Teleportation capability, distillability, and nonlocality on three-qubit states

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Motivation

Teleportation, distillability & nonlocality

- Teleportation: a practical application of quantum entanglement.
- Distillability: an important method to classify quantum entanglement with respect to the usefulness for quantum communication.
- Nonlocality: a physical property to explain the quantum correlation.

Two relations in 2-qubit states [Horodecki *et al.* (1996)]

- If any 2-qubit state is useful for teleportation then it is distillable into a pure entanglement.
- If any 2-qubit state violates the Bell inequality then it is useful for teleportation.

Question & Our Works

Question

What relations among the three features exist for multiqubit states?

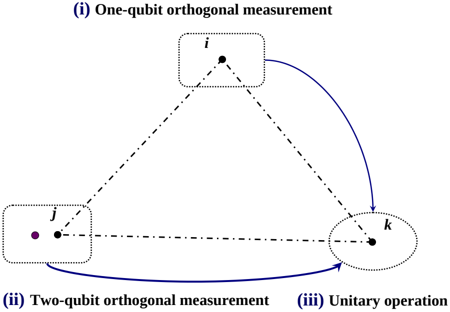
Our Works

- We present teleportation on 3-qubit states, which could be generalized into the multiqubit case.
- We properly define the quantities representing teleportation capability over 3-qubit states, and explicitly compute the quantities.
- We show that there are two relations among teleportation capability, distillability, and nonlocality, which are similar to the 2-qubit case.

A teleportation scheme over a three-qubit state

Our modified HBB protocol

- Original HBB protocol
  - M. Hillery, V. Bužek, & A. Berthiaume, PRA **59**, 1829 (1999).
  - Splitting and reconstruction of quantum information.
- SL, J. Joo, J. Kim, PRA **72**, 024302 (2005).



Teleportation scheme over a general 3-qubit state ρ_{123}

Observable for a one-qubit measurement of the system *i*

$$U_i^\dagger \sigma_3 U_i = U_i^\dagger |0\rangle\langle 0| U_i - U_i^\dagger |1\rangle\langle 1| U_i,$$

where $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$, and U_i is a 2×2 unitary matrix.

The resulting 2-qubit state of the compound system *jk*

$$\begin{aligned} \varrho_{jk}^t &\equiv \frac{\mathrm{tr}_i \left(U_i^\dagger |t\rangle\langle t| U_i \otimes I_{jk} \rho_{123} U_i^\dagger |t\rangle\langle t| U_i \otimes I_{jk} \right)}{\langle t| U_i \rho_i U_i^\dagger |t\rangle} \\ &= \frac{\mathrm{tr}_i \left(|t\rangle\langle t| U_i \otimes I_{jk} \rho_{123} U_i^\dagger |t\rangle\langle t| \otimes I_{jk} \right)}{\langle t| U_i \rho_i U_i^\dagger |t\rangle} \end{aligned}$$

Teleportation capability for 2-qubit states

Teleportation fidelity

- Teleportation fidelity
  - $F(\Lambda_\rho) = \int d\xi \langle\xi|\Lambda_\rho(|\xi\rangle\langle\xi|)|\xi\rangle$, where Λ_ρ is a given teleportation protocol over a 2-qubit state ρ.
- Fully entangled fraction of ρ
  - $f(\rho) = \max \langle e|\rho|e\rangle$, where the maximum is over all maximally entangled states $|e\rangle$ of 2 qubits.
- Maximal fidelity achievable from a given bipartite state ρ
  - $F(\Lambda_\rho) = \frac{2f(\rho)+1}{3}$.

Definition

$F(\Lambda_\rho) > 2/3$ (or $f(\rho) > 1/2$) if and only if ρ is said to be useful for teleportation.

Teleportation capability for 3-qubit states

Maximal teleportation fidelity for 3-qubit states

Let F_i be the maximal teleportation fidelity on the resulting 2-qubit state in the compound system jk after the measurement of the system i

- $f_i = \max_{U_i} \left[\langle 0|U_i \rho_i U_i^\dagger|0\rangle f(\varrho_{jk}^0) + \langle 1|U_i \rho_i U_i^\dagger|1\rangle f(\varrho_{jk}^1) \right].$
- $F_i = \dfrac{2f_i + 1}{3}.$

Definition

A given 3-qubit state ρ_{123} is said to be useful for 3-qubit teleportation if and only if $F_i > 2/3$ (or $f_i > 1/2$) for every $i \in \{1, 2, 3\}$.

General 3-qubit states

- A three-qubit state ρ_{123} can be described as

$$
\begin{aligned}
&\frac{1}{8} I \otimes I \otimes I \\
&+\frac{1}{8}\left(\vec{s}_1 \cdot \vec{\sigma} \otimes I \otimes I + I \otimes \vec{s}_2 \cdot \vec{\sigma} \otimes I + I \otimes I \otimes \vec{s}_3 \cdot \vec{\sigma}\right) \\
&+\frac{1}{8} \sum_{k,l=1}^{3} \left(b_1^{kl} I \otimes \sigma_k \otimes \sigma_l + b_2^{kl} \sigma_k \otimes I \otimes \sigma_l + b_3^{kl} \sigma_k \otimes \sigma_l \otimes I\right) \\
&+\frac{1}{8} \sum_{j,k,l=1}^{3} t^{jkl} \sigma_j \otimes \sigma_k \otimes \sigma_l .
\end{aligned}
$$

For each $i = 1, 2, 3$, let $\mathbf{b}_i$ be a 3×3 real matrix with (k, l)-entry b_i^{kl}. Let $\mathbf{T}_1^j$, $\mathbf{T}_2^k$, and $\mathbf{T}_3^l$ be 3×3 real matrices with (k, l)-entry t^{jkl}, (j, l)-entry t^{jkl}, and (j, k)-entry t^{jkl}, respectively.

Teleportation capability for general 3-qubit states

- $F_i = \dfrac{2f_i + 1}{3}$.
- $f_i = \dfrac{1}{4} + \dfrac{1}{8} \max \left[\|\mathbf{b}_i + \sum_{l=1}^{3} x_l \mathbf{T}_i^l\| + \|\mathbf{b}_i - \sum_{l=1}^{3} x_l \mathbf{T}_i^l\| \right]$, where $\|\cdot\| = \mathrm{tr}|\cdot|$, and the maximum is taken over real numbers x_l satisfying $x_1^2 + x_2^2 + x_3^2 = 1$.
- $f_i = \dfrac{1}{4} + \dfrac{1}{8} \left[\|\mathbf{b}_i + \sum_{l=1}^{3} y_l \mathbf{T}_i^l\| + \|\mathbf{b}_i - \sum_{l=1}^{3} y_l \mathbf{T}_i^l\| \right]$, where $y_l = \|\mathbf{T}_i^l\| / \sqrt{\sum_t \|\mathbf{T}_i^t\|^2}$.

Example: 3-qubit states with 4-parameters

[Dür *et al.*, PRL **83**, 3562 (1999)]

The class of 3-qubit states with 4-parameters

$$\rho_{\mathrm{GHZ}} = \lambda_0^+ |\Psi_0^+\rangle\langle\Psi_0^+| + \lambda_0^- |\Psi_0^-\rangle\langle\Psi_0^-| + \sum_{j=1}^{3} \lambda_j \left(\left|\Psi_j^+\right\rangle\left\langle\Psi_j^+\right| + \left|\Psi_j^-\right\rangle\left\langle\Psi_j^-\right| \right),$$

where $\lambda_0^+ + \lambda_0^- + 2\sum_j \lambda_j = 1$, and $\left|\Psi_j^\pm\right\rangle = (|j\rangle \pm |7-j\rangle)/\sqrt{2}$ are the GHZ-basis states.

- Any of 3-qubit states can be transformed into a state ρ_{GHZ} in the class by LOCC (the so-called depolarizing process).
- If $\lambda_i + \lambda_j \leq 1/4$ for $i,j \in \{1,2,3\}$ then
  - $f_1 = \lambda_0^+ + \lambda_3 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_2$
  - $f_2 = \lambda_0^+ + \lambda_2 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_1 - \lambda_3$
  - $f_3 = \lambda_0^+ + \lambda_1 = 1/2 + (\lambda_0^+ - \lambda_0^-)/2 - \lambda_2 - \lambda_3$

GHZ-distillable 3-qubit states

Proposition (Dür *et al.*, PRL **83**, 3562 (1999))

If a 3-qubit state ρ_{123} has $\rho_{123}^{T_j} < 0$ for all $j = 1, 2, 3$, where T_j represents the partial transposition for the system j, then one can distill a GHZ state from many copies of ρ_{123} by LOCC.

Definition

- If one can distill a GHZ state from many copies of ρ_{123} by LOCC then ρ_{123} is said to be GHZ-distillable.
- $N_j(\rho_{123}) = \left(\|\rho_{123}^{T_j}\| - 1 \right) /2$.

Corollary

A given 3-qubit state ρ_{123} is GHZ-distillable if $N_j(\rho_{123}) > 0$ for all $j = 1, 2, 3$.

Relation between teleportation capability and distillability on 3-qubit states

Theorem (1)

If a 3-qubit state ρ_{123} is useful for 3-qubit teleportation then it is GHZ-distillable.

Proof.

We recall that for any 2-qubit state ρ, $f(\rho) \leq 1/2 + N(\rho)$, where N is the negativity. Then since N is an entanglement monotone,

$$\begin{aligned} f_i &= \max_{U_i} \sum_{t=0}^{1} \langle t|U_i \rho_i U_i^\dagger|t\rangle f\left(\varrho_{jk}^t\right) \leq \max_{U_i} \sum_{t=0}^{1} \langle t|U_i \rho_i U_i^\dagger|t\rangle \left(1/2 + N(\varrho_{jk}^t)\right) \\ &\leq 1/2 + N_j(\rho_{123}), \quad 1/2 + N_k(\rho_{123}), \end{aligned}$$

where i, j and k are distinct in $\{1, 2, 3\}$. □

Example: Converse of Theorem (1) is not true.

Example

There exists a 3-qubit state which is GHZ-distillable but is not useful for 3-qubit teleportation.

- ρ_{GHZ} with $\lambda_0^+ = 0.4$, $\lambda_0^- = 0$, and $\lambda_1 = \lambda_2 = \lambda_3 = 0.1$.
- $N_j(\rho_{\mathrm{GHZ}}) = \max\{0, (\lambda_0^+ - \lambda_0^-)/2 - \lambda_{4-j}\}$.
- $N_1(\rho_{\mathrm{GHZ}}) = N_2(\rho_{\mathrm{GHZ}}) = N_3(\rho_{\mathrm{GHZ}}) = 0.1 > 0$, that is, it is GHZ-distillable.
- Since $f_1 = f_2 = f_3 = 0.5$, it is not useful for 3-qubit teleportation.

Mermin inequality

Mermin inequality on 3-qubit states

Let $\mathcal{B}_M$ be the Mermin operator associated with the Mermin inequality.

- $\mathcal{B}_M = \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma} - \vec{a}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma}$
  $-\vec{b}_1 \cdot \vec{\sigma} \otimes \vec{a}_2 \cdot \vec{\sigma} \otimes \vec{b}_3 \cdot \vec{\sigma} - \vec{b}_1 \cdot \vec{\sigma} \otimes \vec{b}_2 \cdot \vec{\sigma} \otimes \vec{a}_3 \cdot \vec{\sigma}$,
  where $\vec{a}_j$ and $\vec{b}_j$ are unit vectors in $\mathbb{R}^3$.
- For a given 3-qubit state ρ, the Mermin inequality is $\mathrm{tr}\,(\rho \mathcal{B}_M) \leq 2$.

A specific form of Mermin inequality

- Take $\vec{a} = \vec{a}_1 = \vec{a}_2 = \vec{a}_3$ and $\vec{b} = \vec{b}_1 = \vec{b}_2 = \vec{b}_3$.
- Consider the quantity $\max_{\vec{a},\vec{b}} \mathrm{tr}\,(\rho_{123} \mathcal{B}_M)$.

Relation between teleportation capability and nonlocality on 3-qubit states

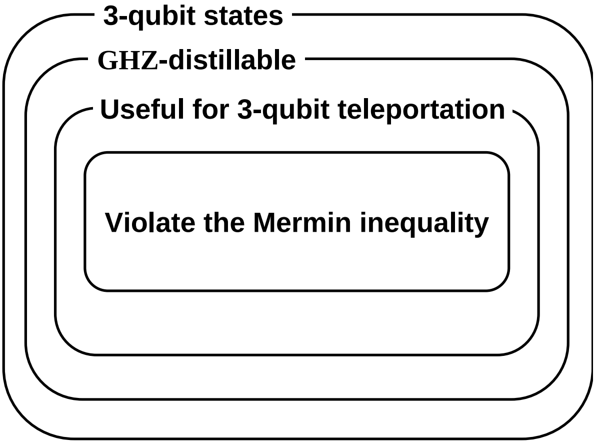
A specific form of Mermin inequality

- Take $\vec{a} = \vec{a}_1 = \vec{a}_2 = \vec{a}_3$ and $\vec{b} = \vec{b}_1 = \vec{b}_2 = \vec{b}_3$.
- Consider the quantity $\max_{\vec{a},\vec{b}} \mathrm{tr}\left(\rho_{123}\mathcal{B}_M\right)$.

Theorem (2)

If a 3-qubit state ρ_{123} violates the Mermin inequality with respect to the quantity $\max_{\vec{a},\vec{b}} \mathrm{tr}\left(\rho_{123}\mathcal{B}_M\right)$, that is, if $\max_{\vec{a},\vec{b}} \mathrm{tr}\left(\rho_{123}\mathcal{B}_M\right) > 2$, then $f_i > 1/2$ for all $i = 1, 2, 3$, and hence it is useful for 3-qubit teleportation.

Relations among teleportation capability, distillability & nonlocality for 3-qubit states



Example: Theorem (2) does not hold in general.

Example

If we consider the Mermin inequality with respect to the quantity $\max_{\vec{a}_j,\vec{b}_k} \mathrm{tr}\,(\rho_{123}\mathcal{B}_M)$, then Theorem (2) does not hold in general.

- $|0\rangle(|00\rangle + |11\rangle)/\sqrt{2}$ violates the Mermin inequality with respect to $\max_{\vec{a}_j,\vec{b}_k} \mathrm{tr}\,(\rho_{123}\mathcal{B}_M)$.
- It is clear that the state is not useful for 3-qubit teleportation since $f_1 = 1$ and $f_2 = 1/2 = f_3$.

Summary

Summary

We have shown that if any 3-qubit state is useful for 3-qubit teleportation then the 3-qubit state is GHZ-distillable, and that if any 3-qubit state violates a specific Mermin inequality then the 3-qubit state is useful for 3-qubit teleportation.

Future Study

- For $n \geq 4$, there exist n-qubit bound entangled states which violate the Mermin inequality [Dür, PRL **87**, 230402 (2001)].
- There exists at least one splitting of the n qubits into two groups such that pure-state entanglement can be distilled [Acín, PRL **88**, 027901 (2002)].
- If we would consider quantum communications between 2 or 3 groups then our results could be generalized into multiqubit cases.