An Elementary Tutorial on Provable Security

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KMS-KIAS Summer Workshop on Cryptography 2007

June 29, 2007

Goal of this talk : Help you to prove the security of your own cryptosystem

We will : introduce necessary backgrounds (for mathematics people)
discuss how to formalize the notion of security
give an example of "proof of security"

We will NOT : give a survey on known results
report nor announce new progresses

Aim : Prove that a cryptosystem is secure

we need technologies! we need definitions!

Roughly speaking, a cryptosystem is said to be secure if it is hard to break the cryptosystem i.e.,

there is no efficient algorithm breaking it. ... (*)

Issues : what do you mean by the terms in (*) ?
Are we really able to prove (*) ?

Outline

- Algorithm and model of computation
- Randomized computation
- Formalization of a cryptosystem and its security
  - security model : definition of security
- How to prove the security
  - reduction method, primitive problems and random oracle model
- A detailed example of security proof
  - Gap Diffie-Hellman signature scheme

Algorithm = a sequence of computational steps transforming the input to the output

(The output is expected to be a solution of a given problem)

Correctness : is the output a solution ?

Efficiency : Time and space complexity (input size)

Asymtotic behaviour of running time $T(n)$

e.g. binary search $T(n) = O(\log n)$

i.e. $\exists C$ s.t. $T(n) \le C \log n$ for all large n

Example : Computation of n^n

input = n, output = n^n

1. $r \leftarrow 1$
2. for $k \leftarrow 1$ to n do
   $r \leftarrow r \cdot k$
3. output r.

Running time $T(n) = O(n)$???

We need a precise def'n of running time !

Model of computation

Def : an algorithm = "Random Access Machine" (RAM)

- register
- memory = an array of integers

addr	content
0	...
1	...
2	...
⋮	...

- I/O tape 0110100... (binary data)
- program = a sequence of "instructions"

- register-memory transfer
- arithmetic operations (+ − • /)
- comparison (< = >) and branching
- tape I/O

Log-cost RAM : (running time of each instruction) $\propto \log |\text{operand}|$ (operand size)

Total running time $T(n) = \sum$ (running time of each instruction executed)

where input size n = length of the input tape (data)

$= \sum \log |x_i|$ if input $= (x_1, x_2, \cdots)$

An algorithm is a polynomial time algorithm if

$T(n) = O(n^c)$ for some const c.

Theorem : Polynomial time RAM $\Leftrightarrow$ polynomial time TM

e.g. input = n

$r \leftarrow 2$

for $i \leftarrow 1$ to n do

$r \leftarrow r \cdot r$

output r

Unit-cost RAM : $T(n) = O(n)$

Log-cost RAM : $T(n) = \sum_{i=1}^{n} O(\log r_i) = \sum_{i=1}^{n} 2^i \geq 2^n$

Note that the output is $2^{2^{n-1}}$, which is too large for a log-cost RAM to construct within time $\leq n$.

Randomized (Probabilistic) algorithm

= RAM + "unbiased coin toss" instruction

= RAM + a random tape ...01101110... random numbers!

Las Vegas algorithm : the output is always correct but the running time may vary even for the same input

.... we think of { expected / worst-case } running time (over all coin tosses & inputs)

Monte Carlo algorithm : even the output may vary,
i.e the output may be incorrect!

e.g. Random guessing : output $\xleftarrow{R}$ {all possible outputs}

We think of : { Success probability = Pr[output is correct]
Advantage = Pr[succ] − (random guessing's Pr[succ]) }

For a decision problem (Yes/No question),
we usually normalize it as : Advantage = $2\Pr[\text{succ}] - 1 \leq 1$

Def A computational problem is **feasible** if :
$\exists$ a polynomial time randomized algorithm solving the given problem with advantage "not too small".

(1) Adv = 1 : Las Vegas

(2) Adv $\geq \frac{1}{2}$ if output = yes : RP (Randomized Poly. time)
= 1 if output = no

(3) Adv $\geq \frac{1}{2}$: BPP (Bounded-error...)

(4) Adv > 0 : PP (Prob. Poly. time)

(5) Adv is non-negligible (as a function of input size) ← This is what we will use!

Def : $\nu(k)$ is negligible if $\forall$ poly $p(k)$, $\nu(k) \leq \frac{1}{p(k)}$ asymptotically

Cryptosystem : a collection of algorithms

e.g. Symmetric Key Cryptosystem = $(\mathcal{E}, \mathcal{D})$

$(K, M) \to \mathcal{E} \to \mathcal{E}(K,M) = C$

s.t. $\mathcal{D}(K, \mathcal{E}(K,M)) = M$

$(K, C) \to \mathcal{D} \to \mathcal{D}(K,C) = M$

[Shannon] This can be secure only if the key size is as large as the message size

(assumming the adversary has unlimited computing power)

Modern Cryptography : based on complexity theory

- Security parameter k $\xrightarrow[\text{Setup}]{}$ Cryptosystem
- Said to be secure if: $\nexists$ poly. time (in k) adversary which "breaks" the system

We want to prove

To obtain meaningful results, we work in the rand. comp. model

e.g. Public Key Cryptosystem = $(\mathcal{K}, \mathcal{E}, \mathcal{D})$

$\mathcal{K}(1^k) = (PK, SK)$ PK = public key, SK = secret key

$\mathcal{E}(PK, M) = C$ s.t. $\mathcal{D}(SK, \mathcal{E}(PK, M)) = M$

$\mathcal{D}(SK, C) = M$

Note : C is not uniquely determined by (PK, M)

What do you mean by "breaking" a cryptosystem?

e.g. In case of PKC, requirements may be ...

- Recovery of SK is infeasible
  .... what about $PK \leftarrow$ random, $\mathcal{E}(PK,M) = M$, $\mathcal{D}(SK,C) = C$?
- Recovery of M (from PK, C) is infeasible
  .... But, an adversary still can recover first bit of M ....

The answer is NOT straightforward at all !

To give a formal definition, we think of the goal and ability of an adversary :

e.g. Goal - indistinguishability

given $m_0, m_1, y = \mathcal{E}(PK, m_b)$ where $b \in \{0,1\}$ hidden, can an adversary find b? (without knowing SK)

Ability - Chosen plaintext attack : $\mathcal{E}(PK,-)$ is available

Chosen ciphertext attack : $\mathcal{E}(PK,-)$, $\mathcal{D}(SK,-)$ before y is given

Adaptively chosen ciphertext attack : $\mathcal{E}(PK,-)$, $\mathcal{D}(SK,-)$ anytime but $\bullet \neq y$

Def: IND-CCA2 security of $(\mathcal{K}, \mathcal{E}, \mathcal{D})$

An adversary A is a pair (A_1, A_2) of algorithms s.t.

$A_1(PK) = (x_0, x_1, s)$ where x_i = message, s = bitstring

$A_2(x_0, x_1, s, y) = d$ where x_i, s = as above, y = ciphertext, $d \in \{0,1\}$

Consider the following game (by a challenger $\mathcal{C}$):

1. $(PK, SK) \leftarrow \mathcal{K}(1^k)$
2. $(x_0, x_1, s) \leftarrow A_1(PK)$ where A_1 can make $\mathcal{D}(SK, -)$ and $\mathcal{E}(PK, -)$ queries
3. $b \xleftarrow{R} \{0,1\}$; $y \leftarrow \mathcal{E}(PK, x_b)$
4. $d \leftarrow A_2(x_0, x_1, s, y)$ where A_1 can make $\mathcal{D}(SK, -)$ (not y) and $\mathcal{E}(PK, -)$ queries

We say A wins the game if $d = b$.

Advantage of $A := 2\Pr[d=b] - 1 \leq 1$

We say $(\mathcal{K}, \mathcal{E}, \mathcal{D})$ is IND-CCA2 secure if

there is no polynomial time $A = (A_1, A_2)$ with non-negligible advantage

Remark: ① IND-CCA2 is the strongest known security notion for PKC

② There are known schemes which are IND-CCA2 secure (under certain assumptions!)

Signature Scheme $= (\mathcal{K}, \mathcal{S}, \mathcal{V})$

$\mathcal{K}(1^k) = (PK, SK)$ ..... Public and secret key

$\mathcal{S}(SK, M) = \sigma$ ..... Signature for the msg M

$\mathcal{V}(PK, \sigma, M) =$ "valid" or "invalid" ... Validity of σ

Def: Security against existential forgery (goal) on adaptively chosen message attack (ability)

Adversary = an algorithm $A(PK) = (\sigma, M)$

Game: 1. $(PK, SK) \leftarrow \mathcal{K}(1^k)$

2. $(\sigma, M) \leftarrow A(PK)$ where A can make $\mathcal{S}(SK, -)$ queries

A wins if $\mathcal{V}(PK, \sigma, M) =$ valid; Advantage $:= \Pr[\text{win}]$

$(\mathcal{K}, \mathcal{S}, \mathcal{V})$ is secure if $\nexists$ poly. time A with non-neg. adv.

ID-Based signature scheme $(\mathcal{K}, \mathcal{E}, \mathcal{S}, \mathcal{V})$

$\mathcal{K}(1^k) =$ (group G, hash $H: \{0,1\}^* \to G$)

$\mathcal{E}(ID) = SK_{ID} \in G$

($PK_{ID} = H(ID)$ plays the role of a pub. key)

$\mathcal{S}(SK_{ID}, M) = \sigma$

$\mathcal{V}(PK_{ID}, M, \sigma) =$ valid or invalid

Game: 1. Run $\mathcal{K}$

2. A makes queries to H, $\mathcal{E}$, $\mathcal{S}$

3. A outputs (ID, M, σ) "chosen ID, chosen msg"

A wins if $\mathcal{V}(PK_{ID}, M, \sigma)$ is valid.

There is no known technology to prove the security without any additional assumption ... what can we do?

Primitive problem = a problem which { is / seems / believed to be } infesible

e.g. RSA inverting — given $x^e \bmod n$, find x
Factoring — given $n = pq$, find p and q
DLP — given P and aP, find a ($a \in \mathbb{Z}$, $P \in G$)
CDHP — given P, aP, bP, find abP
BDHP — given aP, bP, cP, find $e(P,P)^{abc}$
($e : G \times G \to G'$ bilinear)

(G: finite abelian group)

Reduction method

If we are lucky, then we can show:

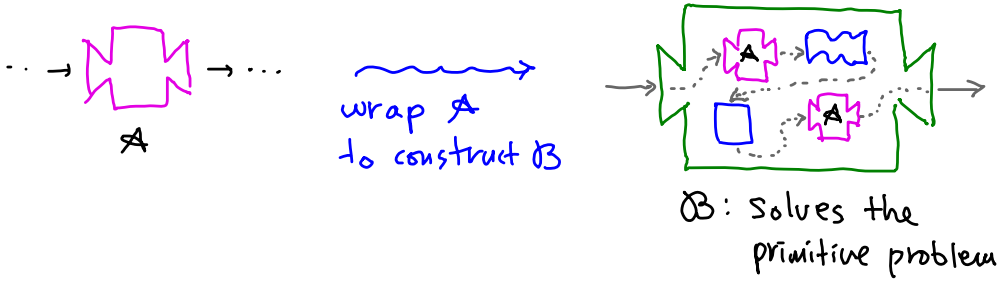
$\exists$ an algorithm breaking the cryptosystem
$\Rightarrow$ $\exists$ an algorithm solving an associated primitive prob.

(algorithms are polynomial time and of non-negligible advantage)

In this case we say: the cryptosystem is provably secure (under the assumption that the primitive problem is hard)

Remark: Usually, the converse is true.

Technique: Given an algorithm $\mathcal{A}$ breaking the cryptosystem, construct an algorithm solving the primitive problem using $\mathcal{A}$ as a black-box tool!



Hash functions play an important role in modern cryptography

A typical usage:

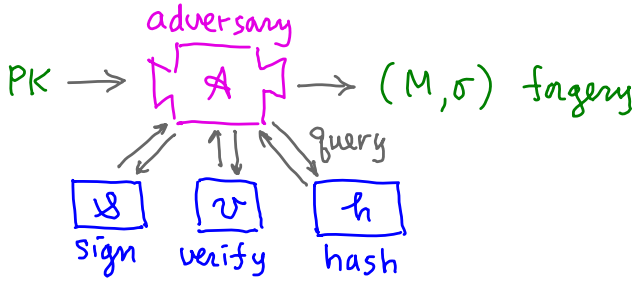
{long messages (bitstrings)} $\xrightarrow{h}$ G = (algebraic object where the primitive problem is defined)

Usually (a pointer to) h is available to all parties as a system parameter so that queries to h can be made:

Random Oracle Model:

assume that h is a random function, i.e.,
for each (new) query, h generates a random output.

e.g. Existential forgery for a signature scheme on adaptively chosen message attack :

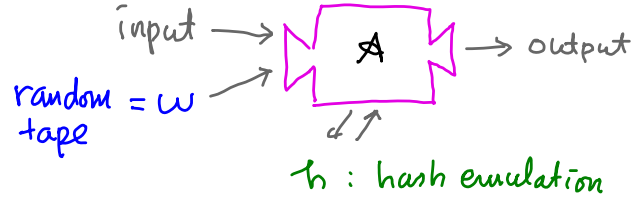


In a security proof, one may "emulate" h (as well as S and V) for one's own purpose, provided that its output is a random distribution.

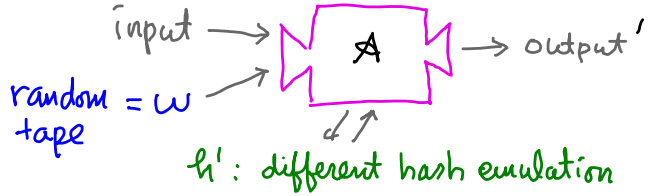
Oracle Replay Attack [Pointcheval-Stern '96]

Idea : "replay" the adversary A with the same random tape but with a differently controlled (emulation of) hash functions

1st execution :



2nd execution :



An example of security proof : GDH-signature scheme [Boneh, Lynn, Shacham]

G = cyclic group with generator g , |G| = p prime

Assume : ① Decision Diffie-Hellman is feasible
i.e. given (g, g^a, g^b, g^c) , we can verify if $c = ab$ (valid DH-tuple)

② Computational Diffie-Hellman is infeasible
i.e., given (g, g^a, g^b) , computing g^{ab} is hard.

Such a group G is called a gap Diffie-Hellman (GDH) group.

GDH-signature scheme

Setup $\mathcal{K}$: choose G (log |G| ∝ security parameter k)
choose a hash function $h: \{0,1\}^* \to G^*$
$x \xleftarrow{R} \mathbb{Z}_p^*$
Output : $PK = g^x$, $SK = x$.

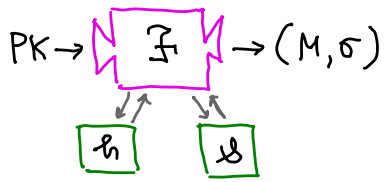
Sign $\mathcal{S}$: given (x, M) ,
output $\sigma = h(M)^x$

Verify $\mathcal{V}$: given (PK, M, σ) ,
output = valid $\Leftrightarrow$ $(g, PK, h(M), \sigma)$ is a valid DH-tuple.
(If $\sigma = h(M)^x$, then it holds!)

Proof of security

Suppose there is a forgery algorithm $\mathcal{F}$ with

- running time = t
- # of h-queries = q_H
- # of $\mathcal{S}$-queries = q_S
- advantage = ε



i.e. $\varepsilon = \Pr\left[V(PK, M, \sigma) = \text{Valid} \;\middle|\; \begin{array}{l} (PK, SK) \leftarrow \mathcal{K} \\ (M, \sigma) \leftarrow \mathcal{F}(PK) \end{array} \right]$

Goal: construct an algorithm solving CDHP
with running time $\leq$ polynomial in t, q_H, q_S
advantage $\geq \varepsilon \cdot \frac{1}{\text{polynomial}}$.

We start with a very simple challenger algorithm, and proceed to more sophisticated ones...

Notation: B_δ = distribution on $\{0,1\}$ given by $\Pr[1] = \delta$, $\Pr[0] = 1-\delta$ $(0 \leq \delta \leq 1)$
(δ will be specified later)
M_i = input (message) to the i^{th} h-query.

We may assume: $\mathcal{F}$ requests $h(M)$ before $\mathcal{S}(SK, M)$.
$\mathcal{F}$ requests h for its output.

Algorithm A_1: input = (g, g^a, g^b)

$PK \leftarrow g^a$
For $i = 1, 2, \cdots, q_H$, pick $s_i \xleftarrow{R} B_\delta$, $r_i \xleftarrow{R} \mathbb{Z}_p^*$
set $h_i \leftarrow g^{r_i}$, $\sigma_i \leftarrow (g^a)^{r_i}$.
Run $\mathcal{F}(PK)$; to the i^{th} h-query, return h_i
to $\mathcal{S}$-query for M_i, return σ_i.
Let (M^*, σ^*) be the output of $\mathcal{F}$ ($M^* = M_{i^*}$ for some i^*)
If $\mathcal{F}$ succeeds, output "success"
otherwise, output "fail".

$(g, PK, h(M_i), \sigma_i) = (g, g^a, g^{r_i}, g^{ar_i})$

Analysis: (M_i, σ_i) is a valid signature
$\mathcal{F}$ cannot distinguish A_1 from the standard challenger
$\therefore \text{Adv}(A_1) = \text{Adv}(\mathcal{F}) = \varepsilon$.

Algorithm A_2

The same as A_1 except the following:
If $\mathcal{F}$ succeeds and $s_{i^*} = 1$, then output "success".
otherwise, output "fail".

Analysis: The choice of s_i is independent of $\mathcal{F}$
$\therefore \text{Adv}(A_2) = \Pr[s_{i^*} = 1] \cdot \text{Adv}(A_1)$
$= \delta \cdot \varepsilon$.

Algorithm A_3

The same as A_2 except:

If $\mathcal{F}$ succeeds and $s_i^* = 1$ and

$\mathcal{F}$ asked for signatures only for M_i s.t. $s_i = 0$,

then output "success"

Otherwise, output "fail"

Analysis: Suppose the j^{th} $\mathcal{S}$-query is on M_{i_j}.

Then $Adv(A_3) = \left(\prod_j \Pr[s_{i_j} = 0]\right) \cdot Adv(A_2)$

$= (1-\frac{1}{S})^{q_S} \cdot \frac{1}{S} \cdot \varepsilon.$

Algorithm A_4

The same as A_3 except:

If $\mathcal{F}$ requests the sign of M_i for which $s_i = 1$,

then output "fail" and exits immediately.

Analysis: Compare runs of A_3 and A_4 with the same input and random tape.

If $\mathcal{S}(SK, M_i)$ is requested and $s_i = 1$ in A_3,

then so is in A_4, and both output fail.

Otherwise, the output of A_3 and A_4 are the same

$\therefore Adv(A_4) = Adv(A_3) = (1-\frac{1}{S})^{q_S} \cdot \frac{1}{S} \cdot \varepsilon.$

Algorithm A_5

Modify A_4 as follows:

In the setup step, let

$h_i \leftarrow g^b g^{r_i}$, $\sigma_i \leftarrow$ "*" if $s_i = 1$

$h_i \leftarrow g^{r_i}$, $\sigma_i \leftarrow (g^b)^{r_i}$ if $s_i = 0$ (as before)

Analysis: The choice of h_i is still random.

Since we don't need σ_i for $s_i = 1$,

the behaviour of $\mathcal{F}$ remains the same.

$\therefore Adv(A_5) = Adv(A_4) = (1-\frac{1}{S})^{q_S} \cdot \frac{1}{S} \cdot \varepsilon.$

Algorithm A_6

Modify A_5 as follows:

When A_5 outputs "success",

A_6 outputs "success" and $\sigma^* / (g^a)^{r_{i^*}}$.

Analysis: Obviously $Adv(A_6) = Adv(A_5) = (1-\frac{1}{S})^{q_S} \cdot \frac{1}{S} \cdot \varepsilon.$

Since $s_{i^*} = 1$, $h_{i^*} = g^b \cdot g^{r_{i^*}}$ by our choice

Since $(g, g^a, h_{i^*}, \sigma^*)$ is a valid DH-tuple,

we have $\sigma^* = (h_{i^*})^a = g^{ab} \cdot g^{a r_{i^*}}$

$\therefore$ output $\sigma^* / (g^a)^{r_{i^*}} = g^{ab}$ as desired !!!

Final Analysis :

$Adv = (1-\zeta)^{q_S} \cdot \zeta \cdot \varepsilon$ is maximized at $\zeta = \frac{1}{q_S+1}$

$$\left(\begin{aligned} \frac{d}{d\zeta} Adv &= -q_S(1-\zeta)^{q_S-1}\zeta\varepsilon + (1-\zeta)^{q_S}\cdot\varepsilon \\ &= (1-\zeta)^{q_S-1}\cdot\varepsilon\cdot(1-\zeta-q_S) = 0 \\ &= 0 \text{ when } \zeta = 1/q_S+1 . \end{aligned} \right)$$

At $\zeta = \frac{1}{q_S+1}$, $Adv = \frac{1}{q_S}\underbrace{\left(1-\frac{1}{q_S+1}\right)^{q_S+1}}_{\to \frac{1}{e} \text{ as } q_S\to\infty} \cdot \varepsilon$

Running time = Running time of $\mathcal{F}_1$ + (const)
+ ($\underbrace{\text{exponentation computing time for } h_i, \sigma_i}_{c\cdot(q_H+q_S)}$)

Thank You!