Quantum Computation and Simulation with Trapped Ions Part II

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OPEN KIAS SCHOOL ON QUANTUM INFORMATION SCIENCE

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Outline

Review of Ion Laser Interaction

- Carrier, Red-Sideband, Blue-Sideband Transition

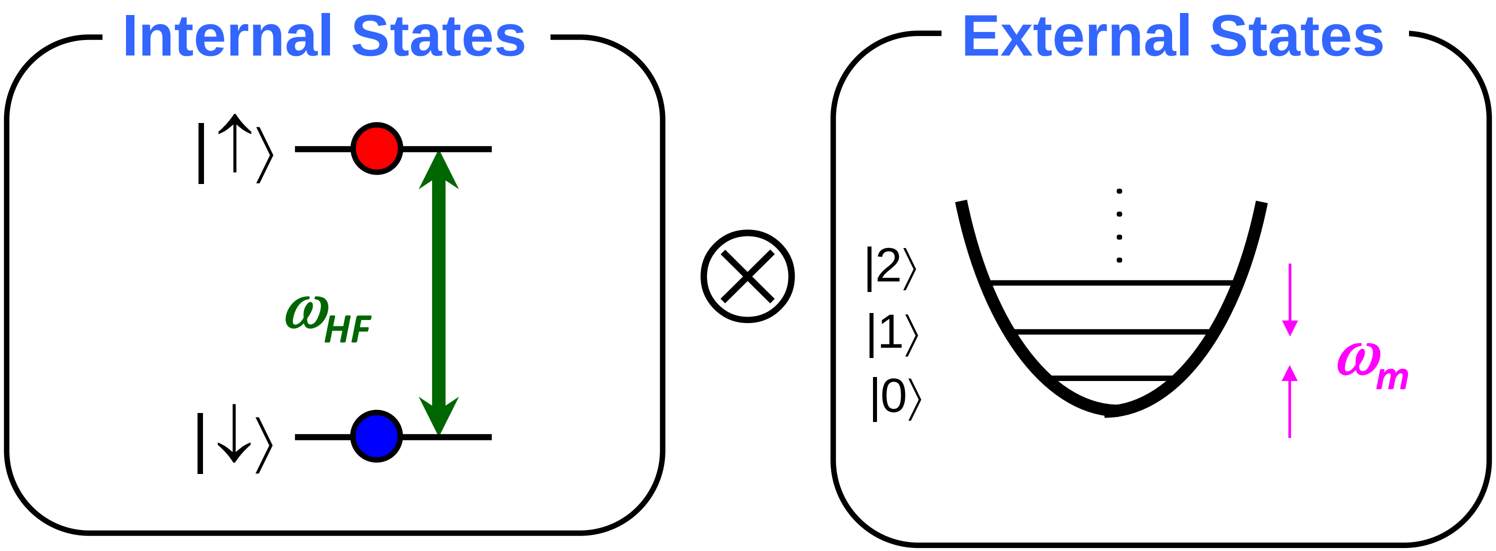
Two Qubit Operation

- State Transfer
- Cirac-Zoller Gate

Quantum State of Motion

- Creation of the State
- Detection of the State

A single ion system



$$H^{(e)} = \frac{\hbar\omega_{HF}}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

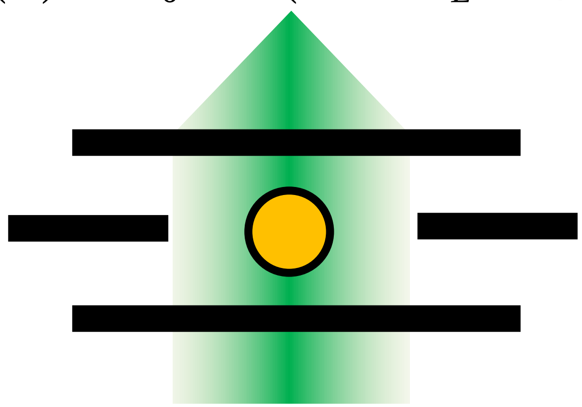
$$= \frac{\hbar\omega_{HF}}{2}\sigma_z$$

$$H^{(m)} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_m^2\hat{x}^2$$

$$= \hbar\omega_m\left(a^+a + \frac{1}{2}\right)$$

Interaction between Laser and an Ion

$$E(\hat{x}) = E_0 \cos(k\hat{x} - \omega_L t + \varphi)$$



$$H^{(i)} = -\hat{\mu} \cdot E(\hat{x})$$

$$= \mu_0 (\hat{\sigma}_+ + \hat{\sigma}_-) \frac{E_0}{2} \left(e^{i(k\hat{x} - \omega_L t + \varphi)} + e^{-i(k\hat{x} - \omega_L t + \varphi)} \right)$$

$$= \frac{\Omega}{2} (\hat{\sigma}_+ + \hat{\sigma}_-) \left(e^{i(k\hat{x} - \omega_L t + \varphi)} + e^{-i(k\hat{x} - \omega_L t + \varphi)} \right)$$

where, $\Omega = \mu_0 E_0 / 2$

Carrier, Red sideband, Blue sideband

$$H_{\text{int}} = (\hbar/2)\Omega\left[\hat{\sigma}_+ e^{i\eta\left(ae^{-i\omega_m t}+a^+e^{i\omega_m t}\right)} e^{-i(\delta t-\varphi)} + \hat{\sigma}_- e^{-i\eta\left(ae^{-i\omega_m t}+a^+e^{i\omega_m t}\right)} e^{i(\delta t-\varphi)}\right]$$

$$\eta\sqrt{n+1} = kx_0\sqrt{n+1} << 1 : \text{Lamb - Dicke limit}$$

Stationary terms of H_{int} at particular values of δ

"CARRIER" $\delta = 0$

$$\boxed{H_{carr} = (\hbar/2)\Omega\left[\hat{\sigma}_+ e^{i\varphi} + \hat{\sigma}_- e^{-i\varphi}\right]}$$

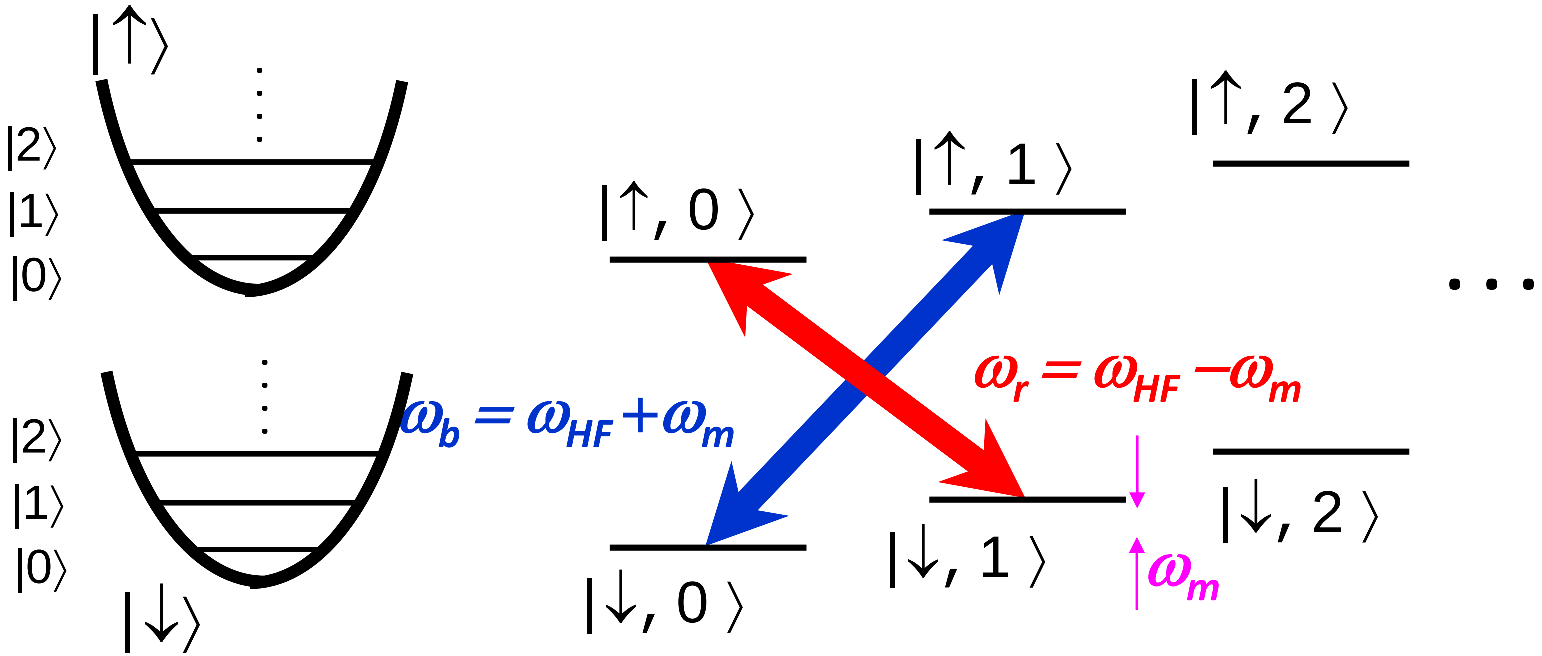
"Red Sideband" $\delta = -\omega_m$

$$\boxed{H_{rsb} = (\hbar/2)\eta\Omega\left[\hat{\sigma}_+ a e^{i\varphi} + \hat{\sigma}_- a^+ e^{-i\varphi}\right]}$$

"Blue Sideband" $\delta = +\omega_m$

$$\boxed{H_{rsb} = (\hbar/2)\eta\Omega\left[\hat{\sigma}_+ a^+ e^{i\varphi} + \hat{\sigma}_- a e^{-i\varphi}\right]}$$

Coupling Between Internal State and Motional Mode

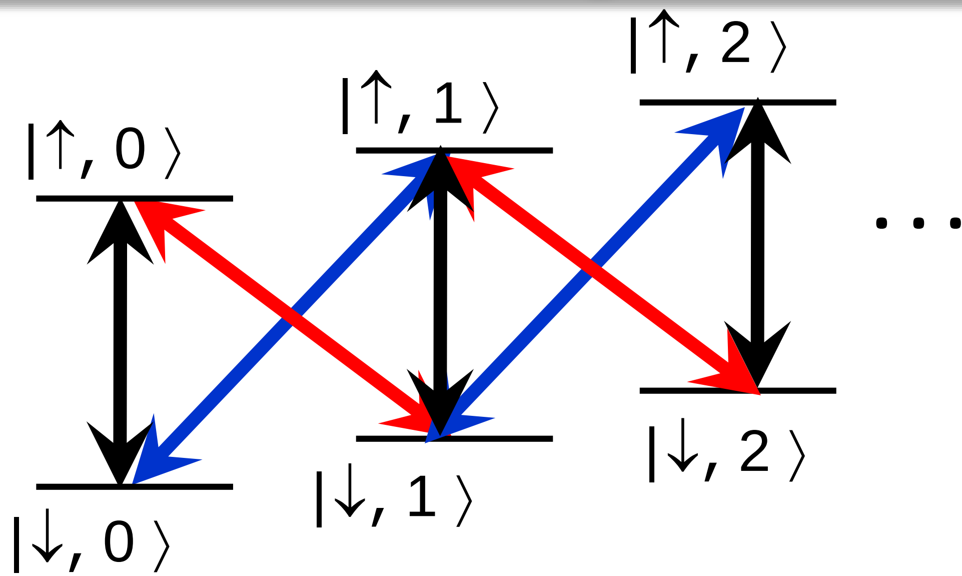


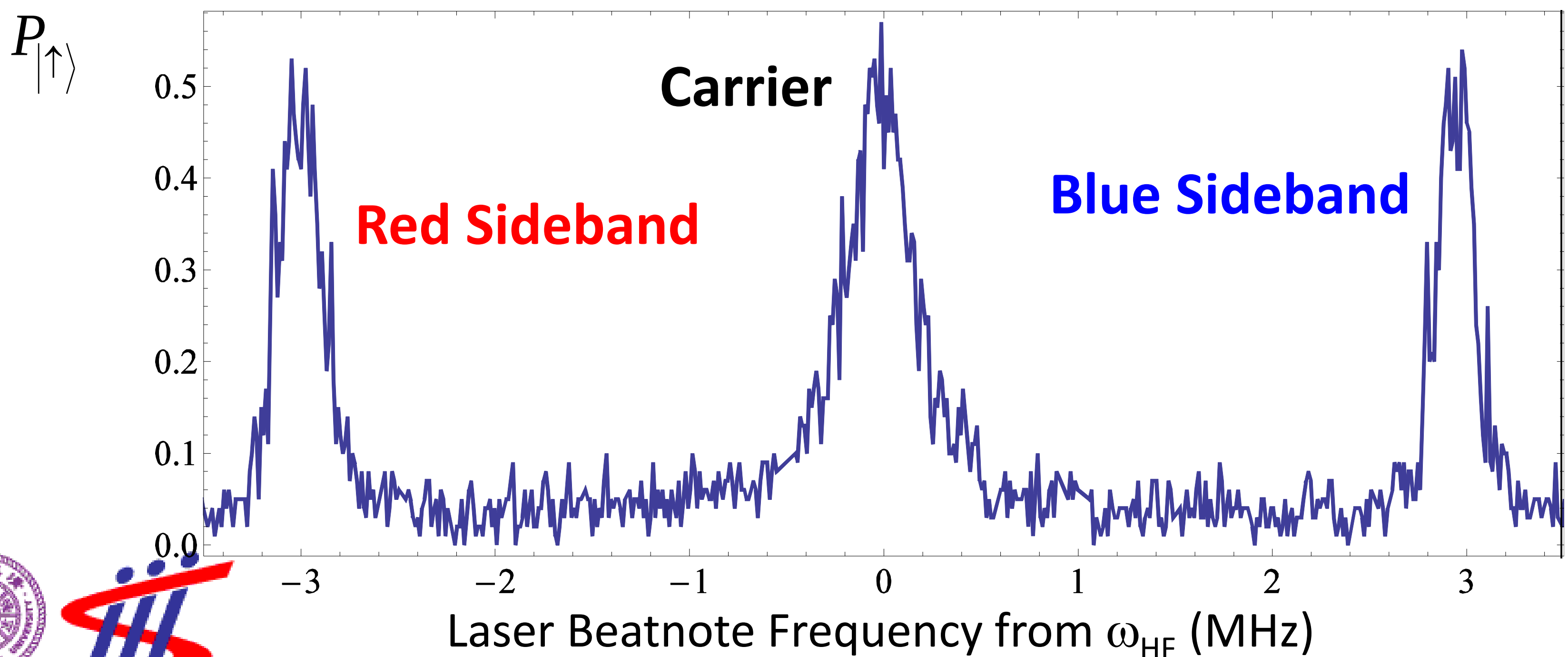
$$H^{(e)} \otimes H^{(m)}$$

$$H_{bsb} = -i\hbar\eta\Omega\sigma^{+}a^{\dagger} + h.c.$$

$$H_{rsb} = -i\hbar\eta\Omega\sigma^{-}a^{\dagger} + h.c.$$

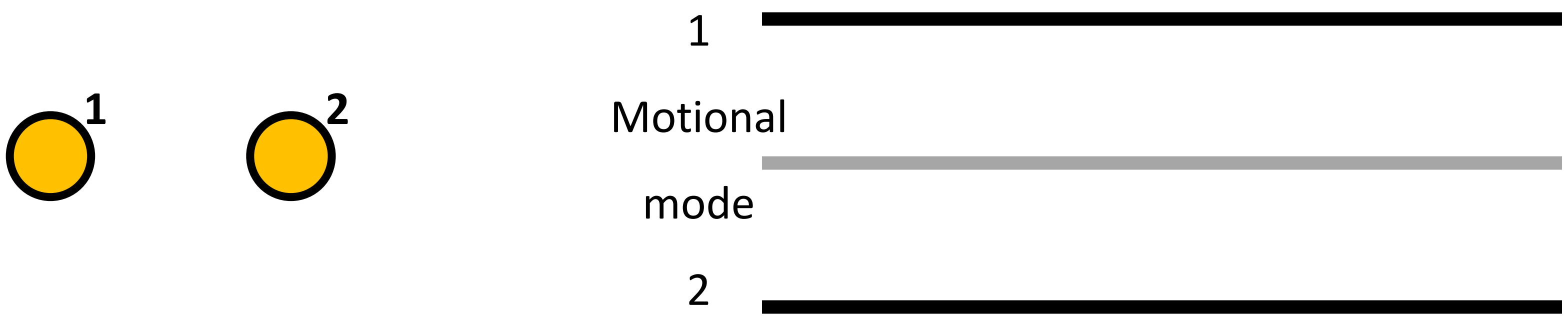
Internal and External Degree of Freedom



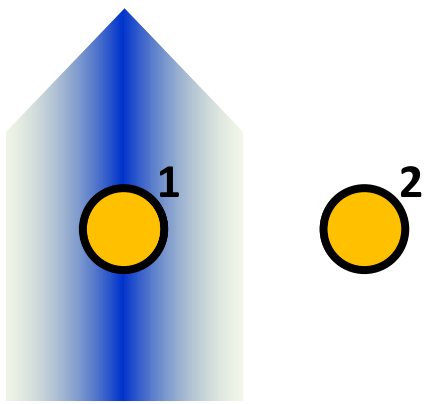


Transfer Quantum Information via Motional Mode

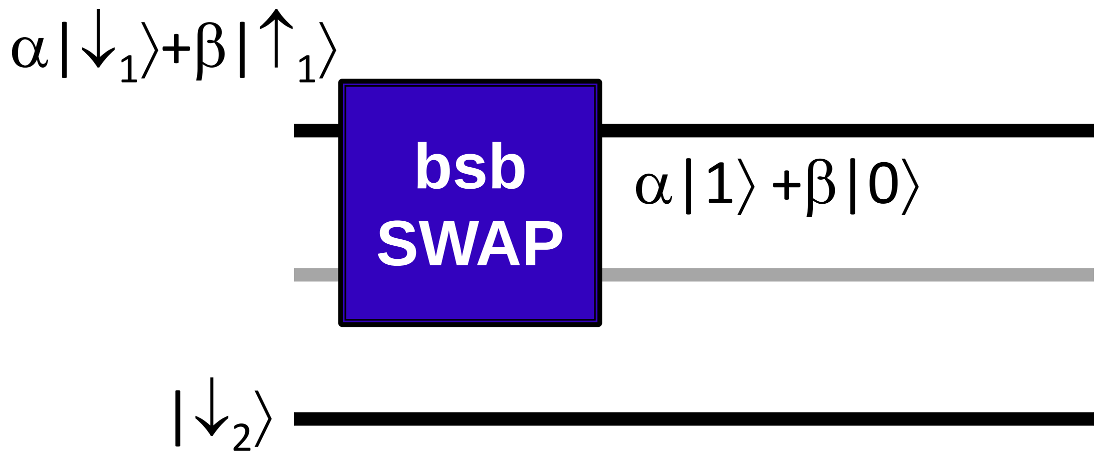
$$\alpha|\downarrow_1\rangle+\beta|\uparrow_1\rangle$$



Transfer Quantum Information via Motional Mode



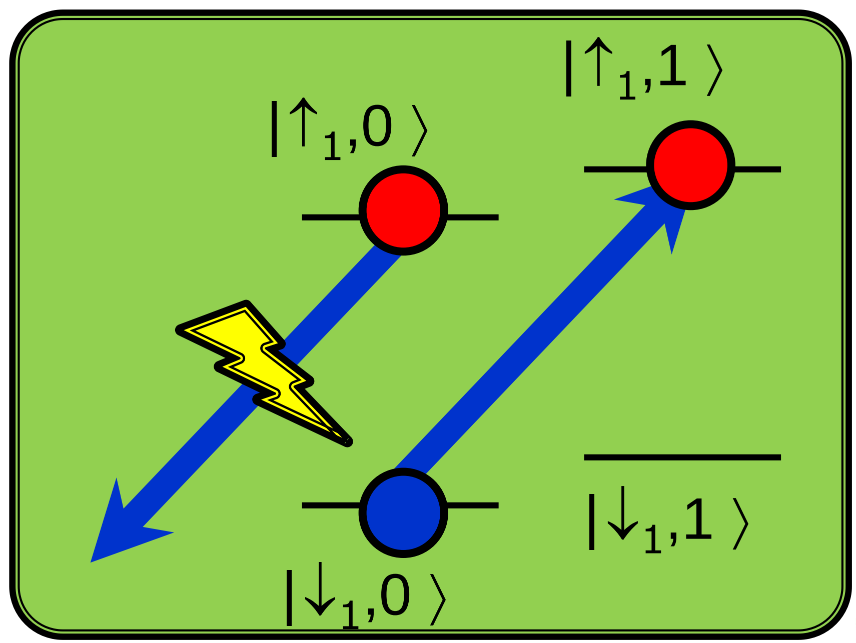
$\omega_b = \omega_{HF} + \omega_m$



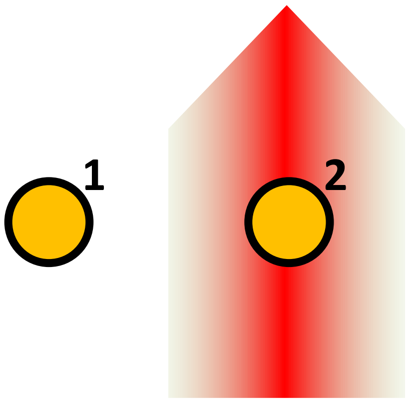
$$(\alpha|\downarrow_1\rangle+\beta|\uparrow_1\rangle)\,|0\rangle|\uparrow_2\rangle$$

$\Big\downarrow$ **bsb SWAP**

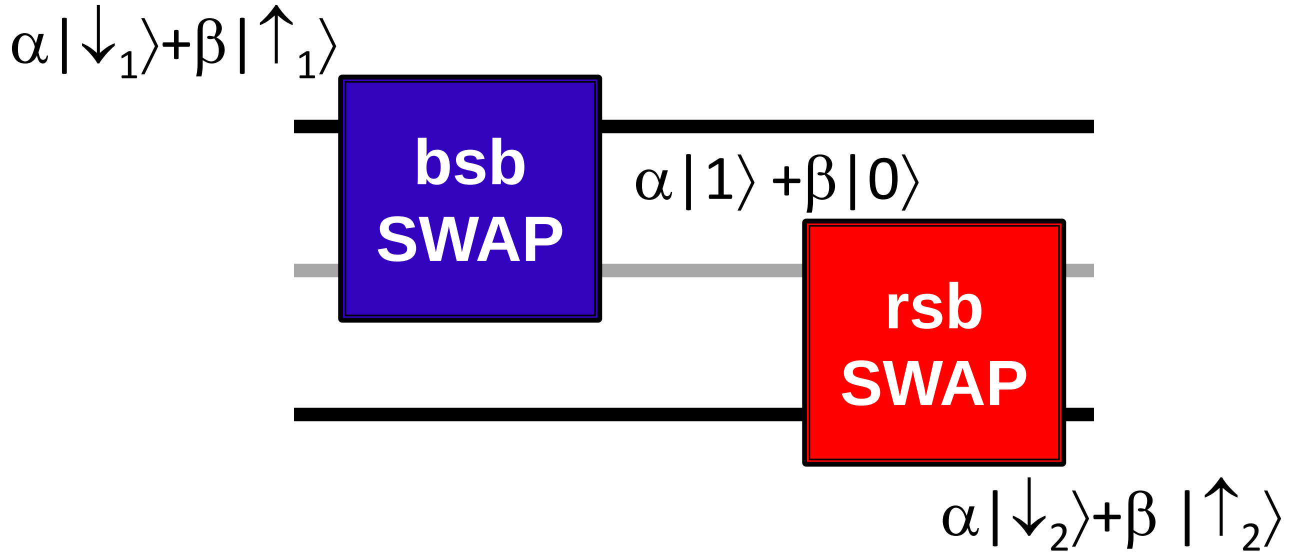
$$|\uparrow_1\rangle\,(\alpha|1\rangle+\beta|0\rangle)\,|\uparrow_2\rangle$$



Transfer Quantum Information via Motional Mode



$\omega_r = \omega_{HF} - \omega_m$



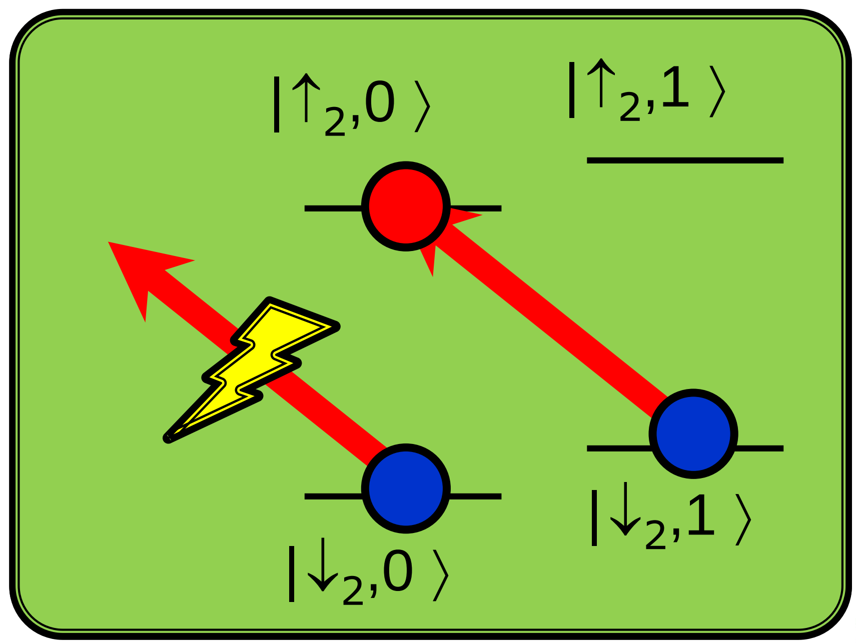
$(\alpha|\downarrow_1\rangle+\beta|\uparrow_1\rangle)\,|0\rangle|\uparrow_2\rangle$

$\downarrow$ **bsb SWAP**

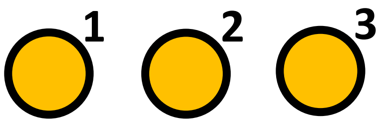
$|\uparrow_1\rangle\,(\alpha|1\rangle+\beta|0\rangle)\,|\uparrow_2\rangle$

$\downarrow$ **rsb SWAP**

$|\downarrow_1\rangle|0\rangle(\alpha|\downarrow_2\rangle+\beta|\uparrow_2\rangle)$



W state generation

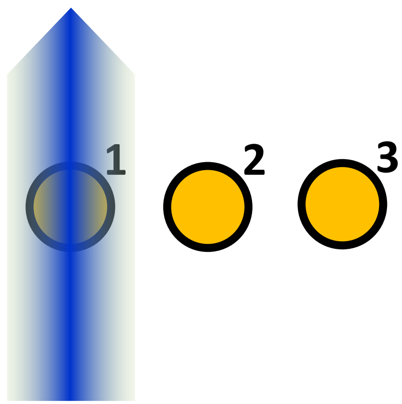


$$\sqrt{\frac{1}{3}}\left(\left|\uparrow_1\uparrow_2\downarrow_3\right\rangle+\left|\uparrow_1\downarrow_2\uparrow_3\right\rangle+\left|\downarrow_1\uparrow_2\uparrow_3\right\rangle\right)$$

$$\left|\uparrow_1\uparrow_2\uparrow_3\right\rangle\left|1\right\rangle$$

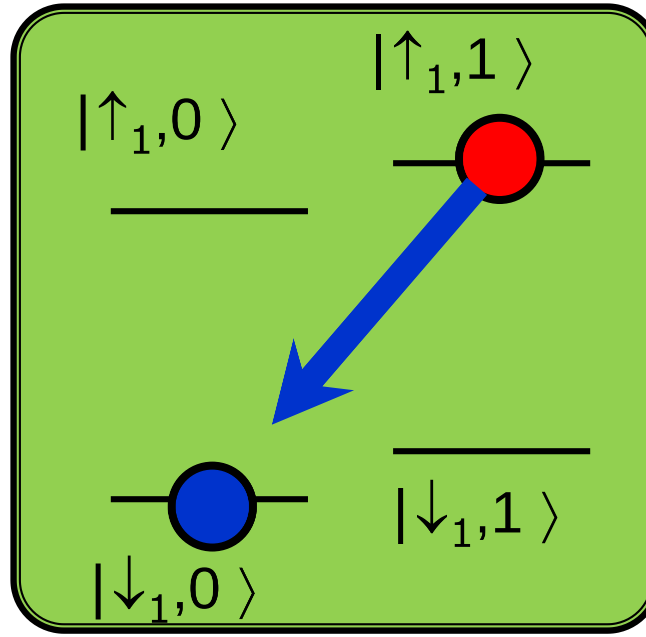
H. Häffner, et al., Nature 438, 639 (2005)

W state generation

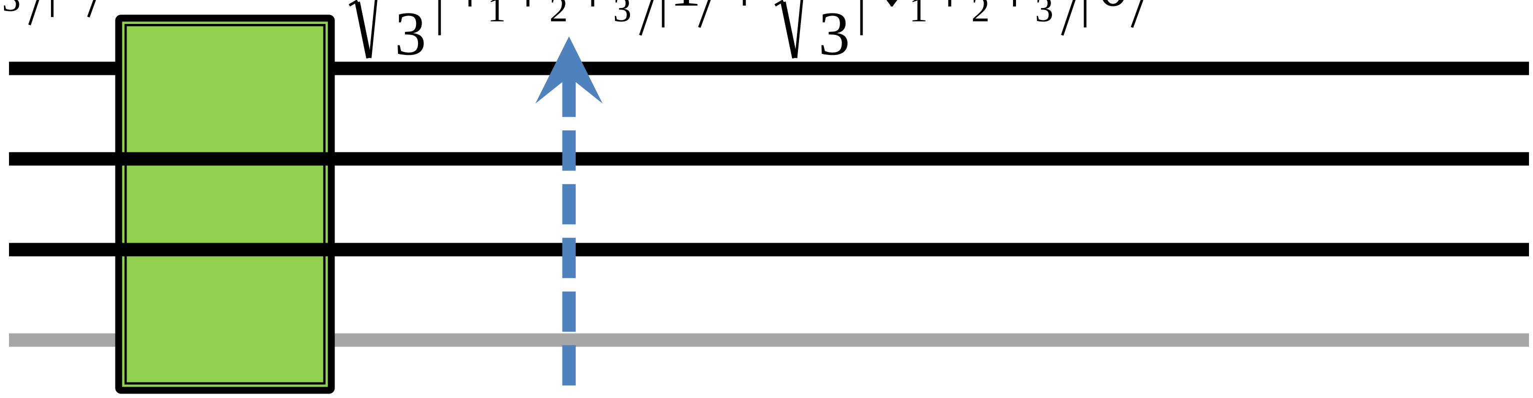


$\omega_b = \omega_{HF} + \omega_m$

$$|\uparrow_1\rangle|1\rangle \rightarrow$$
$$\sqrt{\frac{2}{3}}|\uparrow_1\rangle|1\rangle + \sqrt{\frac{1}{3}}|\downarrow_1\rangle|0\rangle$$

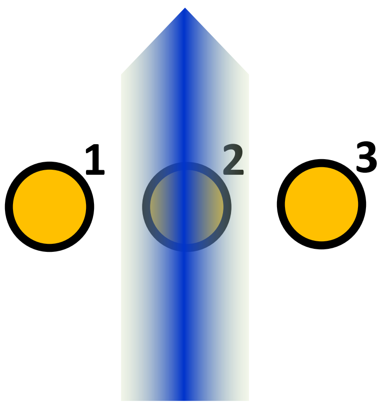


$$|\uparrow_1\uparrow_2\uparrow_3\rangle|1\rangle \qquad \sqrt{\frac{2}{3}}|\uparrow_1\uparrow_2\uparrow_3\rangle|1\rangle + \sqrt{\frac{1}{3}}|\downarrow_1\uparrow_2\uparrow_3\rangle|0\rangle$$



H. Häffner, et al., Nature 438, 639 (2005)

W state generation

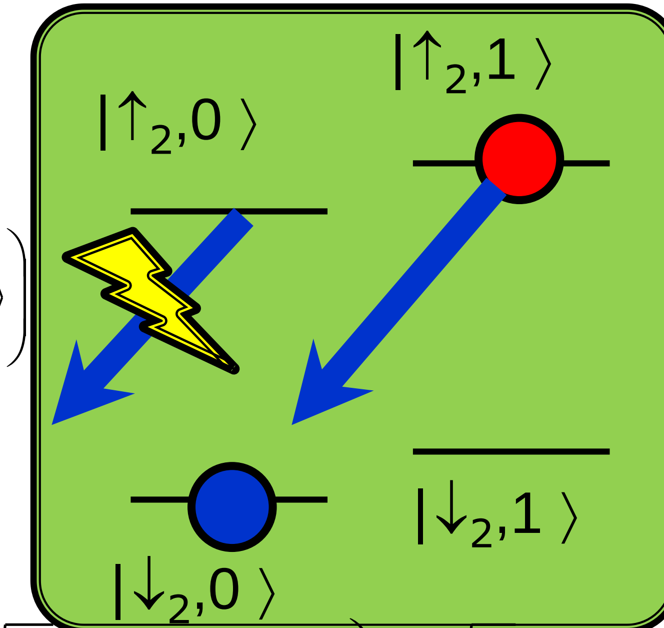


$\omega_b = \omega_{HF} + \omega_m$

$$\sqrt{\frac{2}{3}}|\uparrow_2\rangle|1\rangle \rightarrow$$

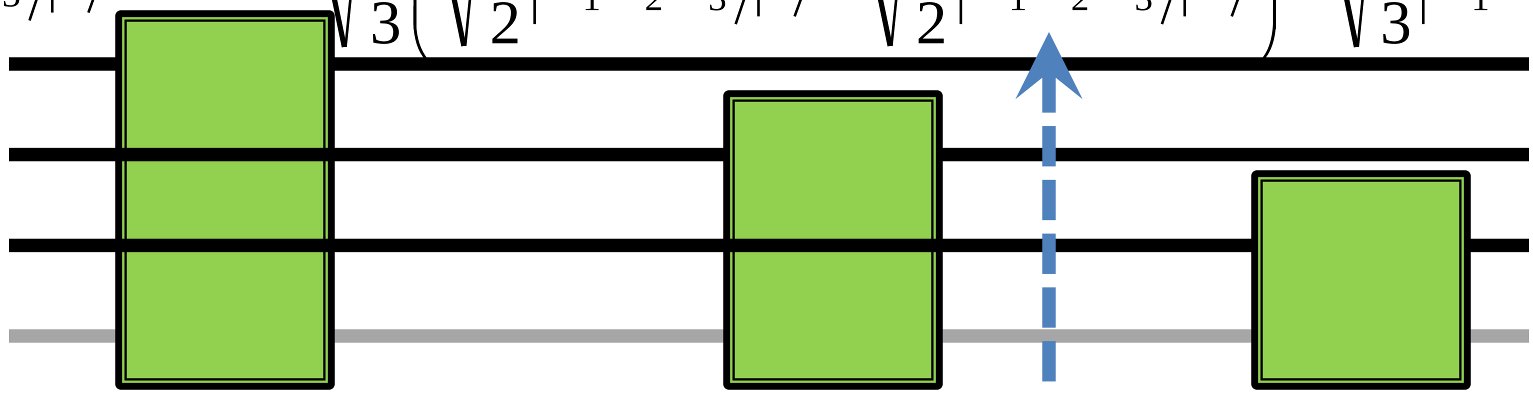
$$\sqrt{\frac{2}{3}}\left(\sqrt{\frac{1}{2}}|\uparrow_2\rangle|1\rangle + \sqrt{\frac{1}{2}}|\downarrow_2\rangle|0\rangle\right)$$

$$\sqrt{\frac{1}{3}}|\uparrow_2\rangle|0\rangle \rightarrow \sqrt{\frac{1}{3}}|\uparrow_2\rangle|0\rangle$$



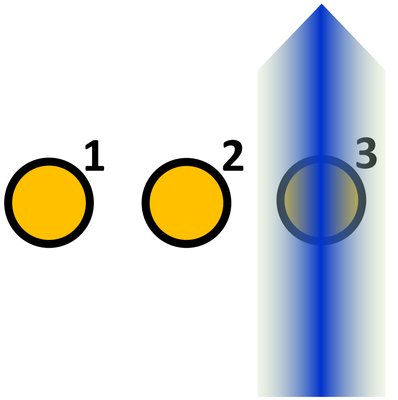
$$|\uparrow_1\uparrow_2\uparrow_3\rangle|1\rangle$$

$$\sqrt{\frac{2}{3}}\left(\sqrt{\frac{1}{2}}|\uparrow_1\uparrow_2\uparrow_3\rangle|1\rangle + \sqrt{\frac{1}{2}}|\uparrow_1\downarrow_2\uparrow_3\rangle|0\rangle\right) + \sqrt{\frac{1}{3}}|\downarrow_1\uparrow_2\uparrow_3\rangle|0\rangle$$



H. Häffner, et al., Nature 438, 639 (2005)

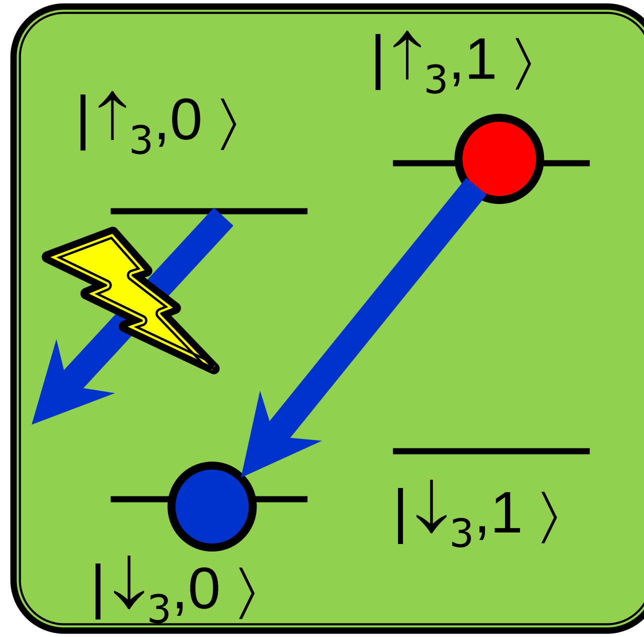
W state generation



$$\omega_b = \omega_{HF} + \omega_m$$

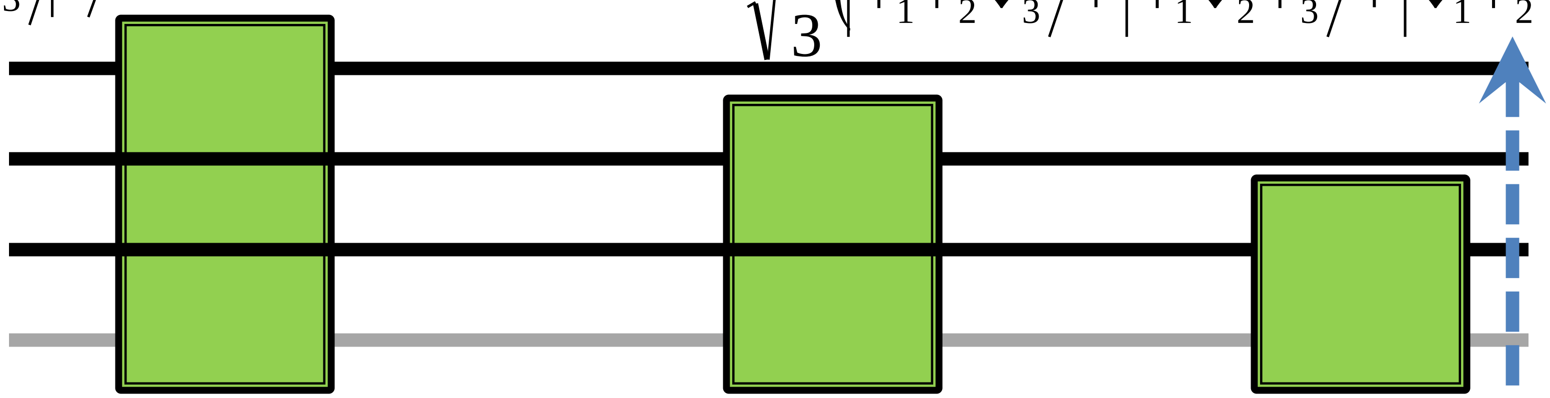
$$\sqrt{\frac{1}{3}}|\uparrow_3\rangle|1\rangle \rightarrow \sqrt{\frac{1}{3}}|\downarrow_3\rangle|0\rangle$$

$$\sqrt{\frac{1}{3}}|\uparrow_3\rangle|0\rangle \rightarrow \sqrt{\frac{1}{3}}|\uparrow_3\rangle|0\rangle$$



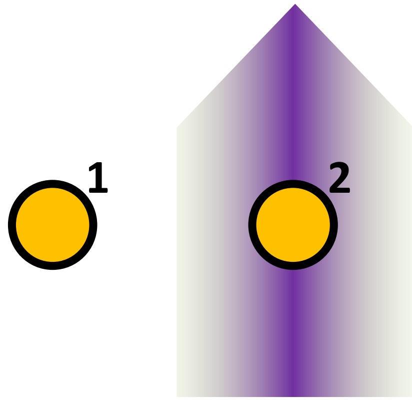
$$|\uparrow_1\uparrow_2\uparrow_3\rangle|1\rangle$$

$$\sqrt{\frac{1}{3}}\Big(|\uparrow_1\uparrow_2\downarrow_3\rangle + |\uparrow_1\downarrow_2\uparrow_3\rangle + |\downarrow_1\uparrow_2\uparrow_3\rangle\Big)|0\rangle$$



H. Häffner, et al., Nature 438, 639 (2005)

Cirac-Zoller (CZ) Gate



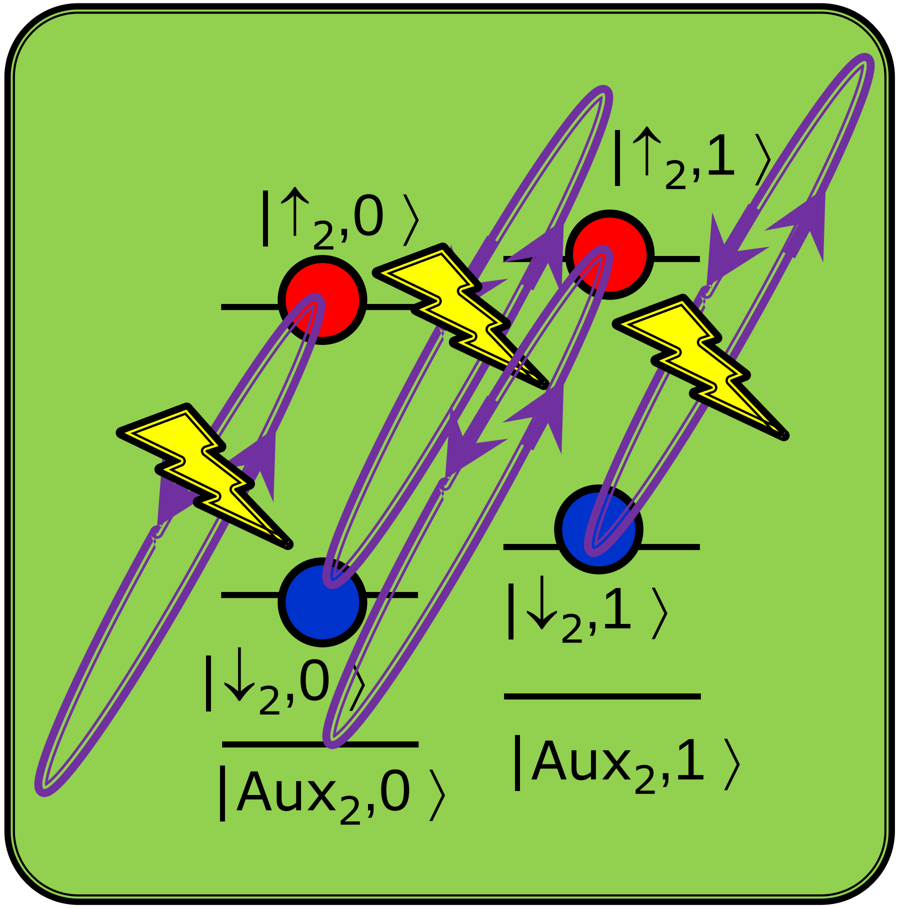
$$\omega_{Ab} = \omega_{HF} + \omega_z + \omega_m$$

$|\downarrow_2,0\rangle \longrightarrow |\downarrow_2,0\rangle$

$|\uparrow_2,0\rangle \longrightarrow |\uparrow_2,0\rangle$

$|\downarrow_2,1\rangle \longrightarrow |\downarrow_2,1\rangle$

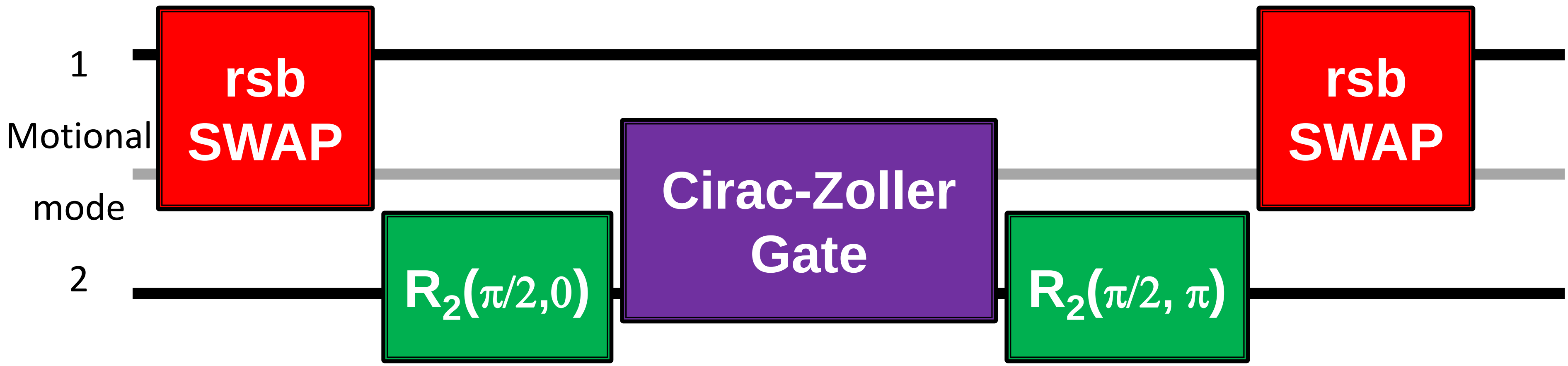
$|\uparrow_2,1\rangle \longrightarrow -|\uparrow_2,1\rangle$



Cirac-Zoller Gate : Controlled-Z gate

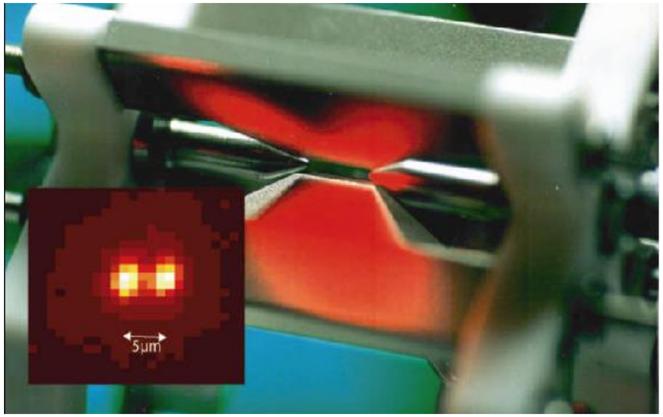
$$
\begin{array}{ccccccc}
\left|\downarrow_1\downarrow_2,0\right\rangle & & \left|\downarrow_1\downarrow_2,0\right\rangle & & \left|\downarrow_1\downarrow_2,0\right\rangle & & \left|\downarrow_1\downarrow_2,0\right\rangle \\
\left|\downarrow_1\uparrow_2,0\right\rangle & & \left|\downarrow_1\uparrow_2,0\right\rangle & & \left|\downarrow_1\uparrow_2,0\right\rangle & & \left|\downarrow_1\uparrow_2,0\right\rangle \\
\left|\uparrow_1\downarrow_2,0\right\rangle & \rightarrow & \left|\downarrow_1\downarrow_2,1\right\rangle & \rightarrow & \left|\downarrow_1\downarrow_2,1\right\rangle & \rightarrow & \left|\uparrow_1\downarrow_2,0\right\rangle \\
\left|\uparrow_1\uparrow_2,0\right\rangle & & \left|\downarrow_1\uparrow_2,1\right\rangle & & -\left|\downarrow_1\uparrow_2,1\right\rangle & & -\left|\uparrow_1\uparrow_2,0\right\rangle
\end{array}
$$

Cirac-Zoller Gate : CNOT gate



$$\begin{aligned}
\left|\downarrow_1\downarrow_2,0\right\rangle &\longrightarrow \left|\downarrow_1\downarrow_2,0\right\rangle \\
\left|\downarrow_1\uparrow_2,0\right\rangle &\longrightarrow \left|\downarrow_1\uparrow_2,0\right\rangle \\
\left|\uparrow_1\downarrow_2,0\right\rangle &\longrightarrow \left|\uparrow_1\uparrow_2,0\right\rangle \\
\left|\uparrow_1\uparrow_2,0\right\rangle &\longrightarrow \left|\uparrow_1\downarrow_2,0\right\rangle
\end{aligned}$$

Cirac-Zoller Gate : Experimental Results



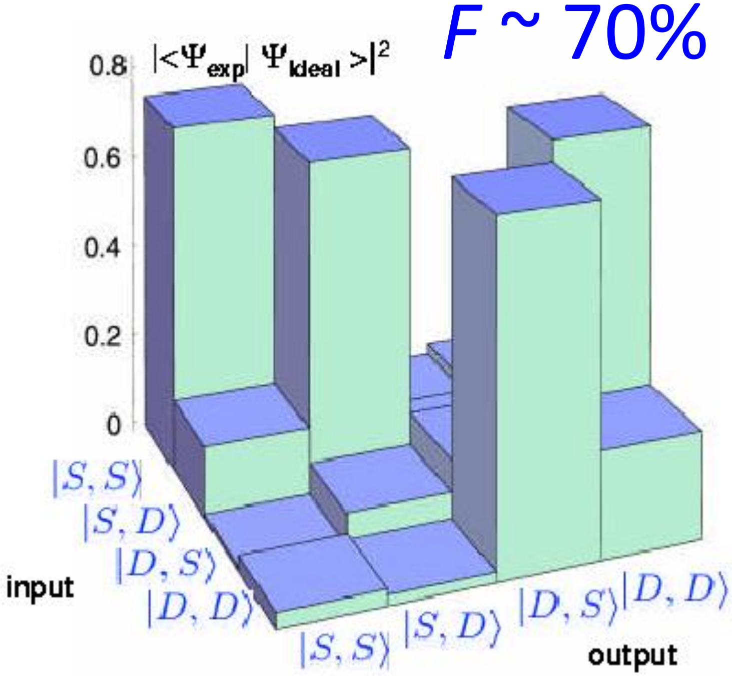
$\pi/2 \bullet$ Controlled-Z $\bullet\ \pi/2$

$\equiv$ Controlled-NOT:

Initial State		Final State	
P(m=1)	P(↑)	P(m=1)	P(↑)
0.02	0.01	0.09	0.16
0.03	0.99	0.04	0.88
0.92	0.05	0.77	0.88
0.94	0.98	0.88	0.19

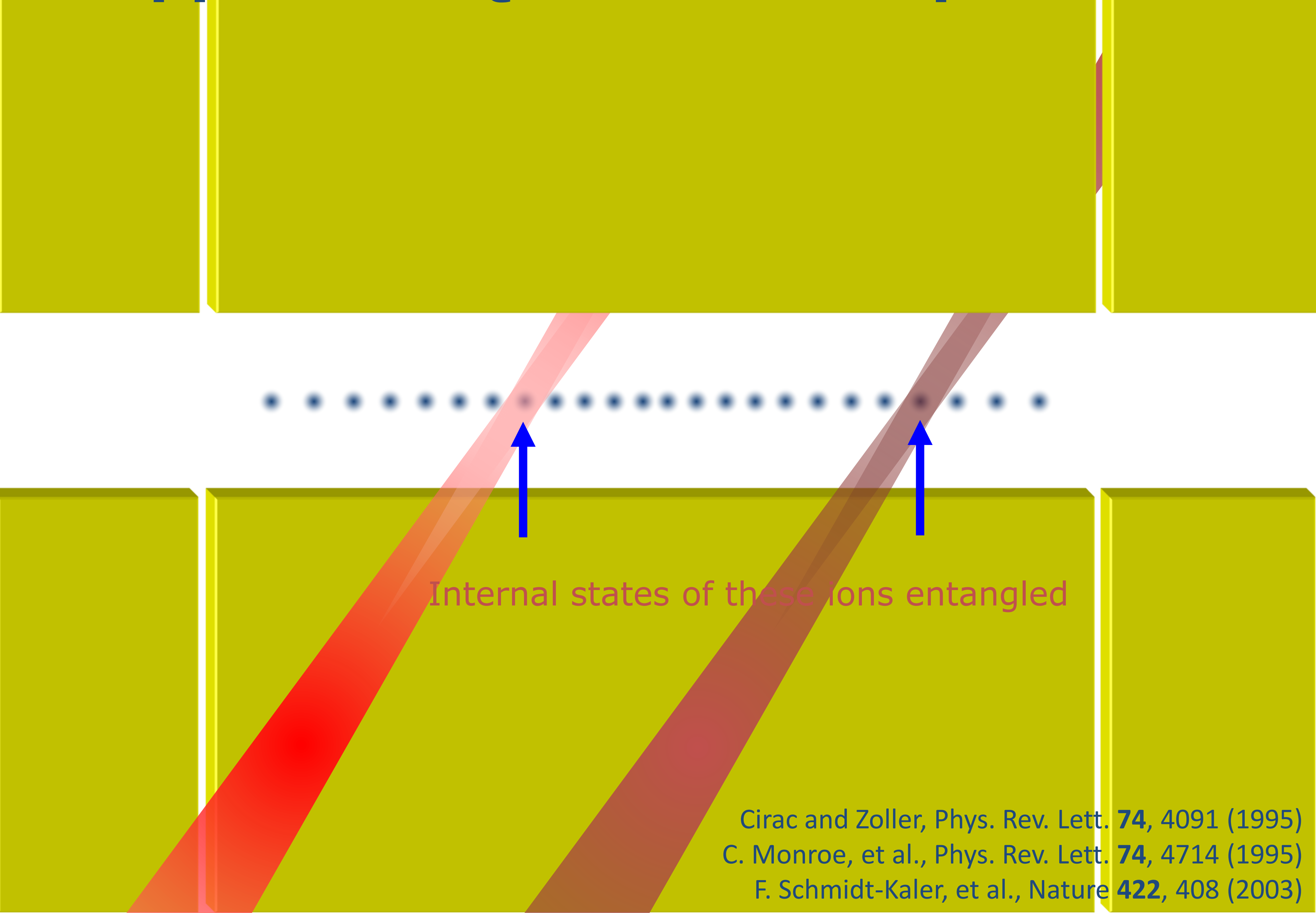
C. Monroe, *et al.*, PRL **75**, 4714 (1995)

F ~ 70%



F. Schmidt-Keler, *et al.*, Nature **422**, 408 (2003)

Trapped Ion Quantum Computer



Cirac and Zoller, Phys. Rev. Lett. **74**, 4091 (1995)
C. Monroe, et al., Phys. Rev. Lett. **74**, 4714 (1995)
F. Schmidt-Kaler, et al., Nature **422**, 408 (2003)

Based on the Sideband Transitions…

- Cirac-Zoller Gate — F. Schmidt-Kaler, et al., Nature **422**, 408 (2003).
- GHZ State Generation — C.F. Roos, et al., Science 304, 1478 (2004).
- W State Generation — H. Haffner, et al., Nature 483, 613 (2005).
- Entanglement Swapping — M. Riebe, et al., Nature Physics **4**, 839 (2008).
- Toffoli Gate — T. Monz, et al., Phys. Rev. Lett. **102**, 040501 (2009).

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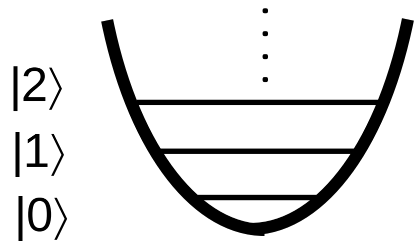
However,

This scheme requires
perfect ground state preparation
perfect individual addressing

Fock State, the number state



$$a^{+}|n\rangle = \sqrt{n+1}|n+1\rangle$$

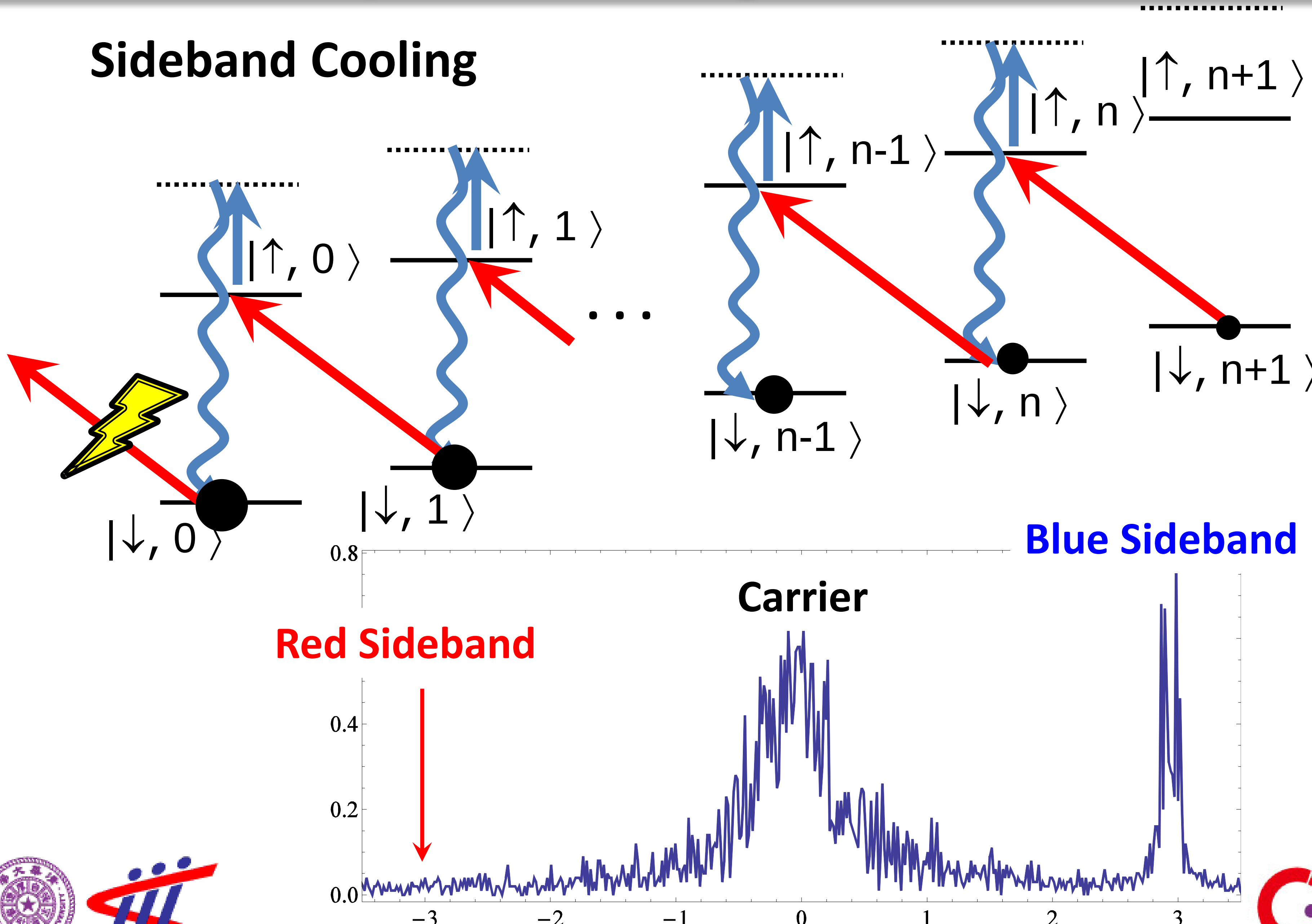
$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{N}|n\rangle = a^{+}a|n\rangle = n|n\rangle$$

Any motional state can be expressed as a superposition of the number state

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$$

Prepare motional ground state

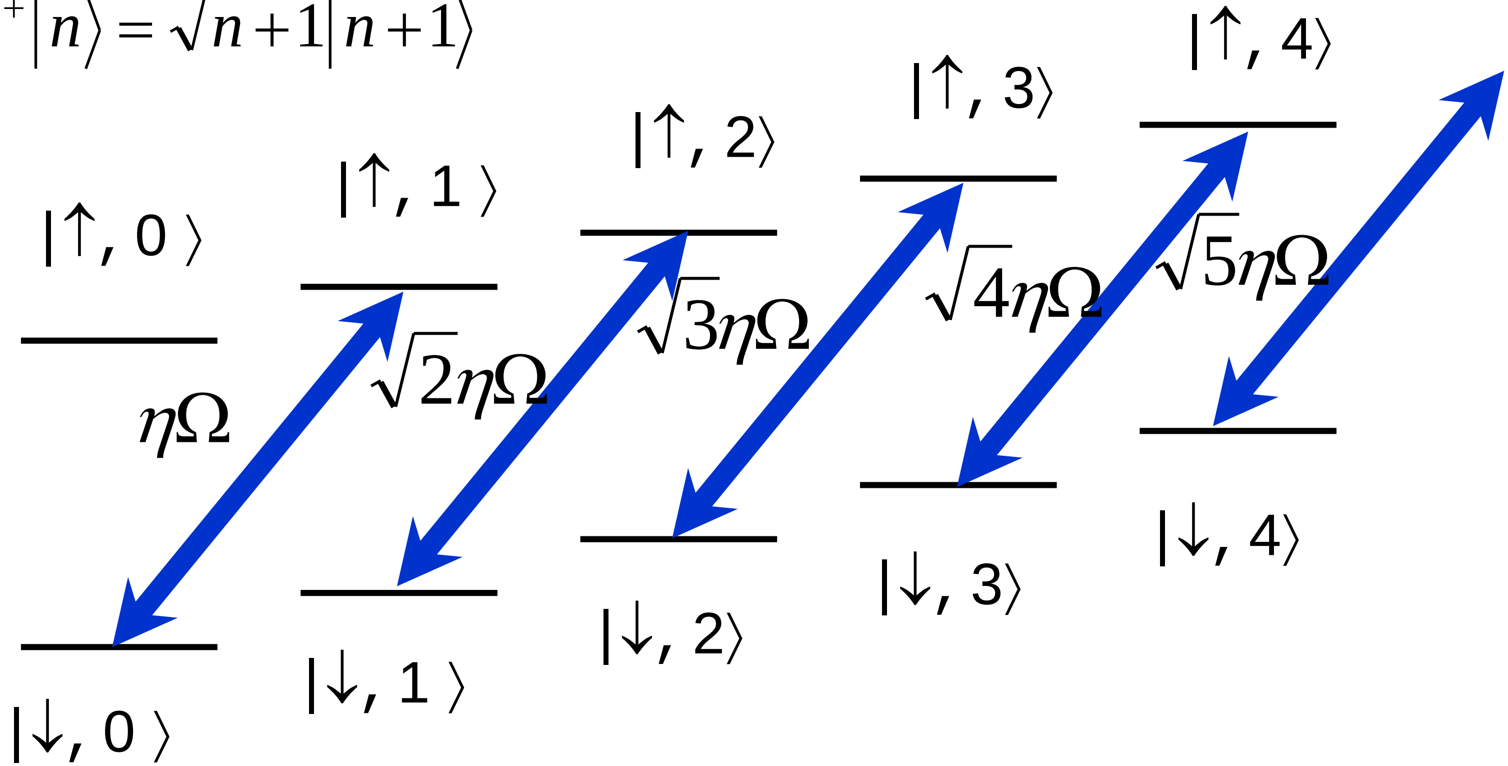


Ion-Motion Coupling: Blue Sideband Transition

"Blue Sideband"

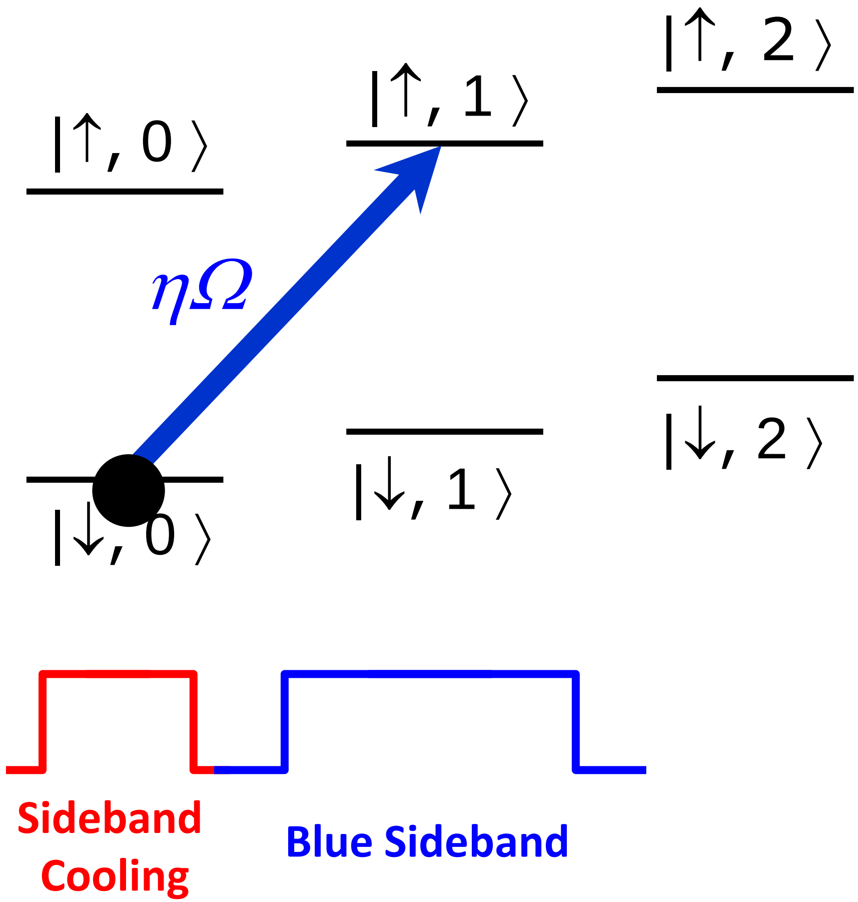
$$\boxed{H_{rsb} = (\hbar/2)\eta\Omega\left[\hat{\sigma}_+ a^+ e^{i\varphi} + \hat{\sigma}\_a e^{-i\varphi}\right]}$$

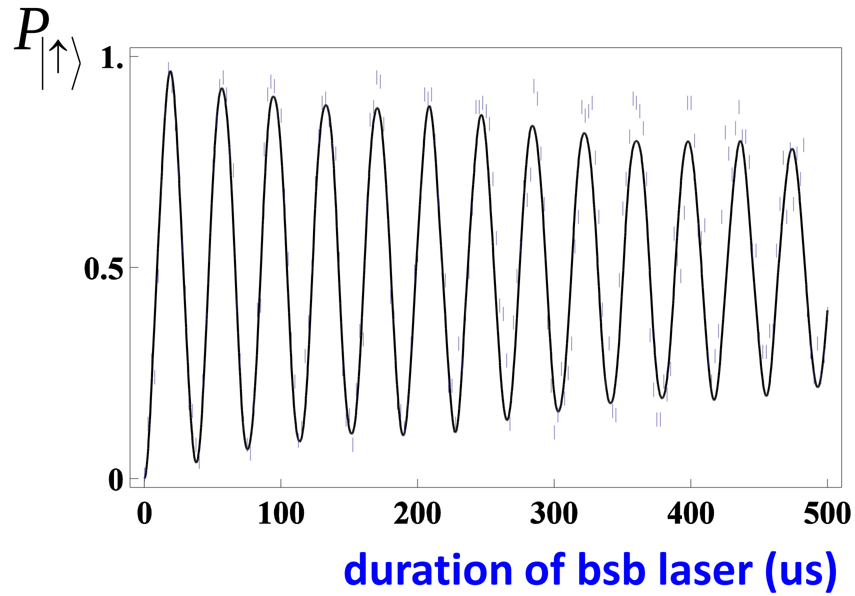
$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

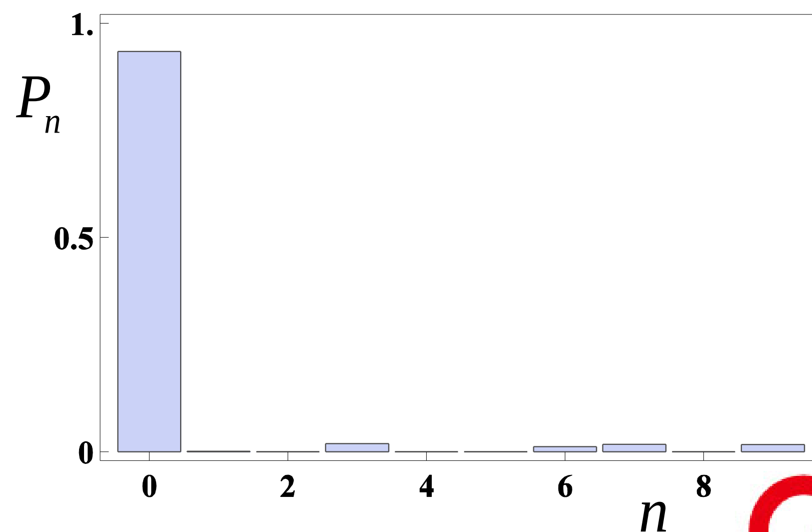


Fock State |n=0⟩ Detection

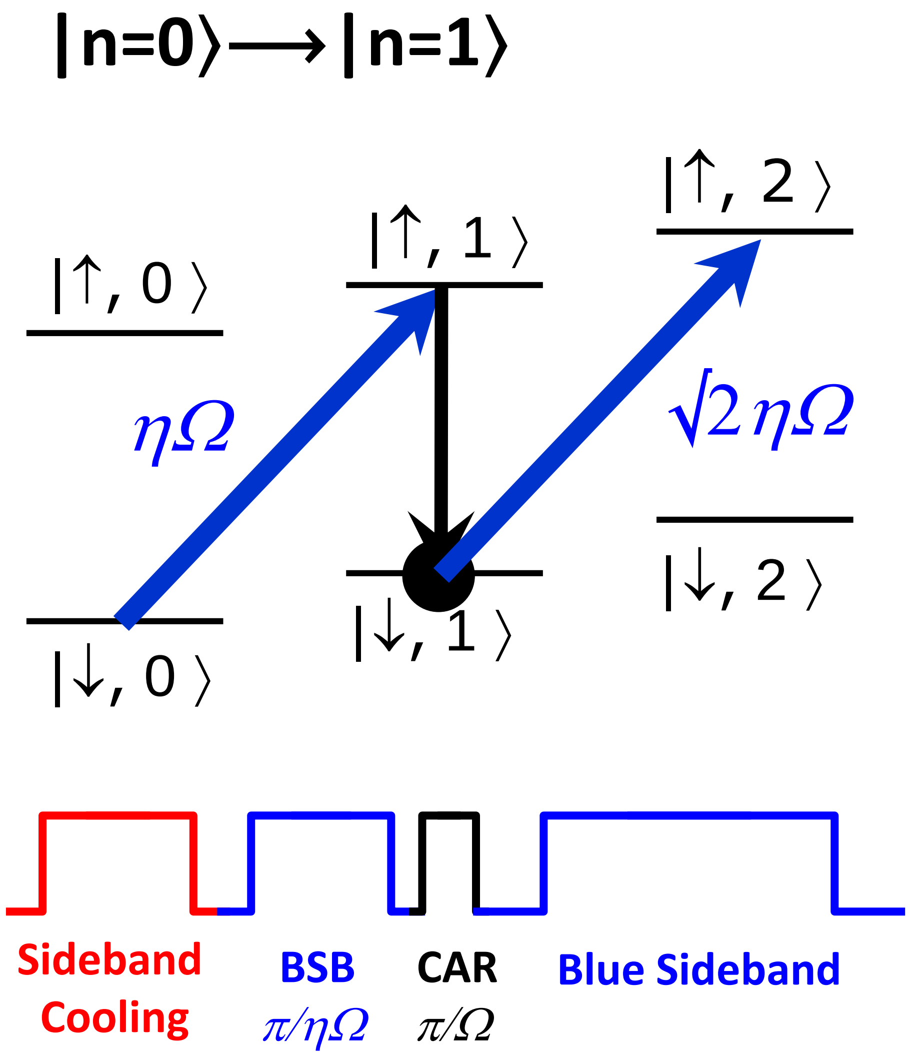
After Sideband Cooling, n=0

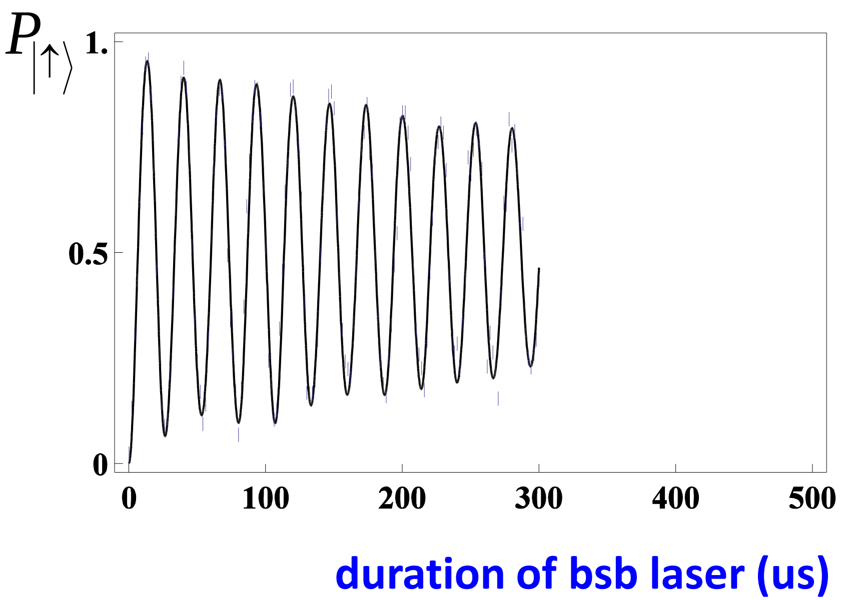


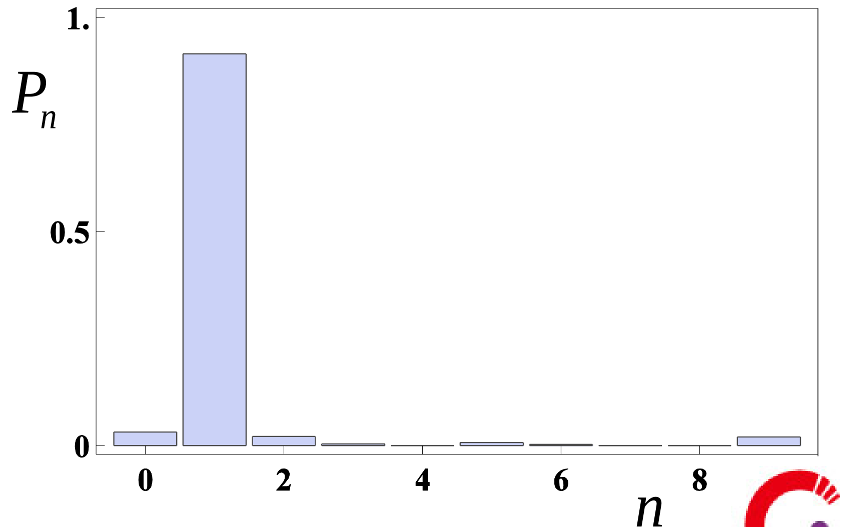




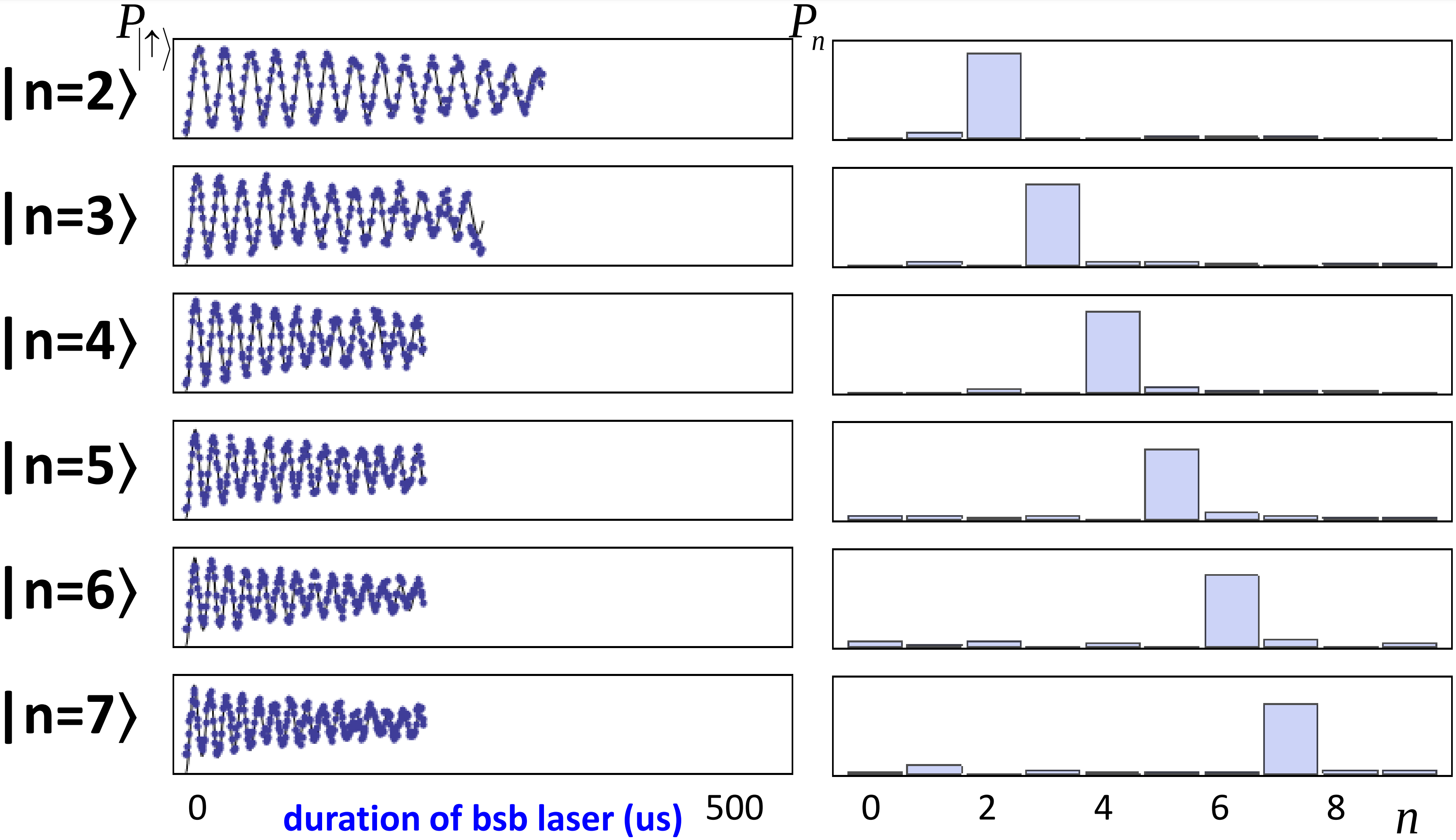
|n=1⟩ State Preparation & Detection







Fock States



$$P_{|\downarrow\rangle} = \frac{1}{2}\left[1 + \sum_{n=0} P(n)\cos\left(\sqrt{n+1}\,\eta\Omega t\right)\right]$$

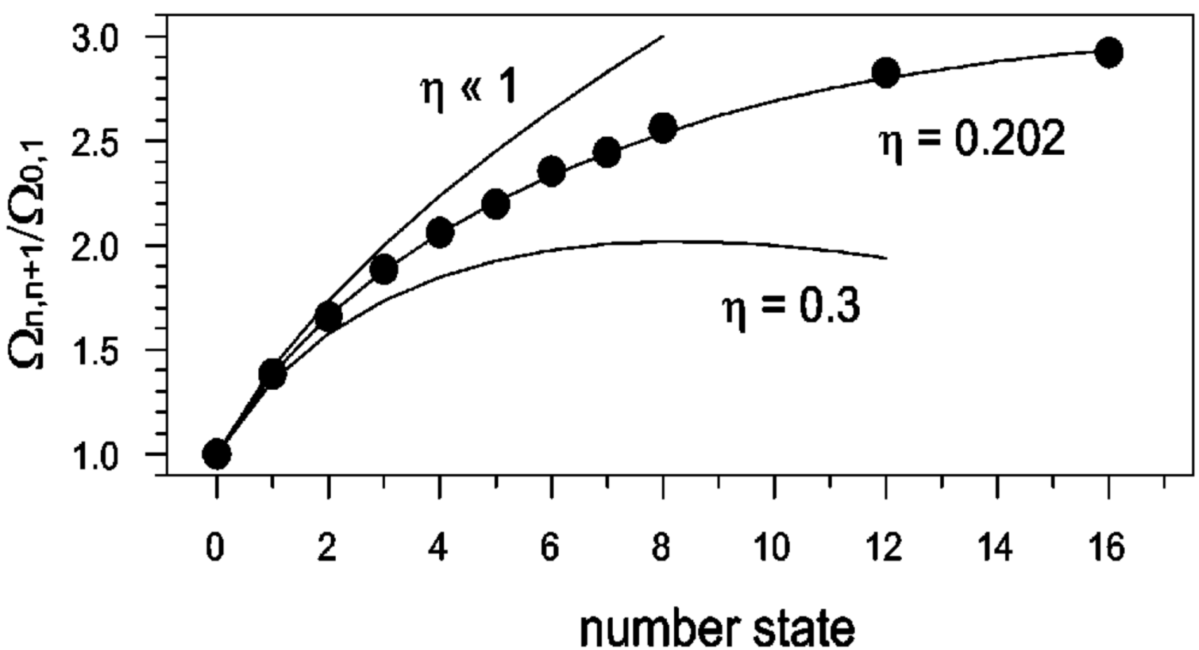
Fock State

D. M. Meekhof, et al., PRL **76**, 1796 (1996)

$$H_{\text{int}} = (\hbar/2)\Omega\hat{\sigma}_+ e^{i\eta\left(ae^{-i\omega_m t}+a^+e^{i\omega_m t}\right)}e^{-i(\delta t+\varphi)} + h.c.$$

$$\Omega_{n,n+s} = \Omega_{n+s,n} = \Omega_0|\langle n+s|e^{i\eta(a+a^\dagger)}|n\rangle|$$

$$= \Omega_0 e^{-\eta^2/2}\eta^{|s|}\sqrt{\frac{n_<!}{n_>!}}L_{n_<}^{|s|}(\eta^2)$$



Thermal State

If the ion is in thermal equilibrium with an external reservoir at temperature T the average weight of the excitation of the state $|n\rangle$ will be proportional to the Boltzmann factor exp[-n$\hbar\omega_m$/(k_BT)], where k_B is the Boltzmann constant.

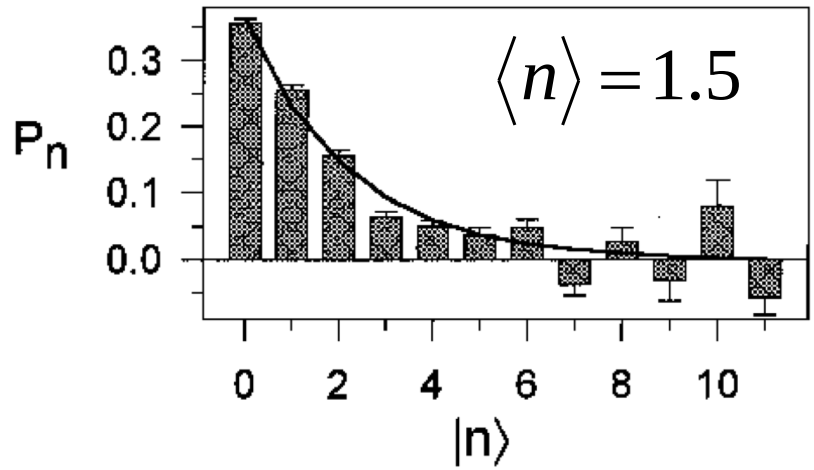
Temperature: $$k_B T = \hbar\omega_m \Big/ \ln\left(1+\frac{1}{\langle n\rangle}\right)$$

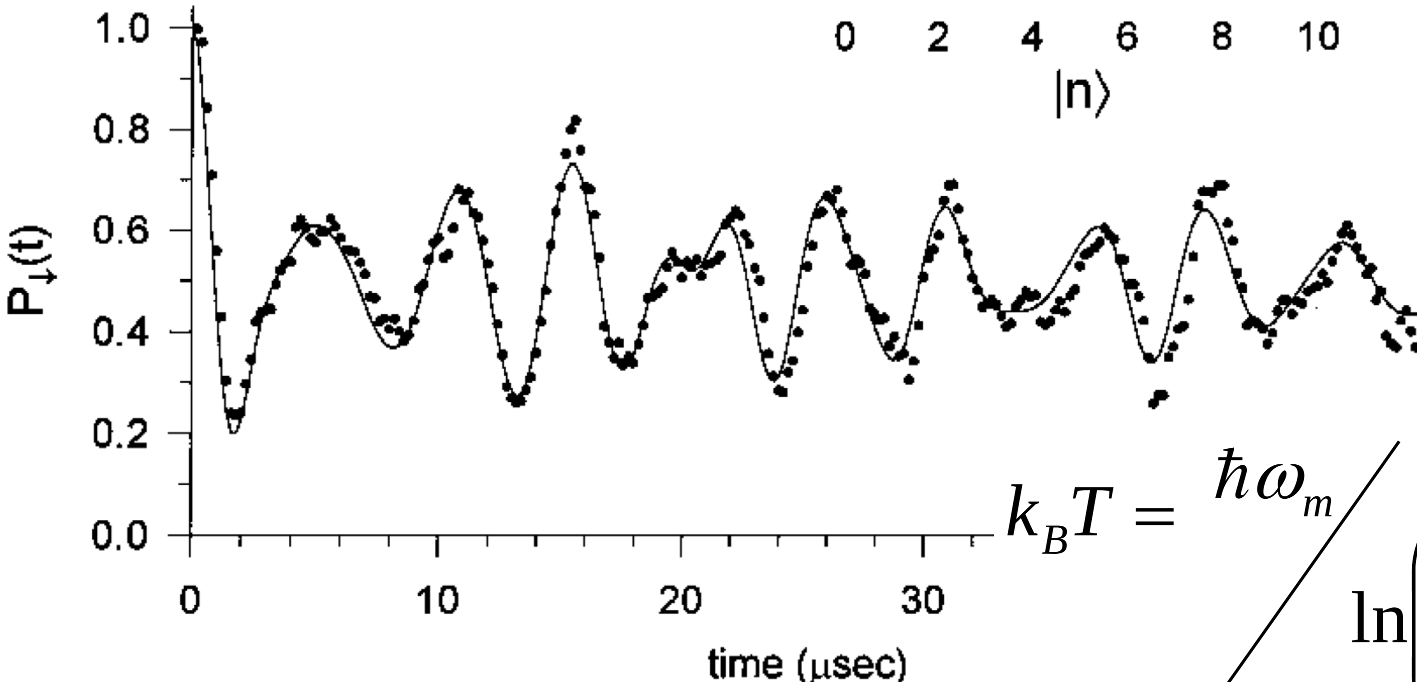
Density matrix $$\rho_{th} = \sum_n P(n)|n\rangle\langle n| = \sum_n \frac{\langle n\rangle^n}{\left(1+\langle n\rangle\right)^{n+1}}|n\rangle\langle n|$$

D. M. Meekhof, et al., PRL **76**, 1796 (1996)

Thermal State

$$P(n) = \frac{\langle n \rangle^n}{\left(1+\langle n \rangle\right)^{n+1}}$$





$$k_B T = \hbar\omega_m \Big/ \ln\left(\frac{\langle n \rangle + 1}{\langle n \rangle}\right)$$

Coherent State

In a static potential harmonic oscillator, a coherent state of motion $|\alpha\rangle$ of the ion corresponds to a Gaussian minimum-uncertainty wave packet in the position representation whose center oscillates classically in the harmonic well and retains its shape. The wave packet has the same shape as the ground-state wave function.

- The eigenstates of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

- In Fock State Basis

$$|\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} \exp\left(-|\alpha|^2/2\right)|n\rangle$$

- Here,

$$\langle n\rangle = |\alpha|^2$$

Displacement Operation

Another popular choice is to represent coherent states as the action of a displacement operator

$$\hat{D}(\alpha) = \exp\left[\alpha a^{+} - \alpha^{*} a\right]$$

on the vacuum state, namely,

$$\hat{D}(\alpha)|0\rangle = |\alpha\rangle$$

The action of successively applied displacement operators is also additive up to phase factors,

$$\hat{D}(\alpha)\hat{D}(\beta)|0\rangle = \hat{D}(\alpha+\beta)e^{1/2\left(\alpha\beta^{*} - \alpha^{*}\beta\right)}$$

Displacement Operation in Motion

Applying both red-sideband and blue-sideband transitions

$$
\begin{aligned}
H_{rsb} + H_{bsb} &= (\hbar/2)\eta\Omega\left[\hat{\sigma}_+ a e^{i\varphi} + \hat{\sigma}_- a^+ e^{-i\varphi}\right] \\
&+ (\hbar/2)\eta\Omega\left[\hat{\sigma}_+ a^+ e^{i\varphi} + \hat{\sigma}_- a e^{-i\varphi}\right] \\
&= (\hbar/2)\eta\Omega\hat{\sigma}_\varphi\left[a^+ + a\right] \quad \text{, where } \hat{\sigma}_\varphi = \hat{\sigma}_+ e^{i\varphi} + \hat{\sigma}_- e^{-i\varphi}
\end{aligned}
$$

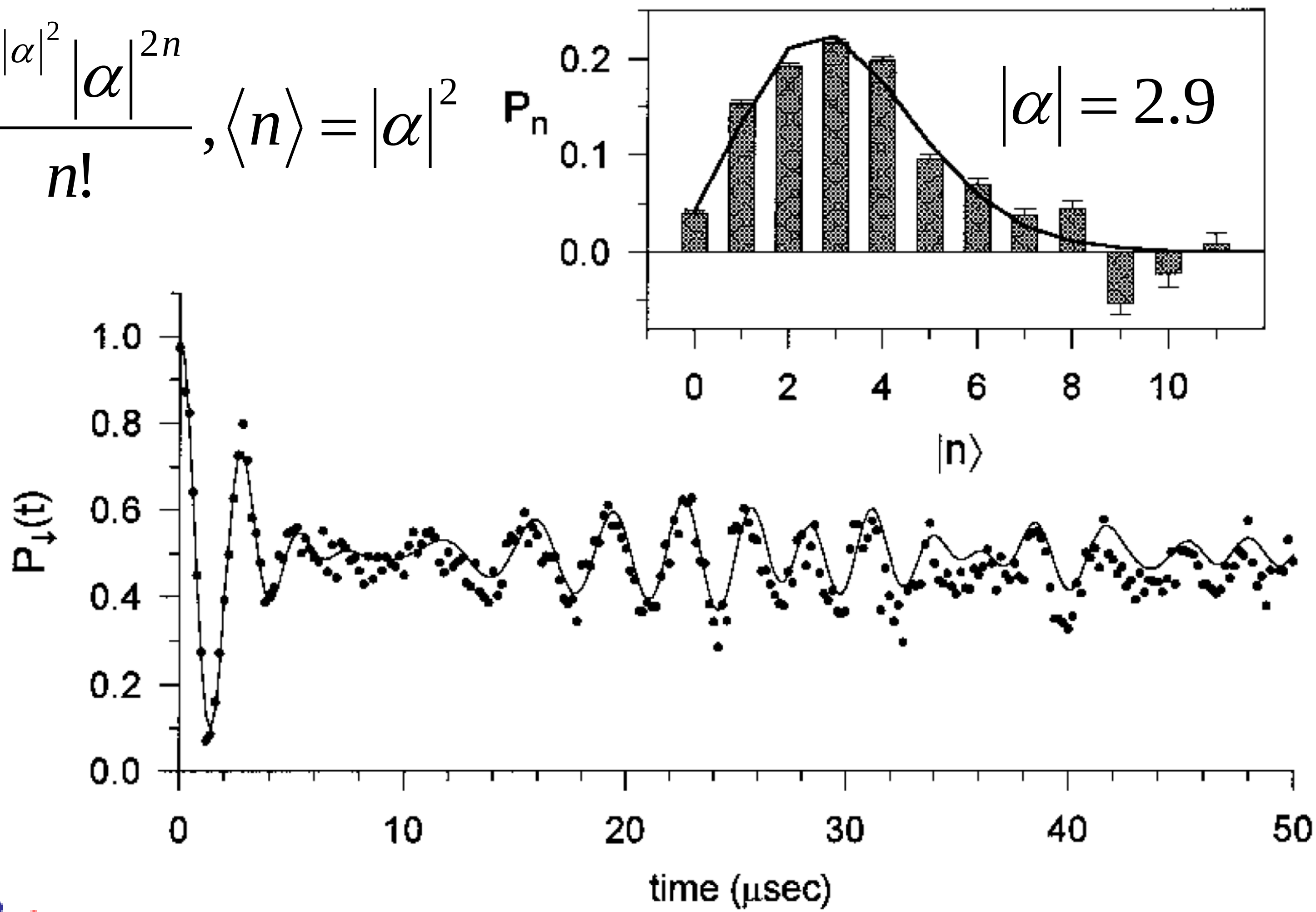
$$
\begin{aligned}
& U_d\left|\downarrow_\varphi\right\rangle|0\rangle = \exp\left[-(i/\hbar)(H_{rsb} + H_{bsb})t\right]\left|\downarrow_\varphi\right\rangle|0\rangle \\
& = \exp\left[-i\alpha\hat{\sigma}_\varphi(a^+ + a)\right]\left|\downarrow_\varphi\right\rangle|0\rangle = \left|\downarrow_\varphi\right\rangle \underbrace{\exp\left[i\alpha(a^+ + a)\right]}_{\text{Displacement Operator}}|0\rangle \\
& = \left|\downarrow_\varphi\right\rangle|\alpha\rangle
\end{aligned}
$$

$$\text{, where } \alpha = \eta\Omega t/2$$

Coherent State of Motion

D. M. Meekhof, et al., PRL **76**, 1796 (1996)

$$P(n) = \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!}, \langle n \rangle = |\alpha|^2$$

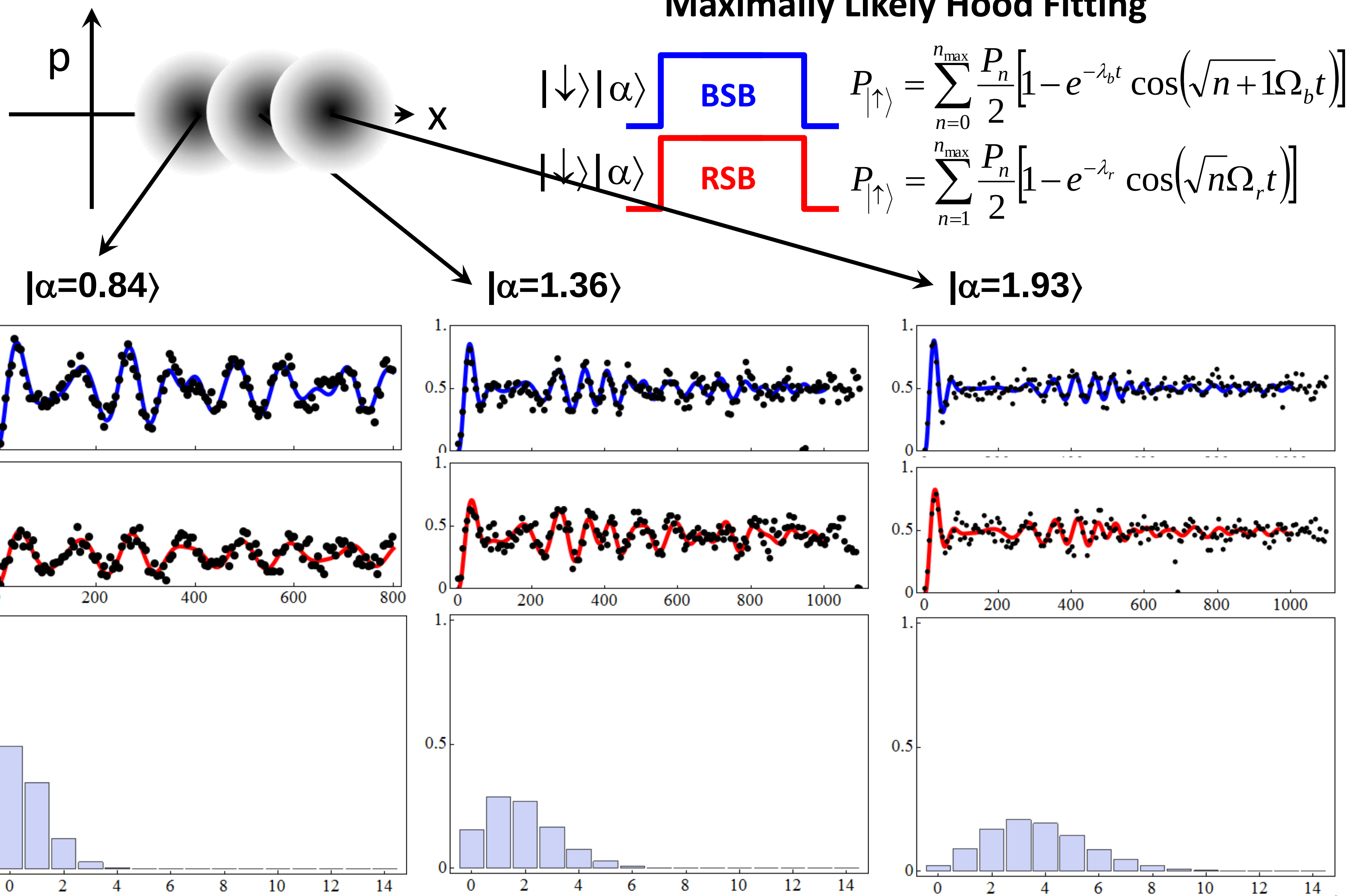


Coherent State Detection

Maximally Likely Hood Fitting

$$P_{|\uparrow\rangle} = \sum_{n=0}^{n_{\max}} \frac{P_n}{2}\left[1 - e^{-\lambda_b t}\cos\left(\sqrt{n+1}\,\Omega_b t\right)\right]$$

$$P_{|\uparrow\rangle} = \sum_{n=1}^{n_{\max}} \frac{P_n}{2}\left[1 - e^{-\lambda_r}\cos\left(\sqrt{n}\,\Omega_r t\right)\right]$$



Squeezed Vacuum State

D. M. Meekhof, et al., PRL **76**, 1796 (1996)

In any quantum state the product of the variance in position and momentum has a lower bound of $\hbar^2/4$, given by the Heisenberg uncertainty relation. If one now "squeezes" the position variance the momentum variance must become wider, so the Heisenberg uncertainty relation is still fulfilled. The coefficients in Fock state basis for the so-called squeezed vacuum state are

$$c_n = \begin{cases} \left(\dfrac{2\sqrt{\beta_s}}{\beta_s+1}\right)^{1/2} \left(\dfrac{\beta_s-1}{\beta_s+1}\right)^{n/2} (-1/2)^{n/2} \dfrac{\sqrt{n!}}{(n/2)!} e^{in\phi}, & n \ \text{even} \\ 0, & n \ \text{odd.} \end{cases}$$

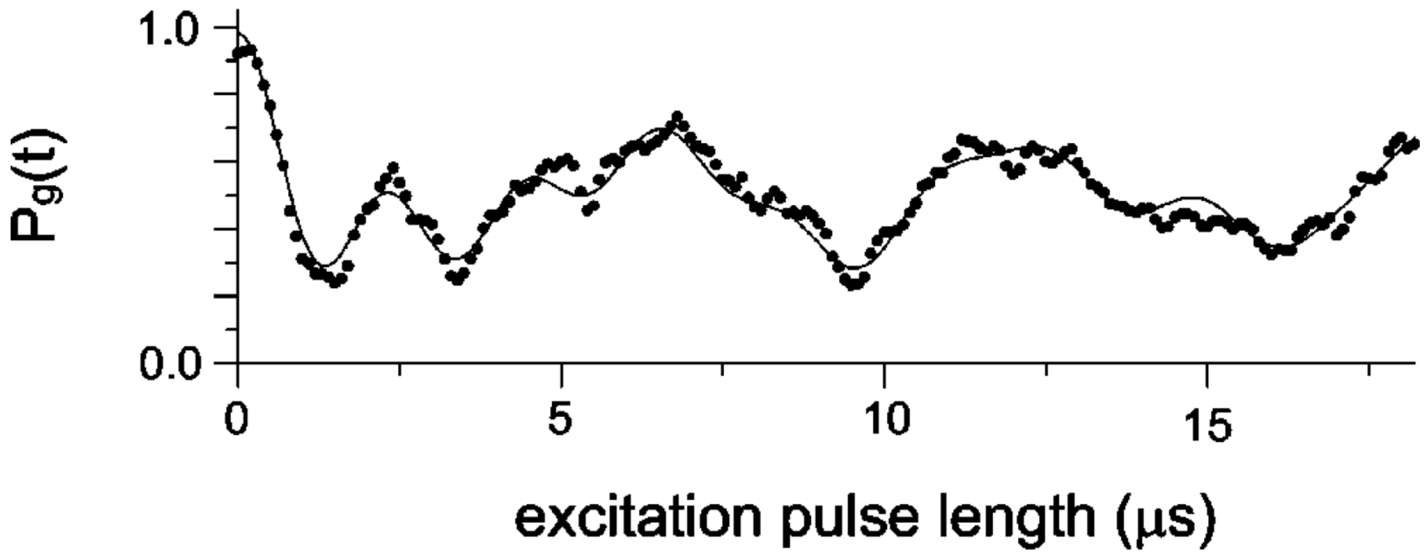
The parameter β_S describes the squeezing of the state, namely, the position variance of the squeezed state is reduced at certain times by $\Delta x_S = \Delta x_0 / \beta_S$, where Δx_0 is the variance of the ground state.

Squeezed Vacuum State of Motion

D. M. Meekhof, et al., PRL **76**, 1796 (1996)

Applying both 2nd order red-sideband and blue-sideband transitions

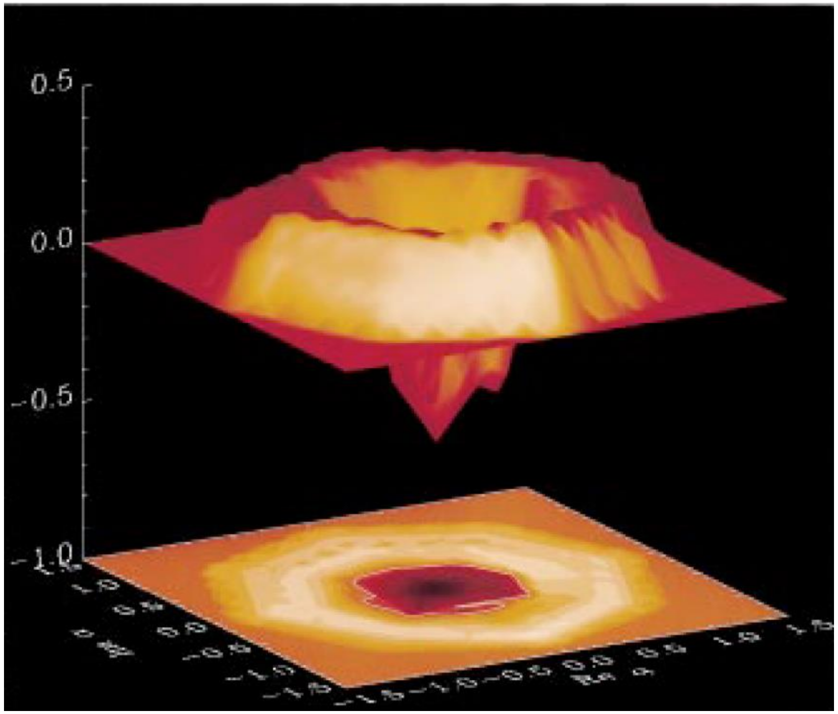
$$H_{squeezing} = (\hbar/2)\eta^2\Omega\hat{\sigma}_\varphi\left[(a^+)^2 + a^2\right]$$



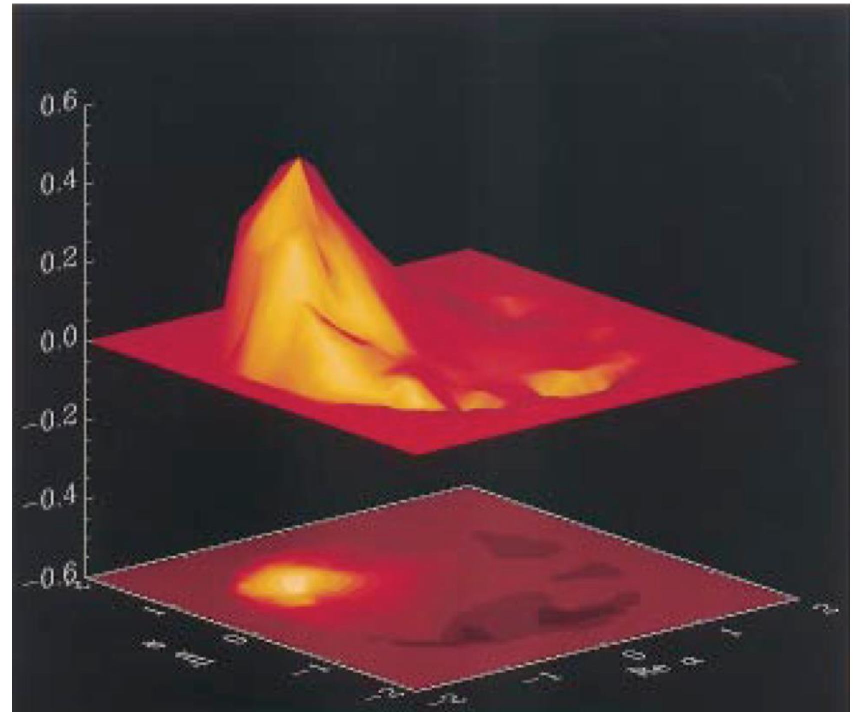
Squeeze parameter β_s=40

Wigner Function

D. Leibfried, et al., PRL **77**, 4821 (1996)



n=1 Fock State



α=1.5 Coherent State

$$W(\alpha)=\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n Q_n(\alpha) \quad , Q_n(\alpha)=\langle n|D^{+}(\alpha)\rho D(\alpha)|n\rangle$$

Ion-Motion Coupling: Schrödinger Cat state

Applying both red-sideband and blue-sideband transitions

$$H_{rb} = H_{rsb} + H_{bsb} = (\hbar/2)\eta\Omega\hat{\sigma}_x\left[a^+ + a\right]$$

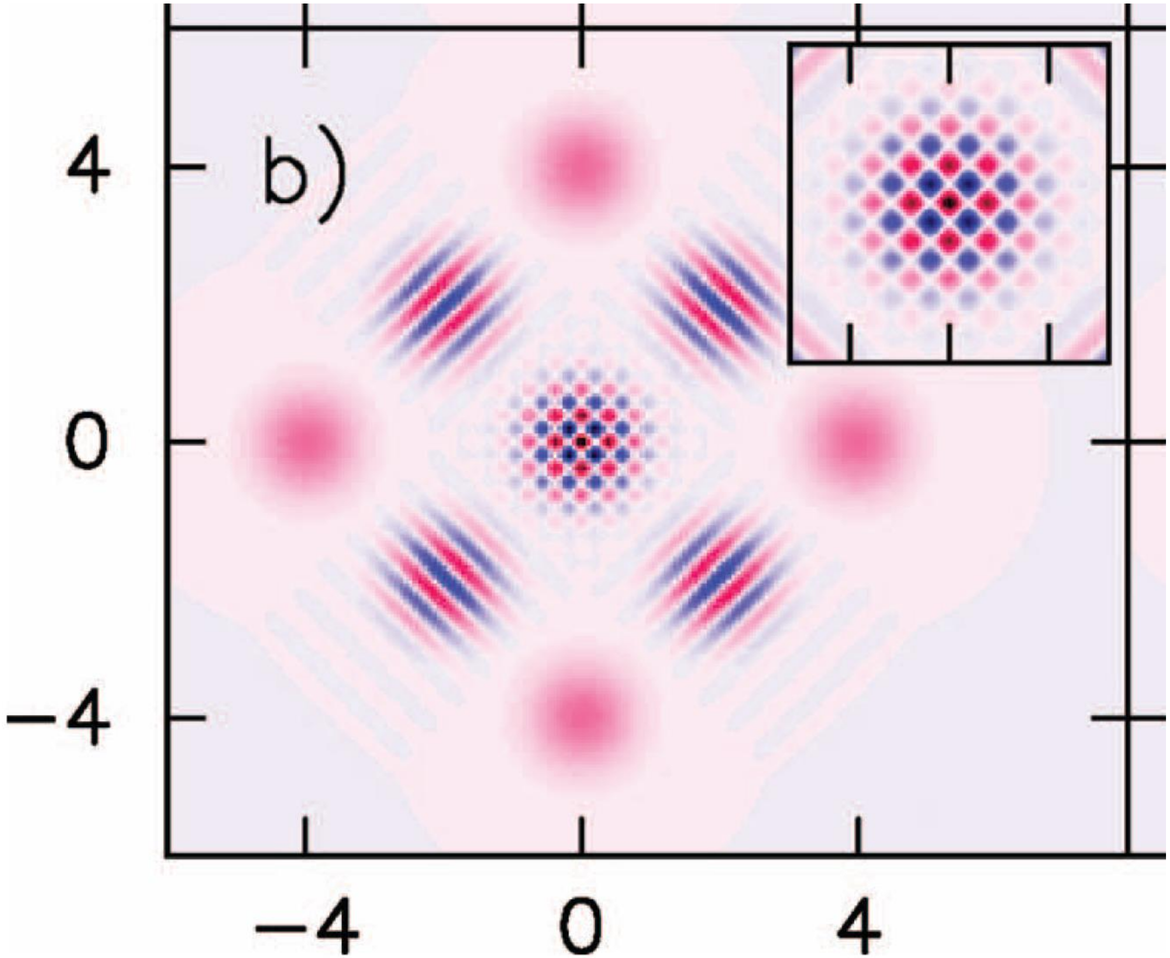
$$U_{rb}\left|\downarrow\right\rangle\left|0\right\rangle = \exp\left[-(i/\hbar)H_{rb}t\right]\left|\downarrow\right\rangle\left|0\right\rangle$$

$$= \exp\left[-i\alpha\hat{\sigma}_x\left(a^+ + a\right)\right]\left|\downarrow\right\rangle\left|0\right\rangle$$

$$= \Big(\underbrace{\left|\downarrow_x\right\rangle\left|\alpha\right\rangle + \left|\uparrow_x\right\rangle\left|-\alpha\right\rangle}_{\text{Schrödinger Cat state}}\Big)/2$$

C. Monroe, et al., Science **272**, 1131 (1996)

Ion-Motion Coupling: sub-Plank Scale



$$|compass\rangle = \left(|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle\right)/2$$

F. Toscano, et al., PRA **73**, 023803 (2006)