V'}{3H^2M_p^2}H\dot{\phi} = -\frac{2}{9}\frac{V'^2}{V}$$

$$\underset{\substack{\shortparallel \\ \frac{V}{M_p^2}}}{\boxed{3H^2}}\ddot{\phi} + \frac{3}{2}\frac{V'^2\dot{\phi}^2}{3M_p^2} = -V''\boxed{H\dot{\phi}} - \frac{V'}{3}$$

$$\dot{\phi}$$

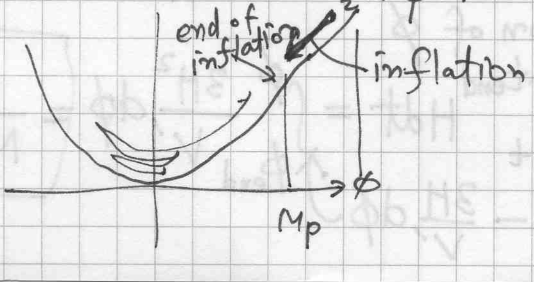
$$3\dot{H} + V'' \ll 9H^2 = \frac{3V}{M_p^2}$$

$$\underset{\epsilon^2}{\boxed{\frac{M_p^2\dot{H}}{V}}} + \underline{\frac{M_p^2}{3}\frac{V''}{V}} \ll 1.$$

$$* \ 3H\dot{\phi} = -V' \ \& \ \epsilon \ll 1 \longrightarrow H^2 = \frac{V(\phi)}{3M_p^2}$$

$$* \text{Example} \quad V(\phi) = \frac{1}{2}m^2\phi^2$$

$$\epsilon = \frac{M_p^2}{2}\left(\frac{m^2\phi}{\frac{1}{2}m^2\phi^2}\right)^2 = \left(\frac{2M_p^2}{\phi^2}\right) \ll 1 \Rightarrow \phi^2 \gg 2M_p^2$$



3.4.1. Relation between inflation and slow-roll

The slow-roll approximation is a sufficient condition for inflation

From $\dot{H} = \frac{\ddot{a}}{a} - H^2$, $\frac{\ddot{a}}{a} = \underbrace{\dot{H} + H^2 > 0}_{\text{Condition for inflation.}}$

① $\dot{H} > 0 \Rightarrow p < -\rho$ exotic matter.

② $\underline{-\frac{\dot{H}}{H^2} < 1}$ $2H\dot{H} = \frac{V'}{3M_p^2}\dot{\phi}$, $2H^2\dot{H} = \frac{V'}{3M_p^2}H\dot{\phi}$

$\dot{H} = \frac{-V'^2}{18M_p^2 H^2}$

$$\frac{V'^2}{18M_p^2 H^4} = \frac{M_p^2}{2}\frac{V'^2}{V^2} = \epsilon$$

<u>sufficient but not necessary</u>

It is possible for inflation to continue even if the slow-roll conditions are violated.
In practice, the amount of inflation that occurs under this circumstances is very small.

Inflation model = potential + <u>a way of ending inflation.</u>

'' $\epsilon(\phi) \gtrsim 1$
or other way.

3.4.2 Amount of inflation

The number of e-foldings $N(t) \equiv \ln\frac{a(t_{end})}{a(t)}$ ← scale factor at the end of inflation

— measures the amount of inflation that still has to occur after time t.

— To solve the horizon & flatness problems, around 70 efoldings are required.

— Use of the comoving Hubble length $\frac{1}{aH}$

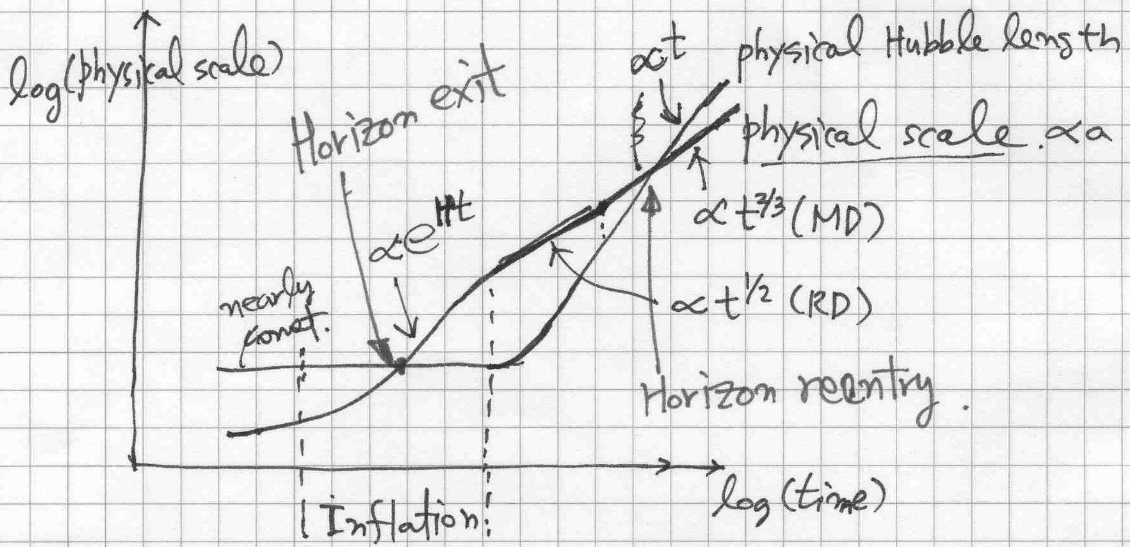
$\tilde{N}(t) = \ln\frac{a(t_{end})H(t_{end})}{a(t)H(t)}$ ← technically more accurate
$H(t) \sim \text{const} \Rightarrow \tilde{N} = N.$

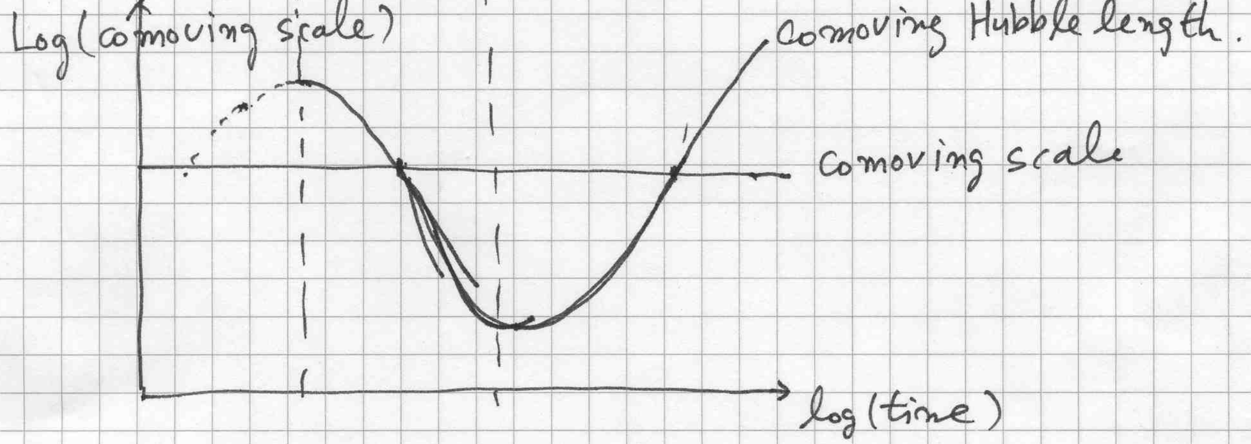
— N as a function of ϕ

$$N = \ln\frac{a(t_{end})}{a(t)} = \int_t^{t_{end}} H\,dt = \int_{\phi_{end}}^{\phi}\frac{3H^2}{V'}d\phi = \boxed{\frac{1}{M_p^2}\int_{\phi_{end}}^{\phi}\frac{V}{V'}d\phi}$$

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{-V'}{3H} = dt = -\frac{3H}{V'}d\phi$$

3.4.1 Evolution of scales





For the comoving scale k t_k

- $\underline{\text{From the time } k=aH}$ to $\underline{\text{the end of inflation}}$ t_{end}
- From the end of inflation ~~until~~ $\underline{\text{HBB is restored}}$ reheating t_{reh}
- RD ← t_{eq}.
- MD

$$\frac{k}{a_0H_0} = \frac{a_kH_k}{a_0H_0} = \boxed{\frac{a_k}{a_{end}}} \cdot \frac{a_{end}}{a_{reh}} \cdot \frac{a_{reh}}{a_{eq}} \frac{a_{eq}}{a_0} \boxed{\frac{H_k}{H_0}}$$

$$N(k) = 62 - \ln \underset{1\sim10^4}{\boxed{\frac{k}{a_0H_0}}} - \ln \frac{10^{16}\,\mathrm{GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{end}^{1/4}} - \frac{1}{3} \ln \frac{V_{end}^{1/4}}{\rho_{reh}^{1/4}}$$