Topological aspects of dual superconductors

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I. Introduction

• topological soliton and knot structure

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II. Model for two-gap superconductors

- **free energy of two-flavor Ginzburg-Landau theory**

$$\begin{aligned} F &= \frac{1}{2m_1}\left|\left(\frac{\hbar}{i}\nabla + \frac{2e}{c}\vec{A}\right)\Psi_1\right|^2 + \frac{1}{2m_2}\left|\left(\frac{\hbar}{i}\nabla - \frac{2e}{c}\vec{A}\right)\Psi_2\right|^2 \\ &+ \frac{1}{8\pi}\vec{B}^2 + V + \eta(\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1) \end{aligned}$$

Ψ_α: order parameters for Cooper pairs of two different flavors ($\alpha = 1, 2$)
V: potential of the form $V(|\Psi_{1,2}|^2) = -b_\alpha|\Psi_\alpha|^2 + \frac{1}{2}c_\alpha|\Psi_\alpha|^4$
η: characteristic of interband Josephson coupling strength

$\eta = 0$ Josephson coupling describes liquid metallic hydrogen which should allow coexistent superconductivity of protonic and electronic Cooper pairs

interband Josephson coupling merely changes energy of knot associated with two-band superconductors

two condensates are characterized by different effective masses m_α, coherence lengths $\xi_\alpha = \hbar/(2m_\alpha b_\alpha)^{1/2}$ and densities $\langle|\Psi_\alpha|^2\rangle = b_\alpha/c_\alpha$

- **modulus field and CP^1 complex fields**

introduce fields ρ and z_α defined as

$$\Psi_\alpha = (2m_\alpha)^{1/2}\rho z_\alpha$$

ρ: modulus field given by condensate densities and masses
$$\rho^2 = \tfrac{1}{2m_1}|\Psi_1|^2 + \tfrac{1}{2m_2}|\Psi_2|^2$$
z_α: CP^1 complex fields chosen to satisfy geometrical constraint
$$z_\alpha^* z_\alpha = |z_1|^2 + |z_2|^2 = 1$$

- **gauge invariant supercurrent**

$$\begin{aligned}\vec{J} &= -\frac{e}{2m_1}\left[\Psi_1^*\left(\frac{\hbar}{i}\nabla + \frac{2e}{c}\vec{A}\right)\Psi_1 - \Psi_1\left(\frac{\hbar}{i}\nabla - \frac{2e}{c}\vec{A}\right)\Psi_1^*\right] \\ &\quad + \frac{e}{2m_2}\left[\Psi_2^*\left(\frac{\hbar}{i}\nabla - \frac{2e}{c}\vec{A}\right)\Psi_2 - \Psi_2\left(\frac{\hbar}{i}\nabla + \frac{2e}{c}\vec{A}\right)\Psi_2^*\right]\end{aligned}$$

which can be rewritten in terms of fields ρ and z_α

$$\vec{J} = -\hbar e\rho^2\left(\vec{C} + \frac{4e}{\hbar c}\vec{A}\right)$$

where

$$\vec{C} = i(\nabla z^\dagger z - z^\dagger \nabla z) = i(z_1\nabla z_1^* - z_1^*\nabla z_1 - z_2\nabla z_2^* + z_2^*\nabla z_2)$$

with $z = (z_1, z_2^*)$

- **dynamical physical fields**

CP^1 model is equivalent to the O(3) NLSM at canonical level

introduce dynamical physical fields n_a $(a = 1, 2, 3)$ which are mappings from space-time manifold (or direct product of compact two-dimensional Riemann surface M^2 and time dimension R^1) to two-sphere S^2

$$n_a : \mathsf{M}^2 \otimes R^1 \to S^2$$

dynamical physical fields of CP^1 model are z_α $(\alpha = 1, 2)$ which map spacetime manifold $\mathsf{M}^2 \otimes R^1$ into S^3

$$z_\alpha : \mathsf{M}^2 \otimes R^1 \to S^3$$

- **Hopf bundle**

S^3 is homeomorphic to SU(2) group manifold and CP^1 model is invariant under local U(1) gauge symmetry for arbitrary spacetime dependent ξ

$$z \longrightarrow e^{i\xi/2} z$$

physical configuration space of CP^1 model is that of gauge orbits which form the coset $S^3/S^1 = S^2 = CP^1$

in order to associate physical fields of CP^1 model with those of O(3) NLSM, exploit projection from S^3 to S^2, namely Hopf bundle

$$n_a = z^\dagger \sigma_a z$$

with Pauli matrices σ_a and n_a fields satisfying constraint $n_a n_a = 1$

- **free energy in terms of ρ and n_a**

$$F = \hbar^2(\nabla\rho)^2 + \frac{1}{4}\hbar^2\rho^2(\nabla n_a)^2 + \frac{1}{4e^2\rho^2}\vec{J}^2 + \frac{1}{8\pi}\vec{B}^2 + V + K\rho^2 n_1,$$

where $K = 2\eta(m_1 m_2)^{1/2}$

introduce gauge invariant vector fields

$$\vec{S} = \frac{1}{\hbar e\rho^2}\vec{J}$$

to yield

$$\begin{aligned} F &= \hbar^2(\nabla\rho)^2 + \frac{1}{4}\hbar^2\rho^2\left[(\nabla n_a)^2 + \vec{S}^2\right] \\ &\quad + \frac{\hbar^2 c^2}{128\pi e^2}\left(\nabla\times\vec{S} + \frac{1}{2}\epsilon_{abc} n_a \nabla n_b \times \nabla n_c\right)^2 + V + K\rho^2 n_1 \end{aligned}$$

III. Meissner effects

- **magnetic field in terms of ρ, n_a and $\vec{S}$**

$$\vec{B} = \nabla \times \vec{A} = -\frac{\hbar c}{4e}\left(\nabla \times \vec{S} + \frac{1}{2}\epsilon_{abc} n_a \nabla n_b \times \nabla n_c\right)$$

$$\nabla \times \vec{C} = \frac{1}{2}\epsilon_{abc} n_a \nabla n_b \times \nabla n_c$$

$$\nabla \times \vec{J} = -\frac{4e^2}{c}\rho^2 \vec{B} + \frac{2}{\rho}\nabla\rho \times \vec{J} - \frac{\hbar e}{2}\rho^2 \epsilon_{abc} n_a \nabla n_b \times \nabla n_c$$

$$\nabla \times \vec{S} = -\frac{4e}{\hbar c}\vec{B} - \frac{1}{2}\epsilon_{abc} n_a \nabla n_b \times \nabla n_c$$

note topological contribution proportional to $\epsilon_{abc} n_a \nabla n_b \times \nabla n_c$ which originates from interactions of Cooper pairs of two different flavors

- **two-gap equations for $\vec{J}$ and $\vec{B}$**

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J}$$

$$\begin{aligned}
\nabla^2 \vec{J} &= \left(\frac{16\pi e^2}{c^2}\rho^2 + \frac{2}{\rho}\nabla^2\rho - \frac{2}{\rho^2}(\nabla\rho)^2\right)\vec{J} + \frac{8e^2}{c}\rho\nabla\rho \times \vec{B} \\
&\quad + \frac{2}{\rho^2}(\nabla\rho \cdot \vec{J})\nabla\rho + \frac{2}{\rho}\left((\nabla\rho \cdot \nabla)\vec{J} - (\vec{J} \cdot \nabla)\nabla\rho\right) \\
&\quad + \frac{\hbar e}{2}\rho^2 \nabla \times (\epsilon_{abc} n_a \nabla n_b \times \nabla n_c) + \hbar e \rho \nabla\rho \times (\epsilon_{abc} n_a \nabla n_b \times \nabla n_c) \\
\nabla^2 \vec{B} &= \frac{16\pi e^2}{c^2}\rho^2 \vec{B} - \frac{8\pi}{c\rho}\nabla\rho \times \vec{J} + \frac{2\pi\hbar e}{c}\rho^2 \epsilon_{abc} n_a \nabla n_b \times \nabla n_c
\end{aligned}$$

note that spatial variation of modulus field $\nabla\rho$ couples $\vec{J}$ and $\vec{B}$ field equations

- **two-gap Meissner effect at low temperature $T < T_c$**

at low temperature $T < T_c$, modulus field ρ varies only very slightly over superconductor to yield

$$\nabla \times \vec{J} = -\frac{4e^2}{c}\rho^2 \vec{B} - \frac{\hbar e}{2}\rho^2 \epsilon_{abc} n_a \nabla n_b \times \nabla n_c$$

so that we can arrive at decoupled equations for $\vec{J}$ and $\vec{B}$

$$\begin{aligned} \nabla^2 \vec{J} &= \frac{16\pi e^2}{c^2}\rho^2 \vec{J} + \frac{\hbar e}{2}\rho^2 \nabla \times (\epsilon_{abc} n_a \nabla n_b \times \nabla n_c), \\ \nabla^2 \vec{B} &= \frac{16\pi e^2}{c^2}\rho^2 \vec{B} + \frac{2\pi \hbar e}{c}\rho^2 \epsilon_{abc} n_a \nabla n_b \times \nabla n_c \end{aligned}$$

note that we have topological contribution with $\epsilon_{abc} n_a \nabla n_b \times \nabla n_c$

- **two-gap London penetration depth**

in London limit when $|\Psi_\alpha| =$ constant and thus $\epsilon_{abc} n_a \nabla n_b \times \nabla n_c = 0$

$$\Lambda = \left(\frac{m_1 c^2}{4\pi e^2 n_{1s}}\right)^{1/2} \left(1 + \frac{m_1 n_{2s}}{m_2 n_{1s}}\right)^{-1/2}$$

where superfluid densities $n_{\alpha s}$ are given by $n_{\alpha s} = 2|\Psi_\alpha|^2$

note that two-gap surface supercurrents screen out applied field to yield two-gap Meissner effect

- **in one-flavor limit with** $n_{2s} = 0$

$$\begin{aligned}
\nabla \times \vec{J} &= -\frac{e^2 n_{1s}}{m_1 c}\vec{B} + \frac{1}{n_{1s}}\nabla n_{1s} \times \vec{J}, \\
\nabla^2 \vec{J} &= \left(\frac{4\pi e^2}{m_1 c^2} n_{1s} + \frac{1}{n_{1s}}\nabla^2 n_{1s} - \frac{1}{n_{1s}^2}(\nabla n_{1s})^2\right)\vec{J} + \frac{e^2}{m_1 c}\nabla n_{1s} \times \vec{B} \\
&\quad + \frac{1}{2n_{1s}^2}\left((\nabla n_{1s} \cdot \vec{J})\nabla n_{1s} + \nabla n_{1s}(\vec{J} \cdot \nabla) n_{1s}\right) \\
&\quad + \frac{1}{n_{1s}}\left((\nabla n_{1s} \cdot \nabla)\vec{J} - (\vec{J} \cdot \nabla)\nabla n_{1s}\right) \\
\nabla^2 \vec{B} &= \frac{4\pi e^2}{m_1 c^2} n_{1s}\vec{B} - \frac{4\pi}{c n_{1s}}\nabla n_{1s} \times \vec{J}
\end{aligned}$$

in more restricted low temperature limit $T < T_c$

$$\nabla \times \vec{J} = -\frac{e^2 n_{1s}}{m_1 c}\vec{B}$$

$$\nabla^2 \vec{J} = \frac{4\pi e^2}{m_1 c^2} n_{1s}\vec{J}$$

$$\nabla^2 \vec{B} = \frac{4\pi e^2}{m_1 c^2} n_{1s}\vec{B}$$

which yield single-gap London penetration depth

$$\Lambda = \left(\frac{m_1 c^2}{4\pi e^2 n_{1s}}\right)^{1/2} = 41.9 \ \left(\frac{r_s}{a_0}\right)^{3/2} \left(\frac{n_e}{n_{1s}}\right)^{1/2} \ \mathring{A}$$

where $r_s = \left(\frac{3}{4\pi n_e}\right)^{1/3}$, a_0 is Bohr radius and n_e is total electron density given by $n_e = n_{1n} + n_{1s}$ with normal (superfluid) electron density n_{1n} (n_{1s})

- **phenomenological two-gap London penetration depth**

$$\Lambda = 41.9 \ \left(\frac{r_s}{a_0}\right)^{3/2} \left(\frac{n_e}{n_{1s}}\right)^{1/2} \left(1 + \frac{m_1 n_{2s}}{m_2 n_{1s}}\right)^{-1/2} \ \mathring{A}$$

in two-gap London penetration depth, with respect to single-gap case we have more degrees of freedom associated with physical parameters m_2 and n_{2s} to adjust theoretical predictions to experimental data for London penetration depth

IV. Flux quantization

- **magnetic flux carried by vortex of superconductor**

consider two-gap superconductor in shape of cylinder-like ring where there exists cavity inside inner radius

in order to evaluate magnetic flux inside two-gap superconductor, we embed within interior of superconducting material contour encircling cavity

at low temperature $T < T_c$, appreciable supercurrents can flow only near surface of superconductor, and modulus field ρ varies only very slightly over two-gap superconductor

$$0 = \oint J = \oint A + \frac{\hbar c}{4e} \oint C = \Phi + \frac{\hbar c}{4e} \oint C$$

Φ: magnetic flux carried by vortex of superconductor

to explicitly evaluate phase effects of two-gap superconductor, we parameterize z_α fields:

$$z_1 = |z_1| e^{i\phi_1} = e^{i\phi_1} \cos\frac{\theta}{2}, \quad z_2 = |z_2| e^{i\phi_2} = e^{i\phi_2} \sin\frac{\theta}{2}$$

to satisfy constraint $z_\alpha^* z_\alpha = |z_1|^2 + |z_2|^2 = 1$

$$\vec{C} = 2(|z_1|^2 \nabla\phi_1 - |z_2|^2 \nabla\phi_2)$$

- **fractional magnetic flux quantization**

order parameters Ψ_α are single-valued in each flavor channels so that their corresponding phases should vary 2π times integers p_α when ring is encircled

$$\oint \nabla\phi_\alpha \cdot \mathrm{d}\vec{l} = 2\pi p_\alpha$$

$$\oint C = 4\pi(|z_1|^2 p_1 - |z_2|^2 p_2)$$

$$\begin{aligned} |\Phi| = \frac{\hbar c}{4e}\oint C &= (|z_1|^2 p_1 - |z_2|^2 p_2)\Phi_0 \\ &= \frac{1}{2}\left(p_1 - p_2 + (p_1 + p_2)n_3\right)\Phi_0 \end{aligned}$$

$\Phi_0 = \frac{hc}{2e} = 2.0679 \times 10^{-7}$ gauss-cm^2: fluxoid

- **two cases of magnetic flux quantization**

in case of $p_1 = p_2 = 1$

$$|\Phi| = n_3\Phi_0 = \Phi_0 \cos\theta$$

such a vortex can possess arbitrary fraction of magnetic flux quantum since $|\Phi|$ depends on parameter $\cos\theta$ measuring relative densities of two condensates in superconductor

in case of $p_1 = -p_2$, magnetic flux is reduced to single-gap magnetic flux quantization

$$|\Phi| = p_1\Phi_0$$

V. Knotted string geometry

- **bundle of two strings**

in Hopf bundle $n_a = z^\dagger \sigma_a z$, n_a remains invariant under U(1) gauge transformation $z \rightarrow e^{i\xi/2} z$

$$z_1 = |z_1| e^{i\phi_1} = e^{i\phi_1} \cos\frac{\theta}{2}, \quad z_2 = |z_2| e^{i\phi_2} = e^{i\phi_2} \sin\frac{\theta}{2}$$

n_a can be rewritten in terms of angles θ and $\beta = \phi_1 + \phi_2$

$$\vec{n} = (\cos\beta \sin\theta, -\sin\beta \sin\theta, \cos\theta)$$

n_a is independent of angle $\alpha = \phi_1 - \phi_2$ so that α can be considered as coordinate generalization of parameter s of string coordinates $\vec{x}(s) \in R^3$, which describe knot structure involved in two-gap superconductor

knot theory in the two-gap superconductor can be constructed in terms of bundle of two strings

U(1) gauge transformation $z \rightarrow e^{i\xi/2} z$ is related with angle α in such a way that

$$\alpha \rightarrow \alpha + \xi$$

to yield reparameterization invariance $s \rightarrow \tilde{s}(s)$

• Hopf invariant associated with knot structure

evaluate Hopf invariant associated with knot structure of two-gap superconductor

$$C = \cos\theta \mathrm{d}\beta + \mathrm{d}\alpha$$

under U(1) gauge transformation $z \to e^{i\xi/2} z$

$$C \to \cos\theta \mathrm{d}\beta + \mathrm{d}(\alpha + \xi)$$

so that C can be identified as U(1) gauge field

exterior derivative of C produces pull-back of area two-form H and dual one-form $G_i = \frac{1}{2}\epsilon_{ijk} H_{jk}$ on two-sphere S^2

$$\begin{aligned} H &= \mathrm{d}C = \frac{1}{2}\vec{n} \cdot \mathrm{d}\vec{n} \wedge \mathrm{d}\vec{n} = \sin\theta \mathrm{d}\beta \wedge \mathrm{d}\theta \\ G &= \frac{1}{2}\sin\theta \mathrm{d}\beta \wedge \mathrm{d}\theta \end{aligned}$$

Hopf invariant Q_H is given by

$$Q_H = \frac{1}{8\pi^2}\int H \wedge C = \frac{1}{8\pi^2}\int \sin\theta \mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\theta$$

if there exists nonvanishing Hopf invariant, bundle of two strings forms a knot

- **curvature and torsion**

to figure out knot structure geometrically, emloy right-handed orthonormal basis defined by $(\vec{n}, \vec{e}_1, \vec{e}_2)$:

$$
\begin{aligned}
\vec{n} &= (\cos\beta\sin\theta, -\sin\beta\sin\theta, \cos\theta) \\
\vec{e}_1 &= (\cos\beta\cos\theta, -\sin\beta\cos\theta, -\sin\theta) \\
\vec{e}_2 &= (\sin\beta, \cos\beta, 0)
\end{aligned}
$$

define with $\vec{e}_\pm = \vec{e}_2 \pm i\vec{e}_1$ curvature and torsion:

$$
\begin{aligned}
\kappa_i^\pm &= \frac{1}{2}e^{\pm\alpha}\vec{e}_\pm \cdot \partial_i\vec{n} = \frac{1}{2}e^{\pm\alpha}(-\sin\theta\partial_i\beta \pm i\partial_i\theta), \\
\tau_i &= \frac{i}{2}\vec{e}_- \cdot (\partial_i + i\partial_i\alpha)\vec{e}_+ = \cos\theta\partial_i\beta - \partial_i\alpha
\end{aligned}
$$

curvature $\kappa_i^\pm$ and torsion τ_i are invariant under U(1)×U(1) gauge transformations: $z \to e^{i\xi/2}z$ and $\alpha \to \alpha + \xi$

curvature $\kappa_i^\pm$ and torsion τ_i are not independent to yield flatness relations between them

$$
\mathrm{d}\tau + 2i\kappa^+ \wedge \kappa^- = 0, \quad \mathrm{d}\kappa^\pm \pm i\tau \wedge \kappa^\pm = 0
$$

knotted stringy structures of two-gap superconductors are constructed only in terms of CP^1 complex fields z_α in order parameters Ψ_α, since modulus field ρ associated with condensate densities does not play a central role in geometrical arguments involved in topological knots of the system

VI. Conclusions

- **Ginzburg-Landau theory for two-gap superconductors**
- **Meissner effects**
- **two-gap London penetration depth**
- **fractional flux quantization**
- **knotted string geometry**