# Fractional Skyrions

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# Topological Objects

- $U(1)$ vortex $\qquad \pi_1(S^1)$
- baby skyrmions $\qquad \pi_2(S^2),\ \pi_2(CP_N), \ldots$
- monopoles $\qquad \pi_2(S^2) \quad \pi_3(G)\ \times$
- skyrmions $\qquad \pi_3(S^3) \quad \pi_3(G)$

  ↖ Lie group
- instanton $\qquad \pi_3(S^3)$

- Introduce Additional Structure To

Fractionalize These Objects

## SU(2) instantons on $\underline{R^3 \times S^1}$

- $x^4 \sim x^4 + 2\pi R$ , $A^a_\mu(\vec{x}, x^4) = A^a_\mu(\vec{x}, x^4 + 2\pi R)$

  periodic

- Allowed gauge transf :

  small : single valued & trivial

  large : $g = e^{i\frac{x^4}{R}\cdot\frac{\sigma^3}{2}}$

  double valued, leaving $A_\mu$ singlevalued

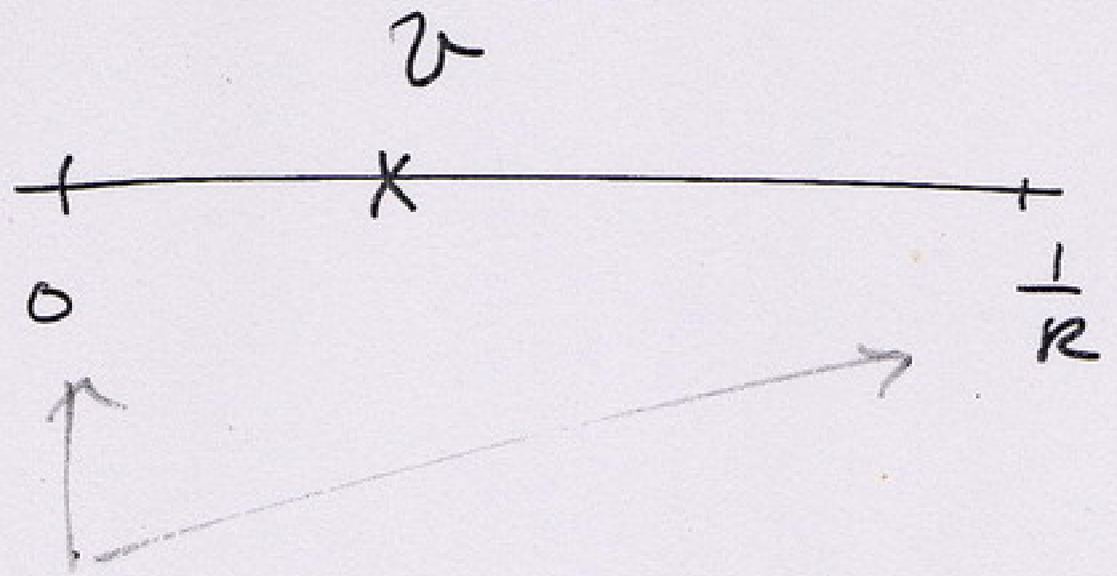
  $$A_4 \longrightarrow A_4 + \frac{1}{R}\frac{\sigma^3}{2}$$

- Wilson loop expectation value $\mathrm{Tr}\, e^{i\oint dx^\mu A_\mu}$

  $$\langle A_4 \rangle = \frac{1}{2}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$$

  $g = e^{i\pi\frac{\sigma_2}{2}}$

  $v \to v + \frac{1}{R}$ , $\quad v \to -v$




symmetric phase

Spontaneous symmetry breaking of SU(2) to U(1)

$\boxed{SU(2)\ \text{Instanton}}$ on $R^4$

- $F^a_{\mu\nu} = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma}$

$$A^a_\mu = i \bar{\eta}^a_{\mu\nu} \sigma^a \partial_\nu \ln\left(1 + \frac{\lambda}{x^2}\right)$$

- 8 zero modes = 4 center of mass position + 1 scale + 3 gauge direction

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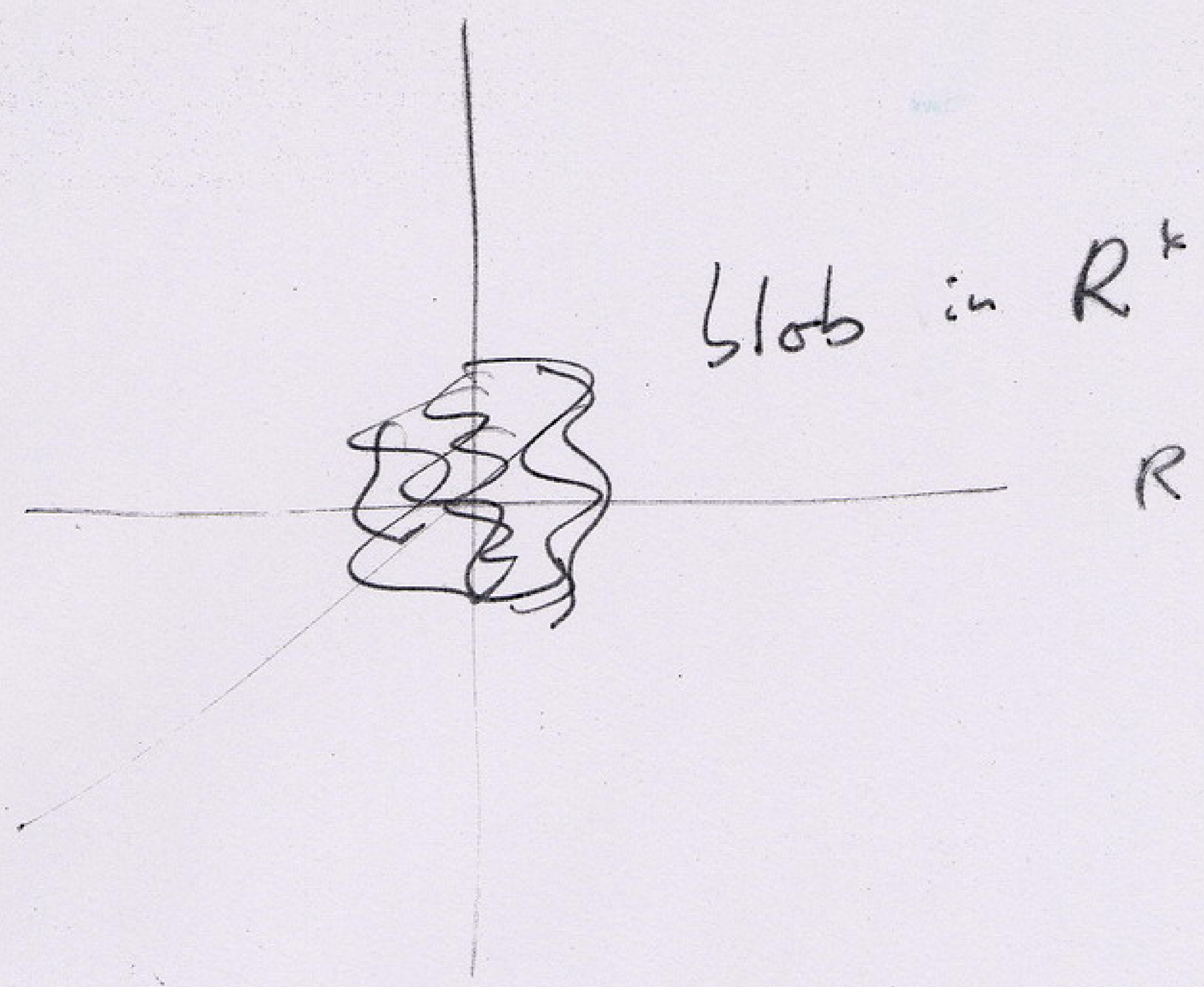
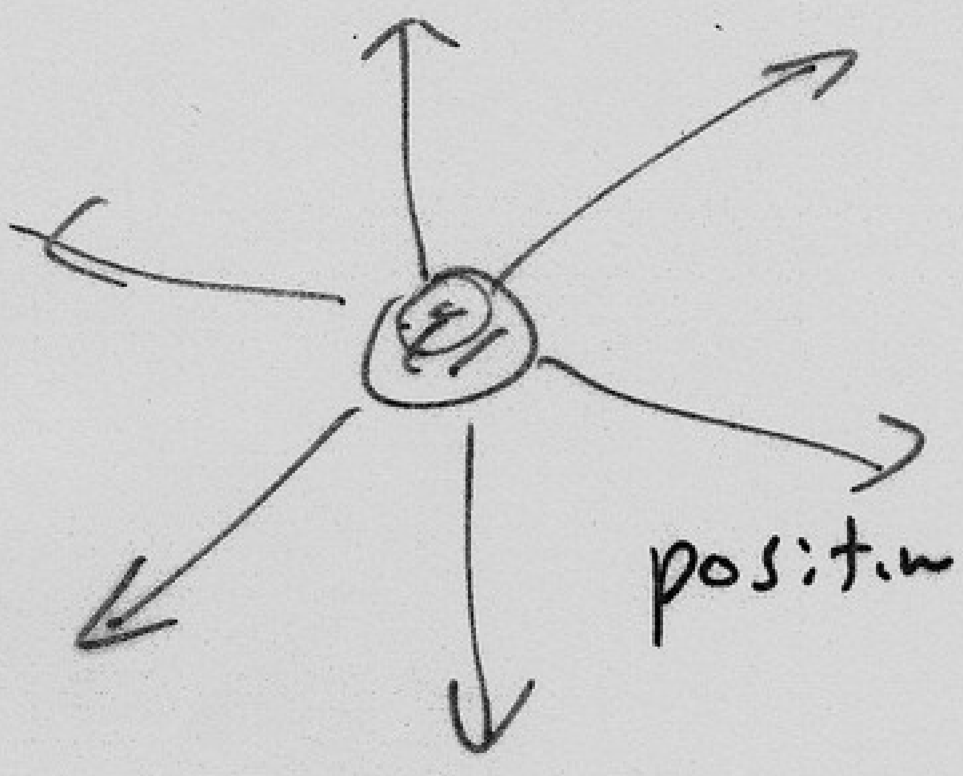



- Action = $\frac{1}{4e^2} \int d^4x\, \mathrm{tr}\, F_{\mu\nu}^2 \;=\; \frac{8\pi}{e^2}$

• <u>Monopoles</u> $\phi = A_4$

• $F_{ij} = \epsilon_{ijk} D_k \phi$ $v$: symmetry breaking scale

self dual



position flux

core size $\frac{1}{v}$

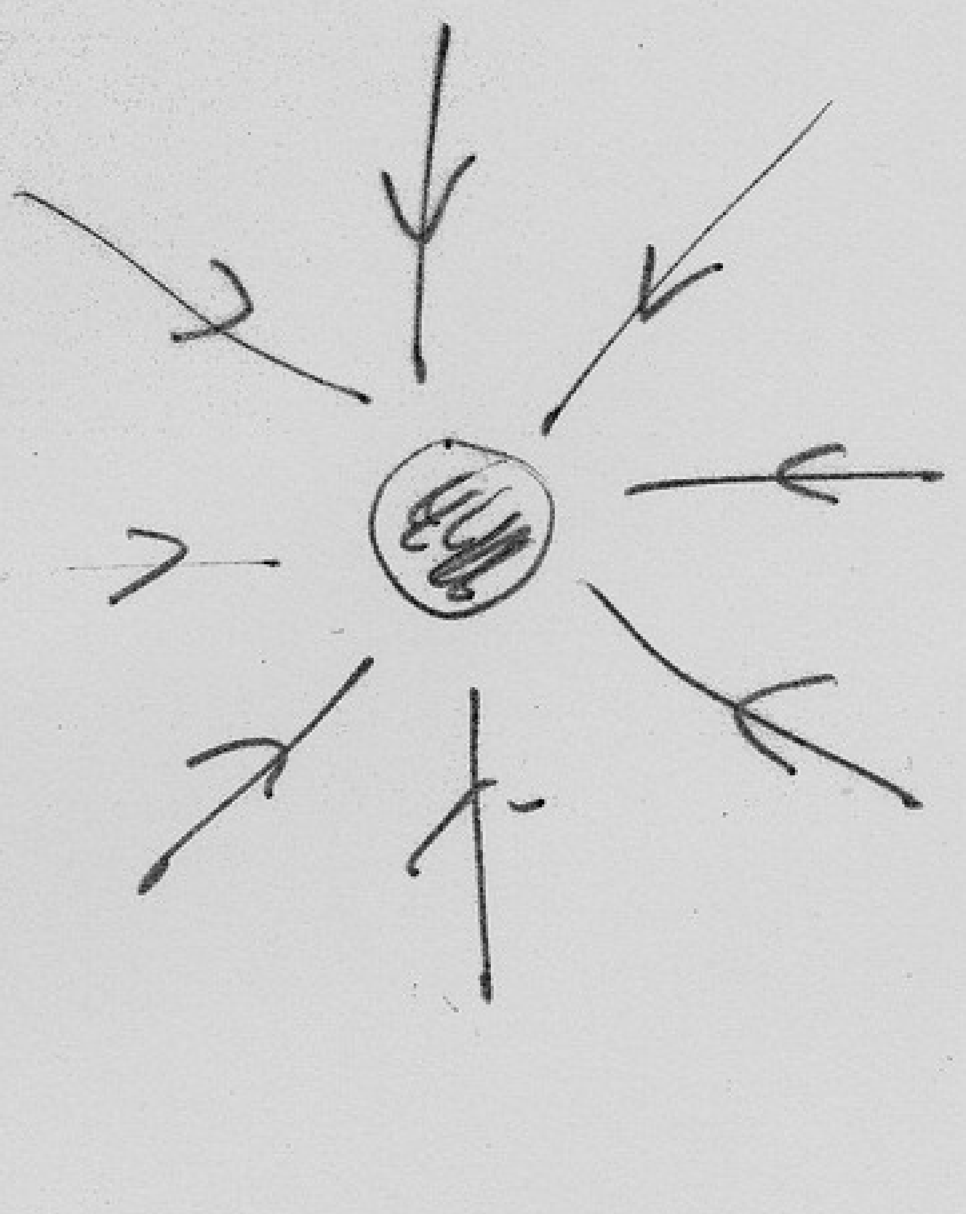
mass $\frac{4\pi}{e^2} v \cdot 2\pi R$

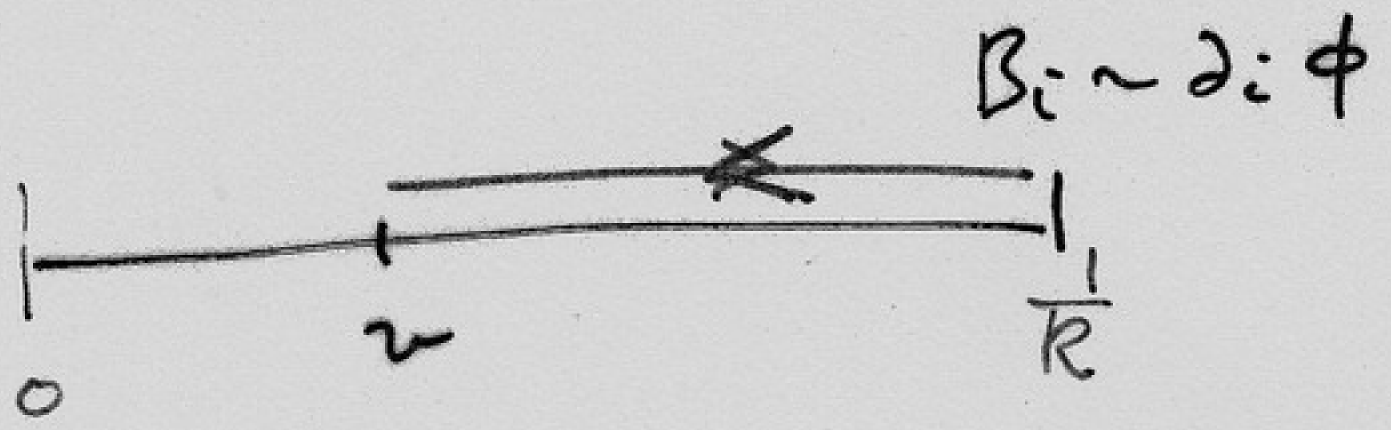
4-zero modes

• $v = \frac{1}{R}$ is also symmetric phase

$v \equiv A_4 \rightarrow$ $x^4$-independent monopole

$v \rightarrow -v \rightarrow \frac{1}{R} - v \rightarrow -v \rightarrow v$





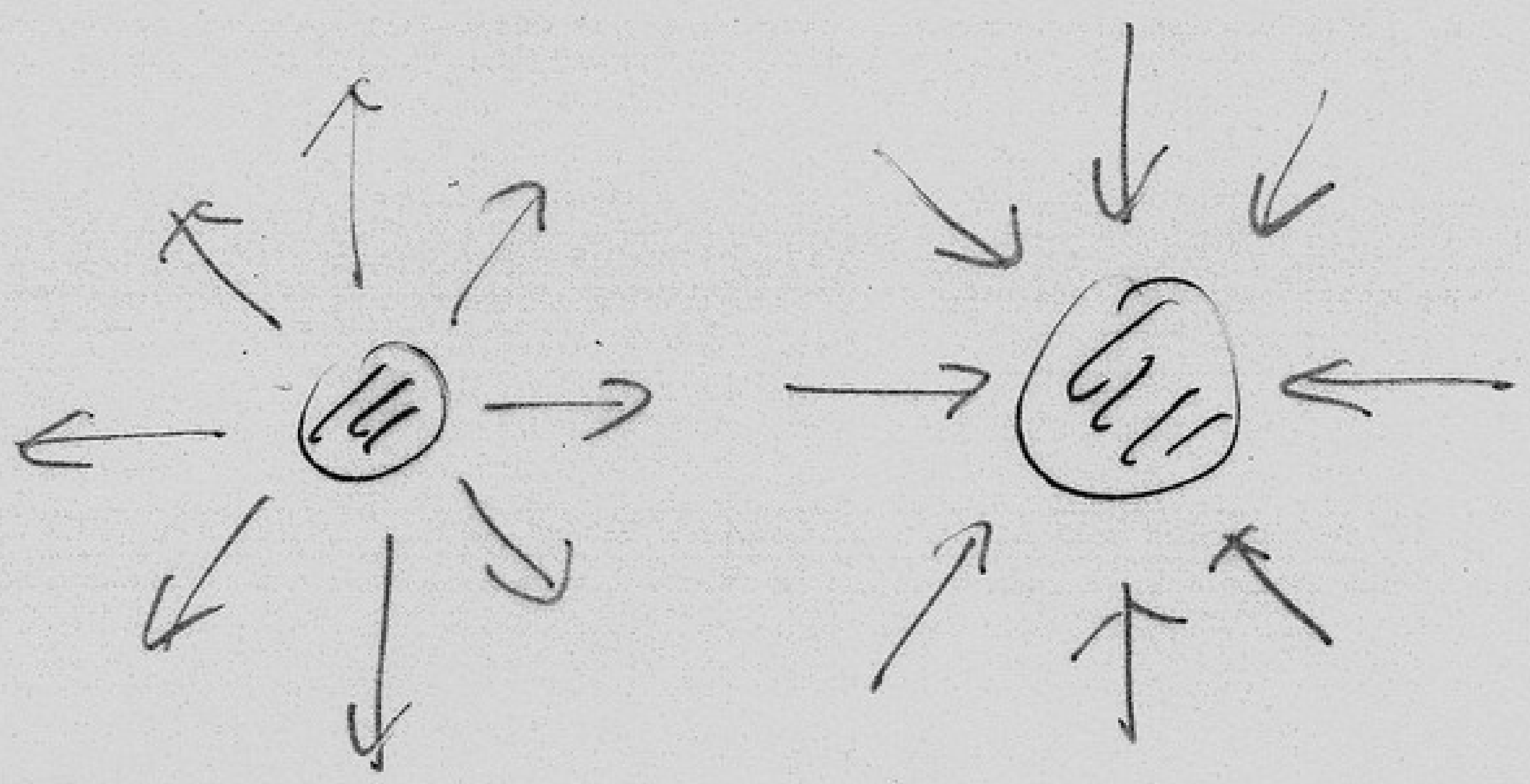

core size $\frac{1}{\frac{1}{R} - v}$

mass $\frac{4\pi}{e^2}\left(\frac{1}{R} - v\right) \cdot 2\pi R$

4-zero modes

<u>KK monopole</u>

• Composite $x^4$-dependence on $R^3 \times S^1$

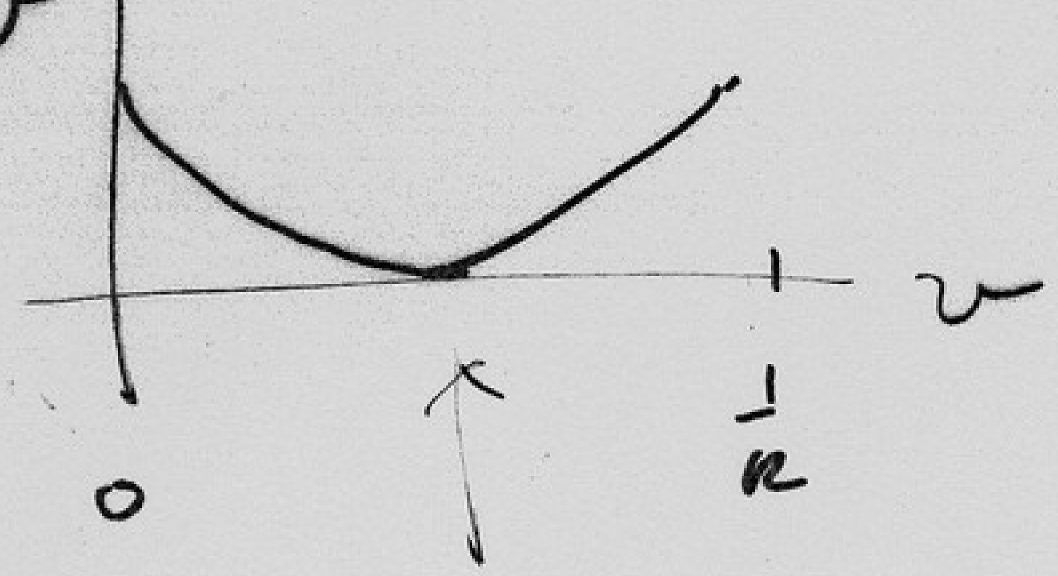


action = mass : $2\pi R \left(\frac{4\pi}{e^2}\right) \cdot \left( v + \left( \frac{1}{R} - v \right) \right) = \frac{8\pi^2}{e^2}$

# of zero modes = $4 + 4 = 8$

· Instanton physics $R^{1+2} \times S^1$

$$V_{eff} = K e^{\frac{8\pi^2}{e^2} v} + K' e^{\frac{8\pi^2}{e^2} \left( \frac{1}{R} - v \right)}$$



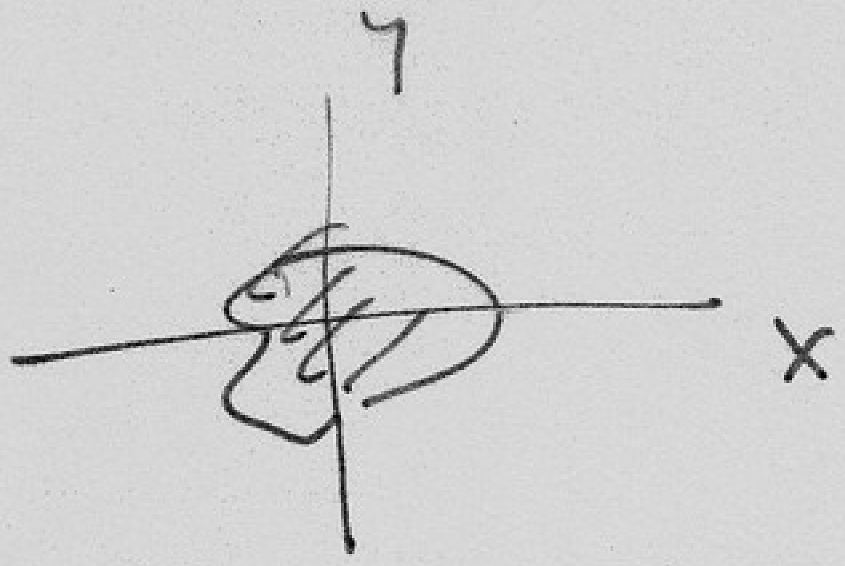

confining phase

tony

- $\boxed{\text{Vortex on } R^2}$

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}^2 + |D_\mu \phi|^2 - U(|\phi|^2)$$

$$\langle \phi \rangle = v, \quad \phi \sim e^{i\varphi} v \qquad r, \varphi$$




- core size $\frac{1}{v}$
- magnetic flux: $\frac{2\pi}{e}$
-

- two flavor

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_0 \phi_f|^2 - |D_1 \phi_f|^2 - |(D_2 - i m_f)\phi_f|^2$$

$$- \lambda \left( |\phi_1|^2 + |\phi_2|^2 - v^2 \right)^2 + \cdots$$

BPS

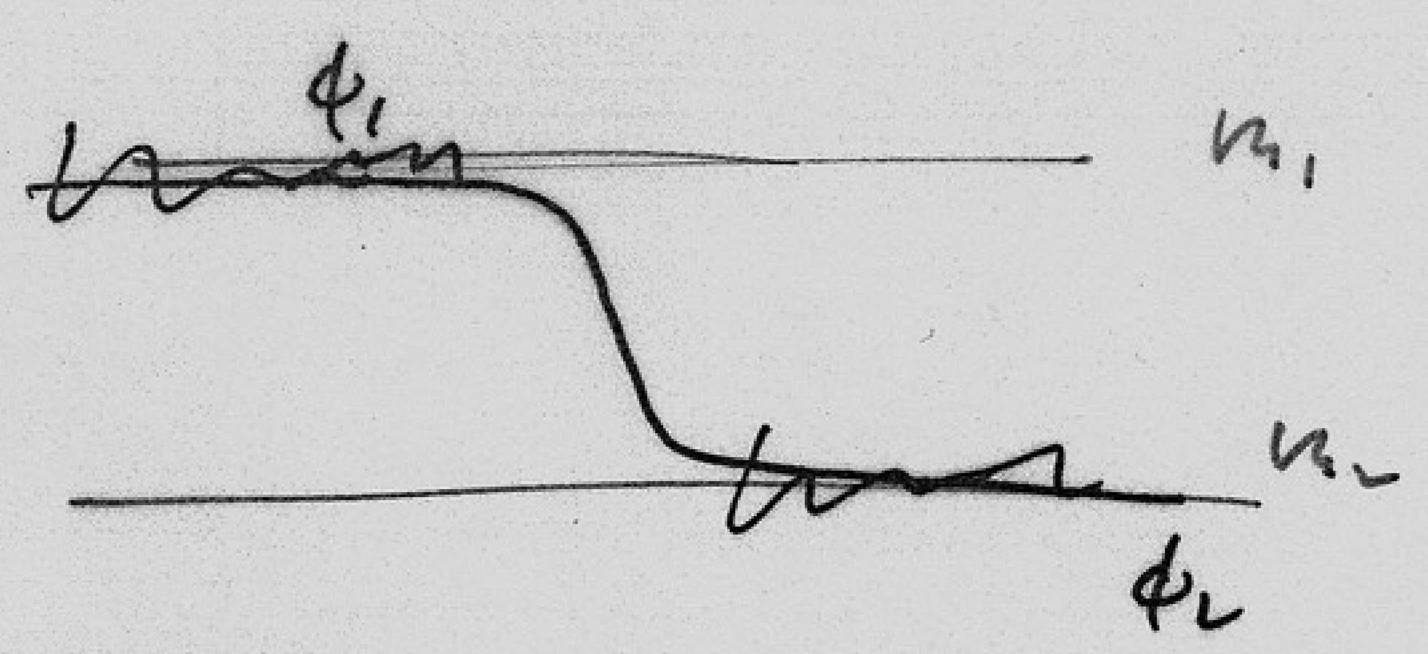
$$\left.\begin{aligned} F_{12} &= v^2 - |\phi_1|^2 - |\phi_2|^2 \\ D_1 \phi_f &+ i(D_2 - i m_f)\phi_f = 0 \end{aligned}\right\}$$

<u>Vacuum</u>

$$\begin{cases} A_2 = m_1, & \phi_1 = v, \; \phi_2 = 0 \\ A_2 = m_2, & \phi_1 = 0, \; \phi_2 = v \end{cases}$$

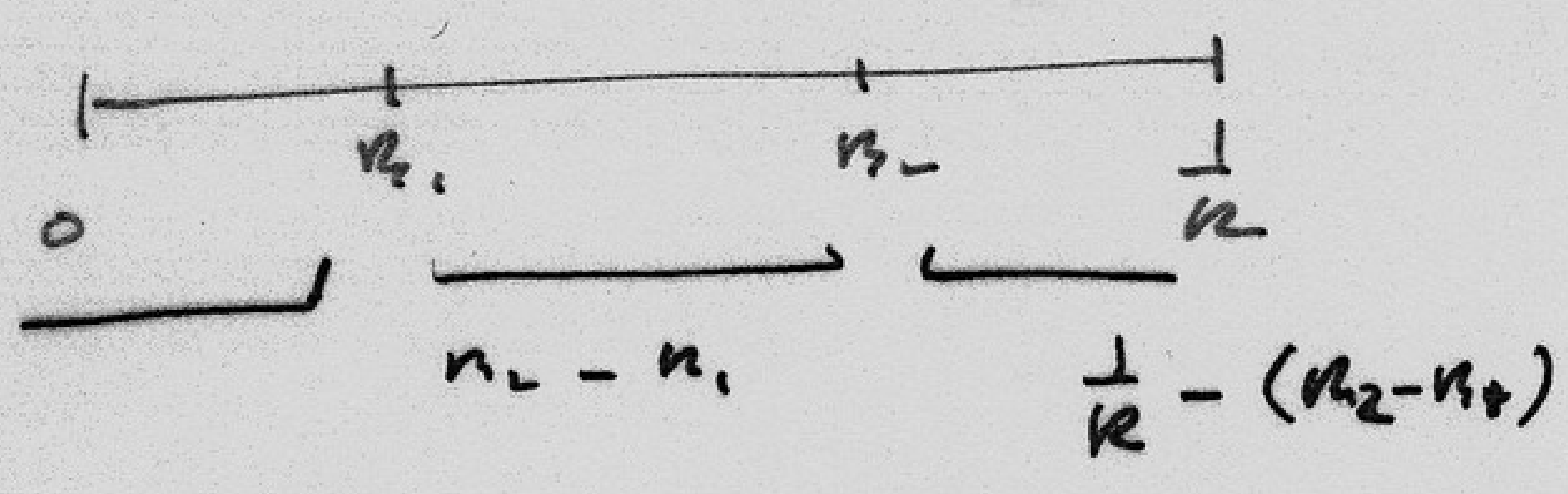
- Domain wall

$$\left.\begin{aligned} \partial_1 A_2 &= v^2 - |\phi_1|^2 - |\phi_2|^2 \\ \partial_1 \phi_f &+ i(A_2 - i n_f)\phi_f = 0 \end{aligned}\right\}$$

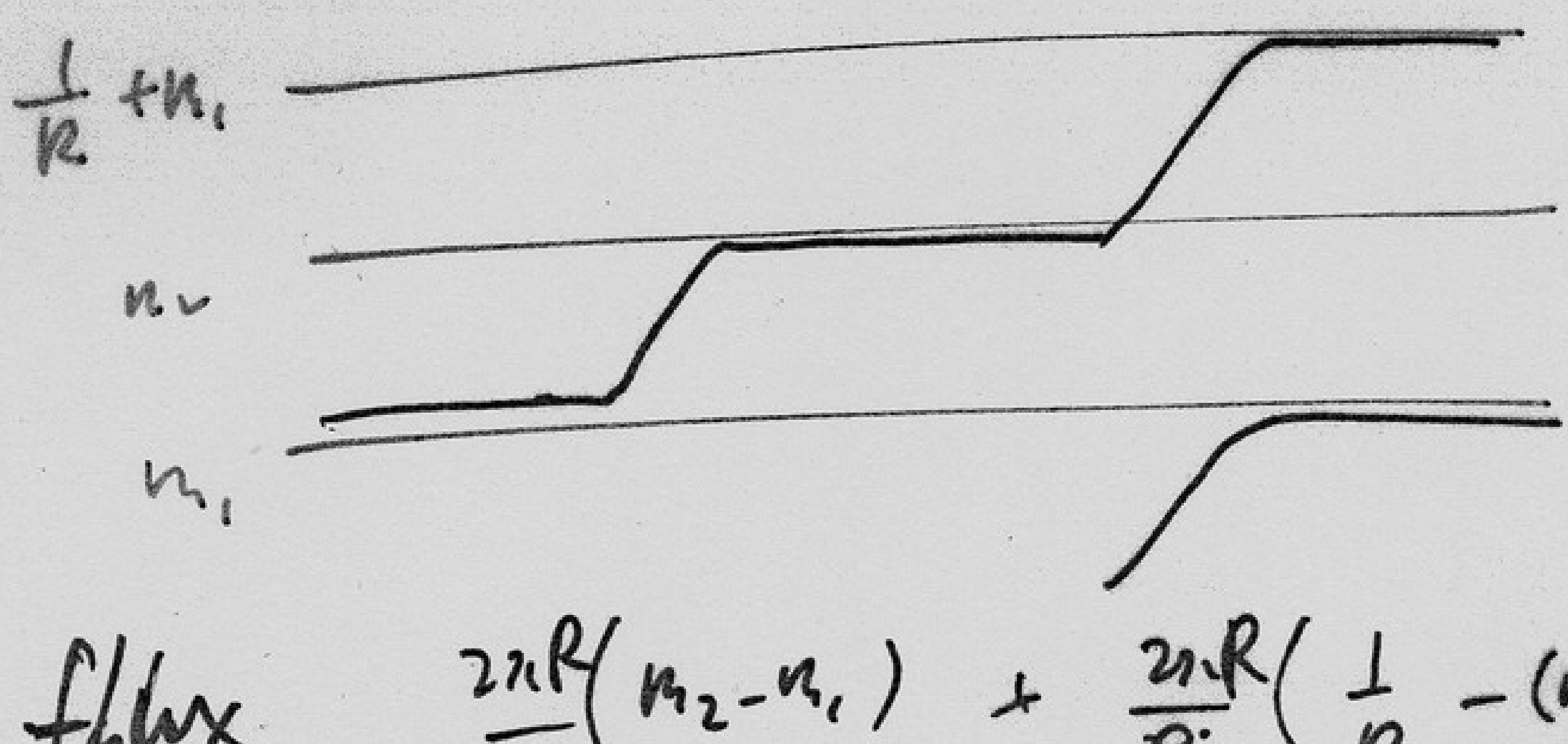



- Space : $\quad x^2 \sim x^2 + 2\pi R$

$$0 < n_1 < n_2 < \frac{1}{R}$$




composite domain wall




flux $\quad \frac{2\pi R}{e}(n_2 - n_1) + \frac{2\pi R}{e}\left(\frac{1}{R} - (n_2 - n_1)\right) = \frac{2\pi}{e}$

# Baby Skyrmion

- $\phi$ : unit vect. in $R^3$ : $S^2$

$$K_\alpha = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma}\, \phi \cdot \partial^\beta \phi \times \partial^\gamma \phi$$ : topological current

baby skyrmion

- $\phi^a \sim \bar{Z} \sigma^a Z$ — $Z \sim \begin{pmatrix} 1 \\ x+iy \end{pmatrix}$

- Symmetry $SO(3) \xrightarrow{\phantom{xx}} U(1)$

break by a reference vector $\hat{n} = (0,0,1)$

gauge $U(1)$

$$D_\alpha \phi^a = \partial_\alpha \phi + A_\alpha\, n \times \phi$$

- gauge inv conserved charge

$$K_\alpha = \frac{1}{8\pi} \epsilon_{\alpha\beta\gamma} \left( \phi \cdot D^\beta \phi \times D^\gamma \phi + F^{\beta\gamma} (v - n\cdot\phi) \right)$$

$\uparrow$ real free parameter

$$= \frac{1}{8\pi} K_\alpha + \frac{1}{4\pi} \epsilon_{\alpha\beta\gamma} \partial^\beta \left( (v - n\cdot\phi) A^\gamma \right)$$

- Model

$$-\frac{1}{4e^2}F_{\mu\nu}^2 + \frac{1}{2e^2}(\partial_\mu N)^2 + \frac{1}{2}(D_\mu\phi)^2 - U$$

$$U = \frac{e^2}{2}(v - n\cdot\phi)^2 + \frac{1}{2}N^2(n\times\phi)^2$$

$v=1$ skyrmions

- BPS

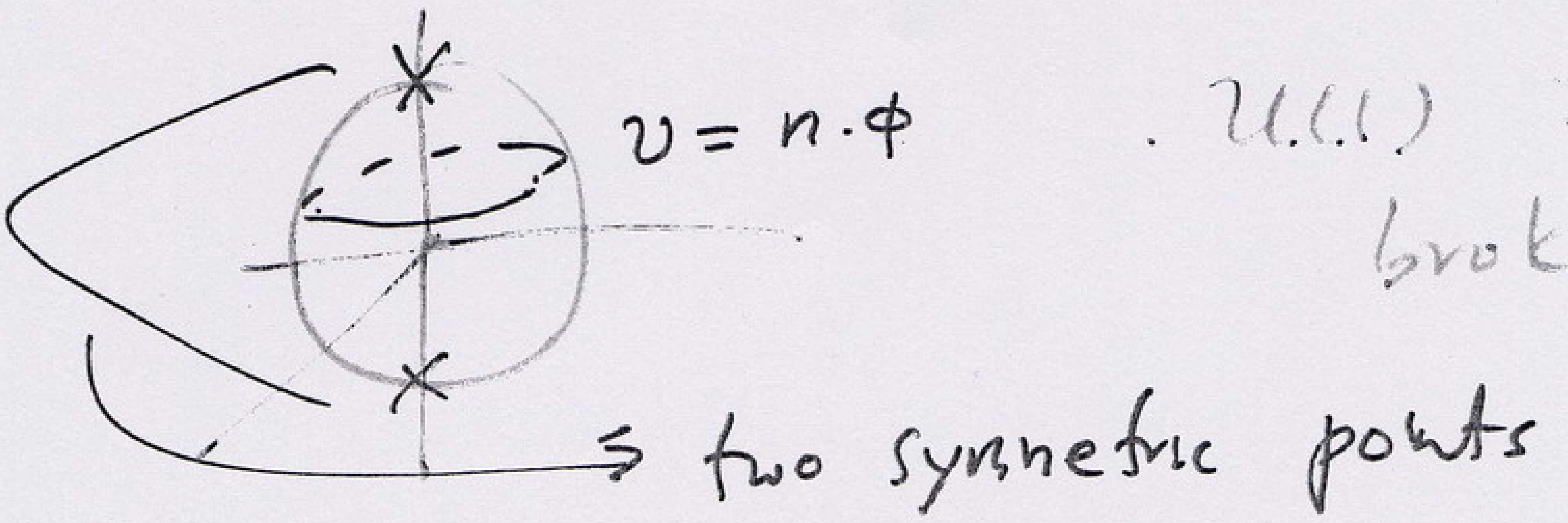
$$\begin{cases} D_1\phi \pm \phi\times D_2\phi = 0 \\ \frac{1}{e^2}F_{12} \mp e^2(v - n\cdot\phi) \end{cases}$$

- Topological charge $T = \int d^2x\, K^0$

Energy $\geq 4\pi|T|$

- Vacuum $v = n\phi \quad -1 < v < 1$

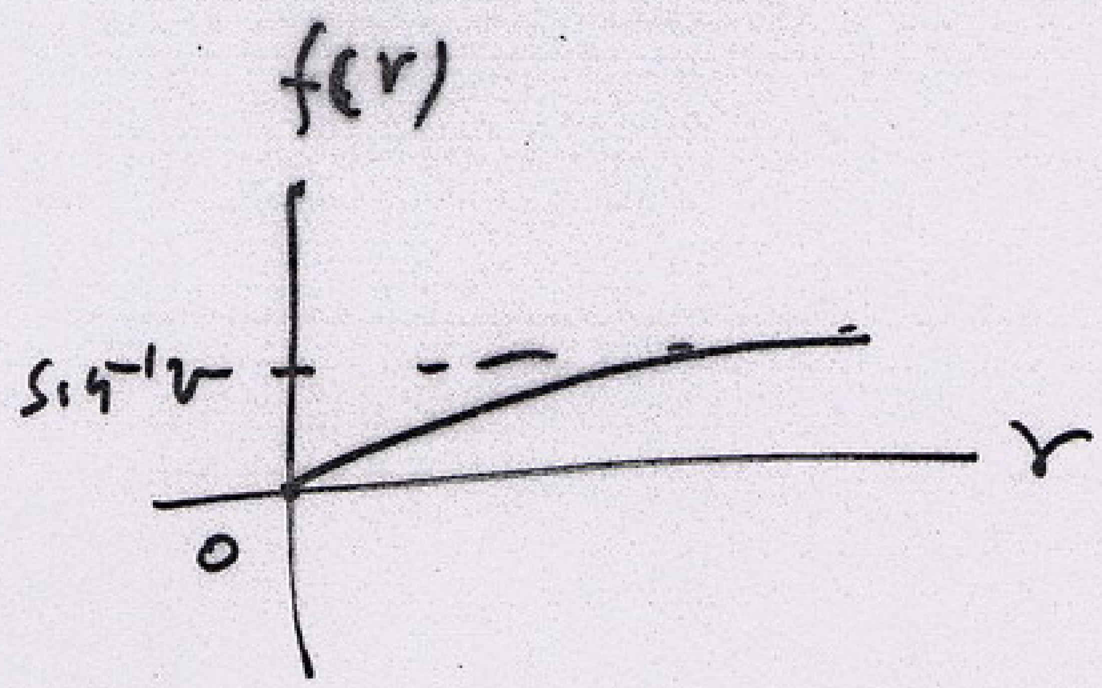
$(0,0,1)$

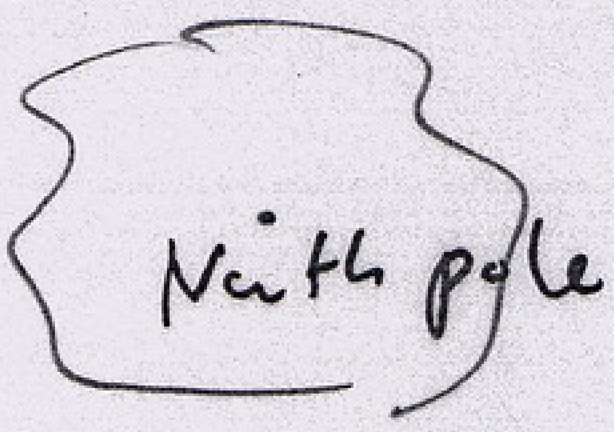



. $U(1)$ : spontaneously broken

- Rotational Symmetry

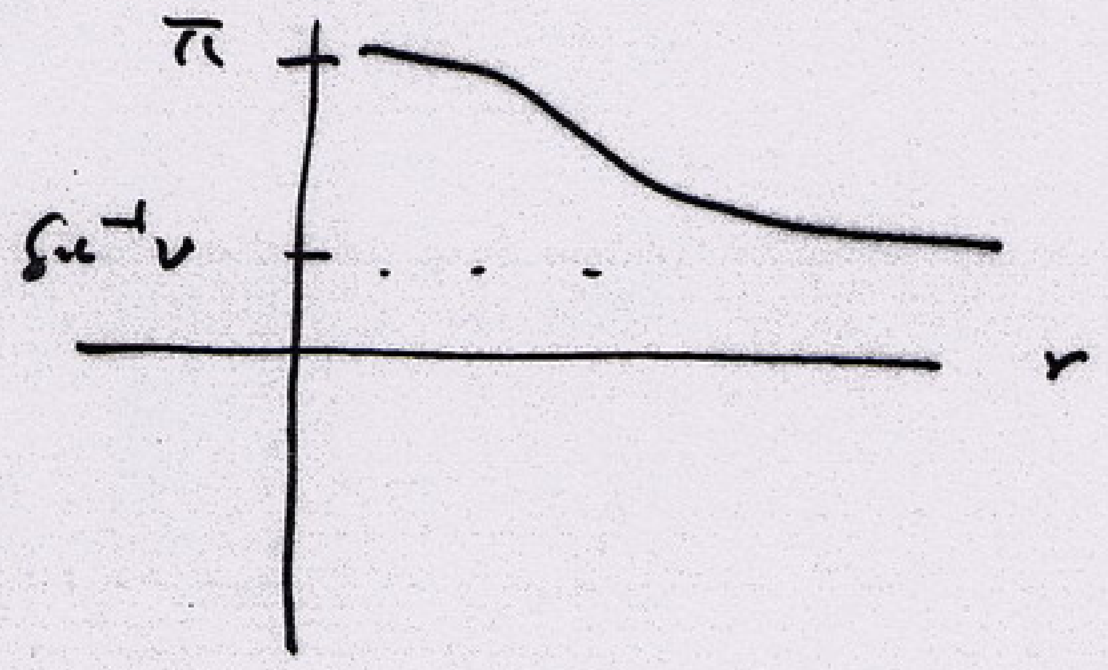
$$\phi = (\sin f(r) \cos n\varphi,\ \sin f(r) \sin n\varphi,\ \cos f(r))$$

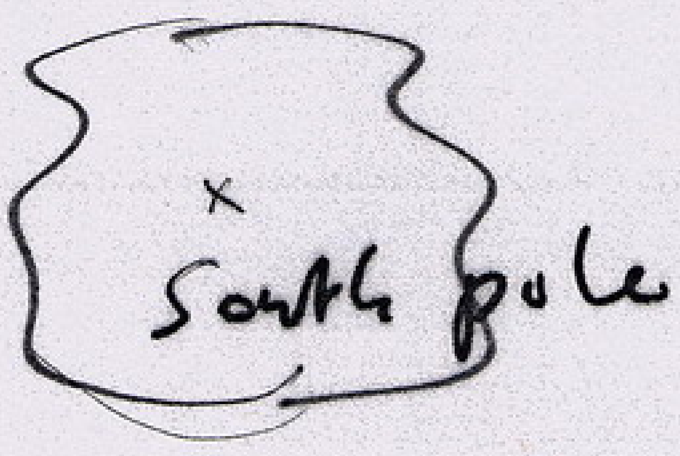



flux

$$T = \frac{n}{2}(1 - v) \qquad 2\pi$$




flux

$$T = \frac{n}{2}(1 + v) \qquad -2\pi$$

- Together

north pole

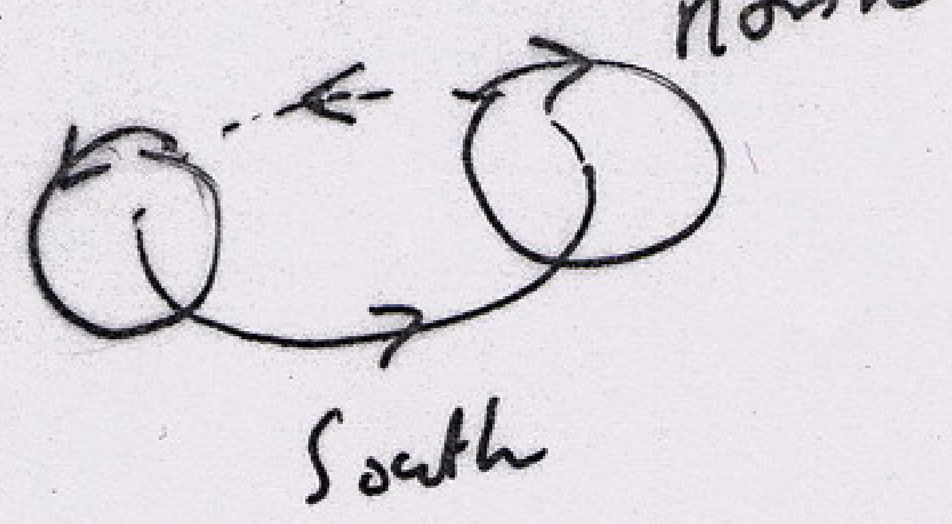
south pole

$$\left\{ \begin{array}{lllll} T = \frac{n}{2}(1-v) & + & T = \frac{n}{2}(1+v) & = n \\ \text{flux} \quad 2\pi & + & -2\pi & = 0 \end{array} \right\}$$

Total :

- Hopfion $\phi : \mathbb{R}^3 \to S^2$

+ Skyrmion term



orikaye, hatu, tchrakian

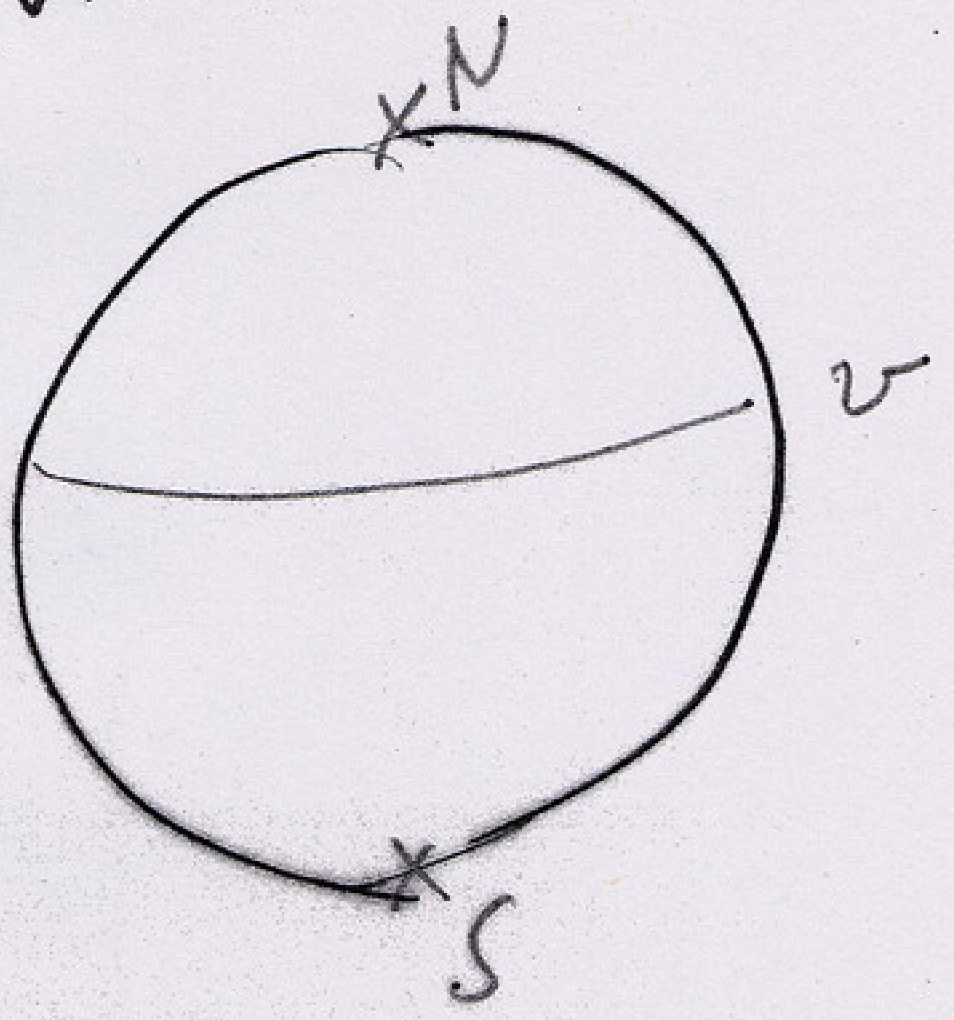
# Skyrmion :

- $U : R^3 \to SU(2) = S^3$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} : \text{unit vect}$$

- vacuum

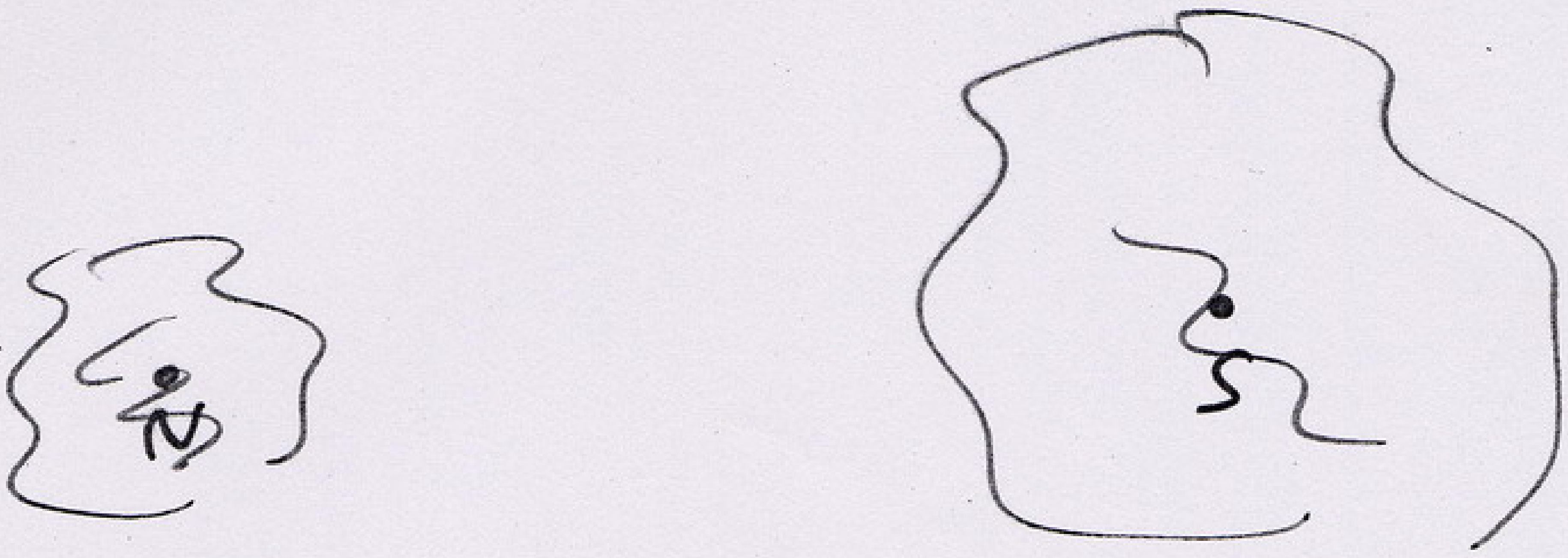
$$V \propto (\phi_0^2 - v^2)^2$$




- gauge $\quad SU(2) \quad \left\{ \begin{matrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{matrix} \right.$ adjoint of $SU(2)$

$\phi_4$

- monopoles




| | | | total |
|---|---|---|---|
| size | $\frac{1}{1-v}$ | $\frac{1}{1+v}$ | |
| flux | $4\pi/e$ | $-4\pi/e$ | 0 |
| topological | $1-v$ | $1+v$ | 1 |

# Conclusion

- One can split many topological objects

- challenge

  - physical implication

  - split $CP^N$ baby skyrmion, $SU(N)$ Skyrmion