

Noncommutative Geometry, Effective Field Theory and Skyrmions in Quantum Hall Systems

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Introduction

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Theoretical predictions can be checked experimentally

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 \Rightarrow Single electron condensation

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- Fractional QH effects, Higher Landau levels, Edge phenomena,

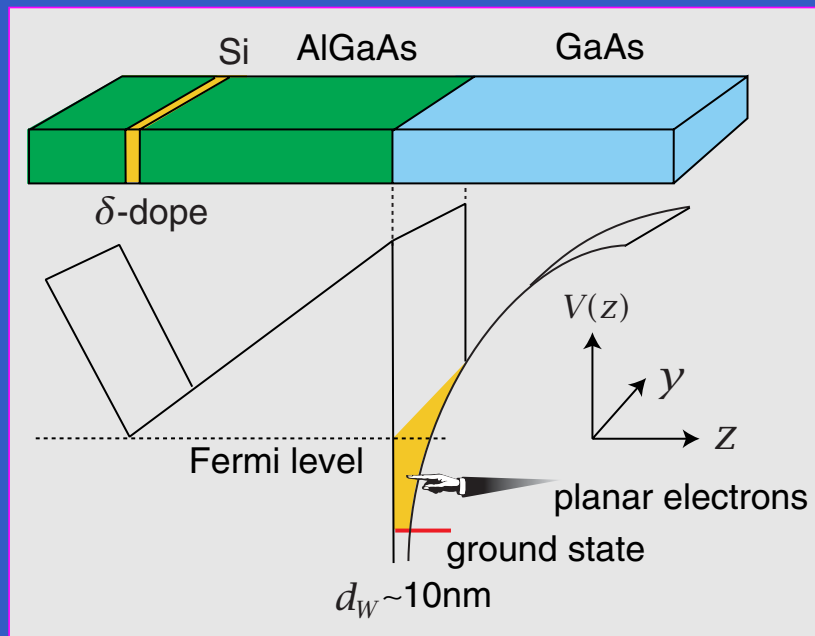
Two-Dimensional World

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- A world of planar electrons emerges between two different semiconductors

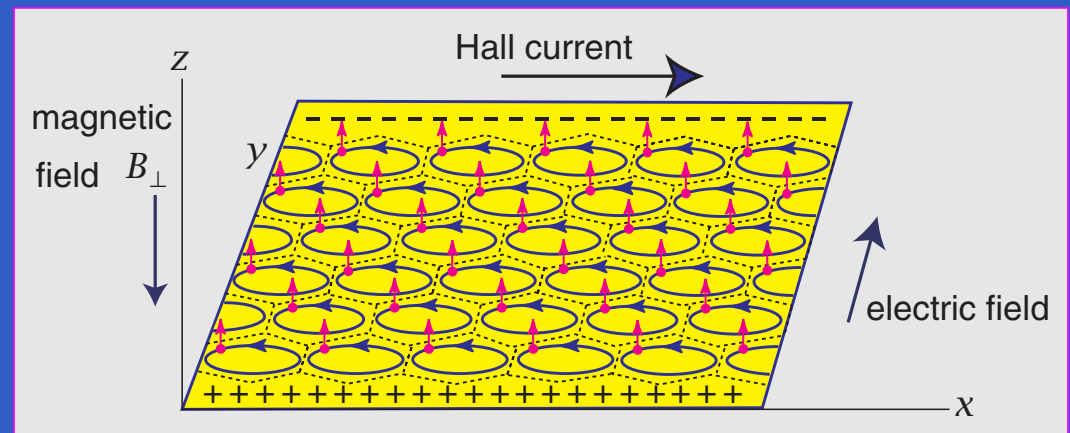


- A electron makes cyclotron motion occupying an area $2\pi\ell_B^2$

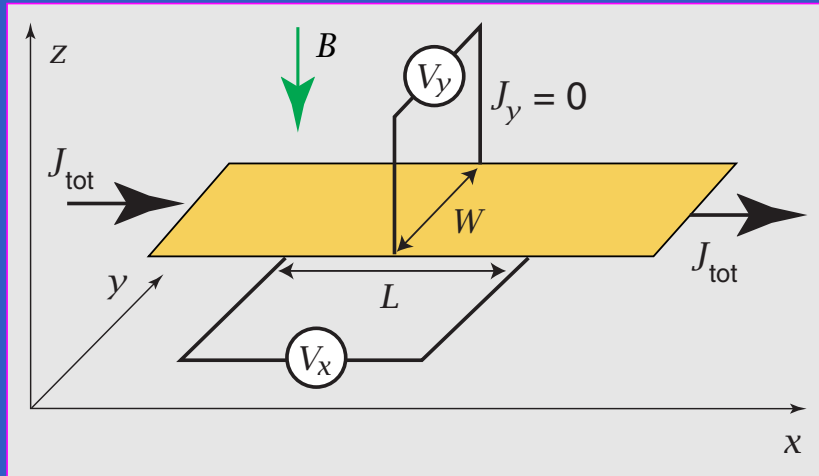
\Rightarrow Landau site

- $\ell_B = \sqrt{\hbar/eB_\perp}$

- Hall current by Lorentz force



Classical and Quantum Hall Effects



Diagonal resistivity

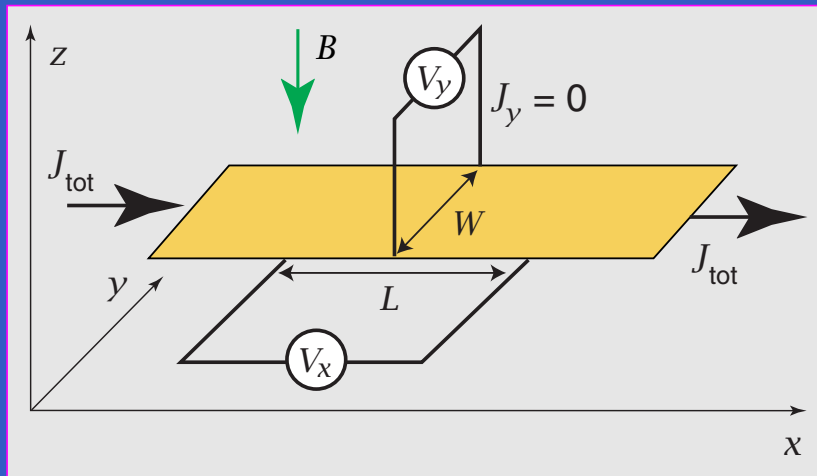
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Hall resistivity

$$R_{xy} = \frac{V_y}{J_{\text{total}}} = \frac{1}{\nu} \frac{2\pi\hbar}{e^2} \propto B_{\perp}$$

$$\text{with } \nu = \frac{2\pi\hbar\rho_0}{eB_{\perp}} \equiv \frac{N}{N_{\Phi}} \Rightarrow \frac{n}{m} \quad (\text{quantized: } \underline{\text{this is the QH effect}})$$

Classical and Quantum Hall Effects



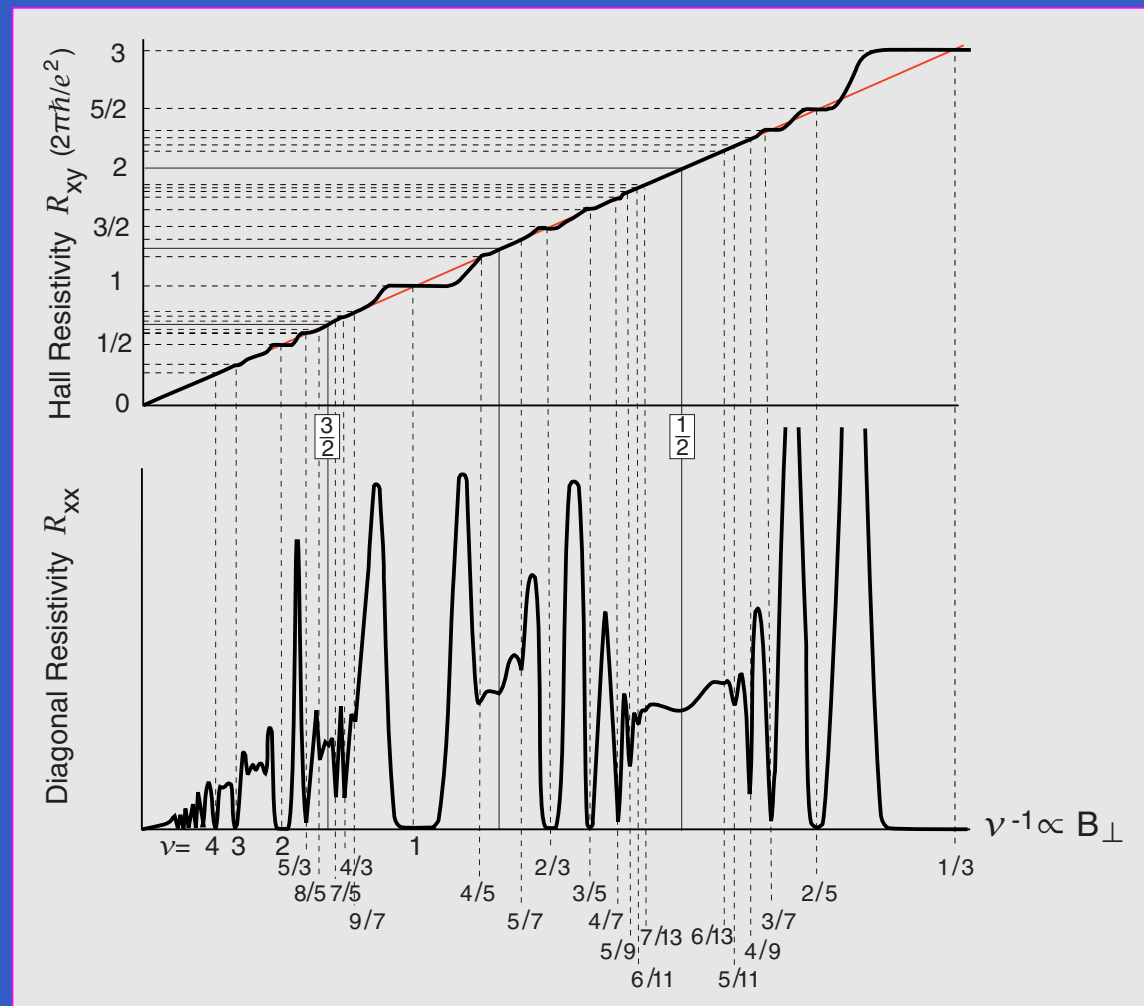
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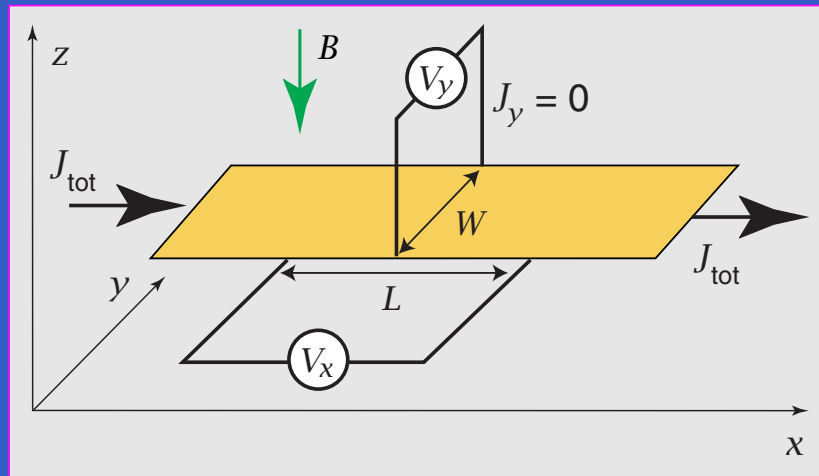
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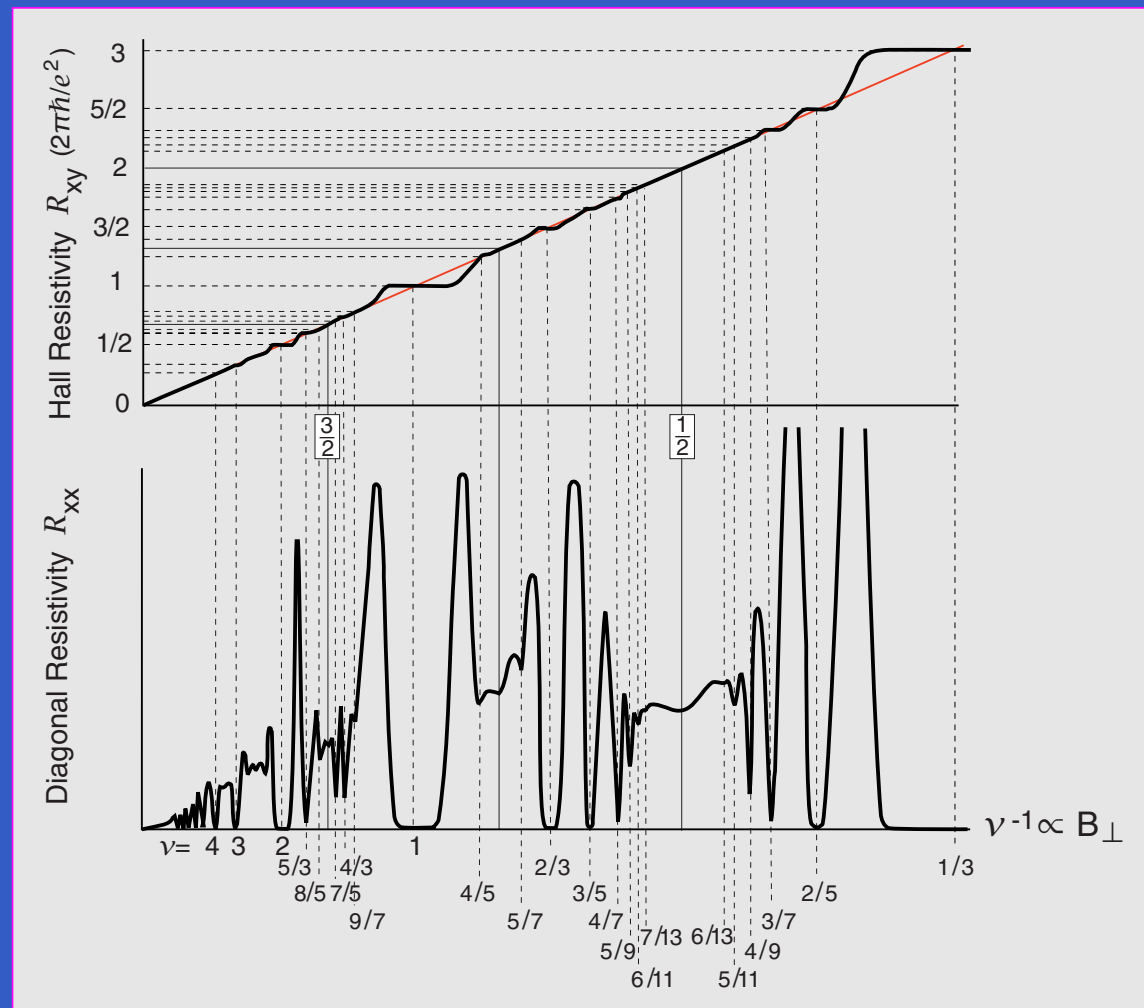
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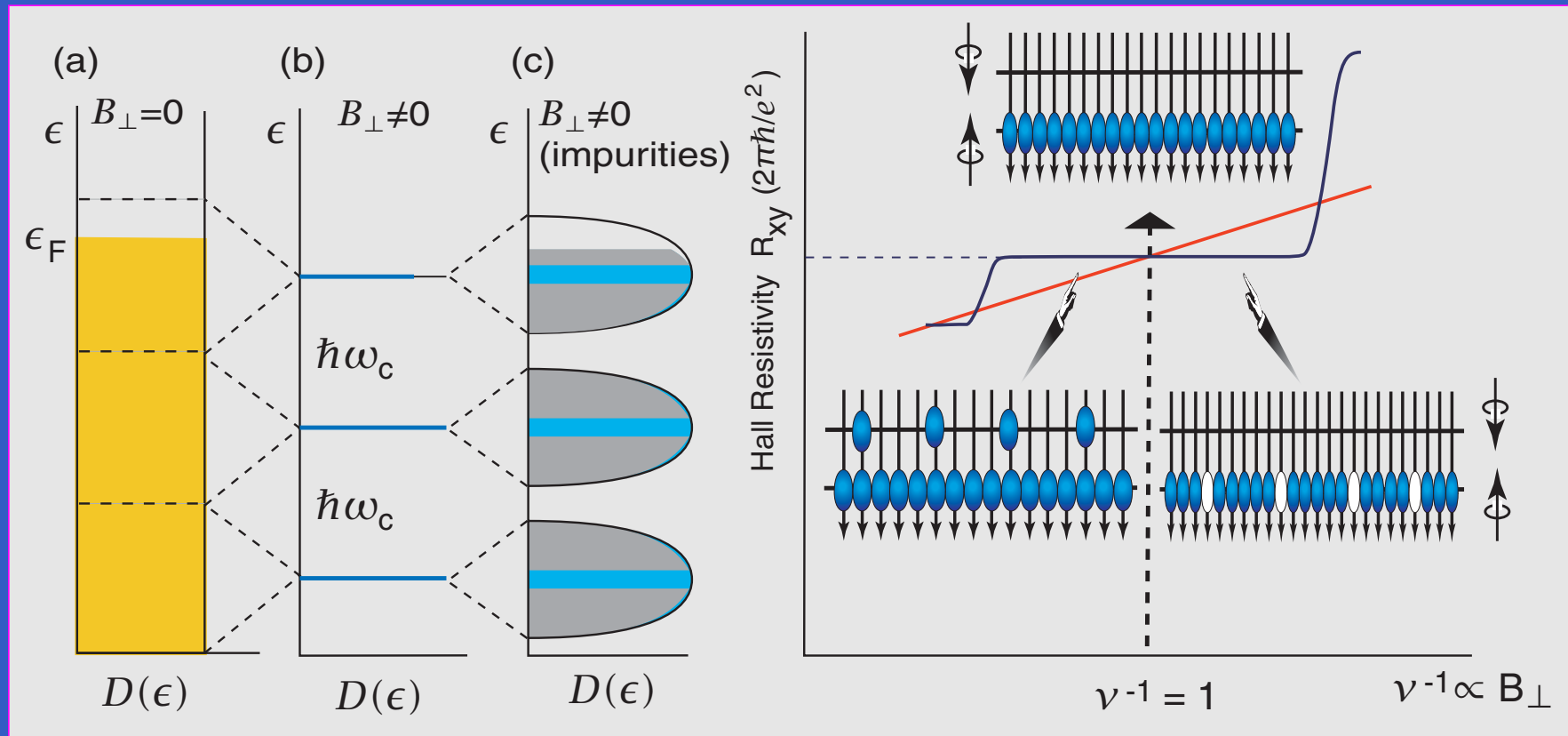
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Noncommutative world develops in each plateau

Hall Plateau

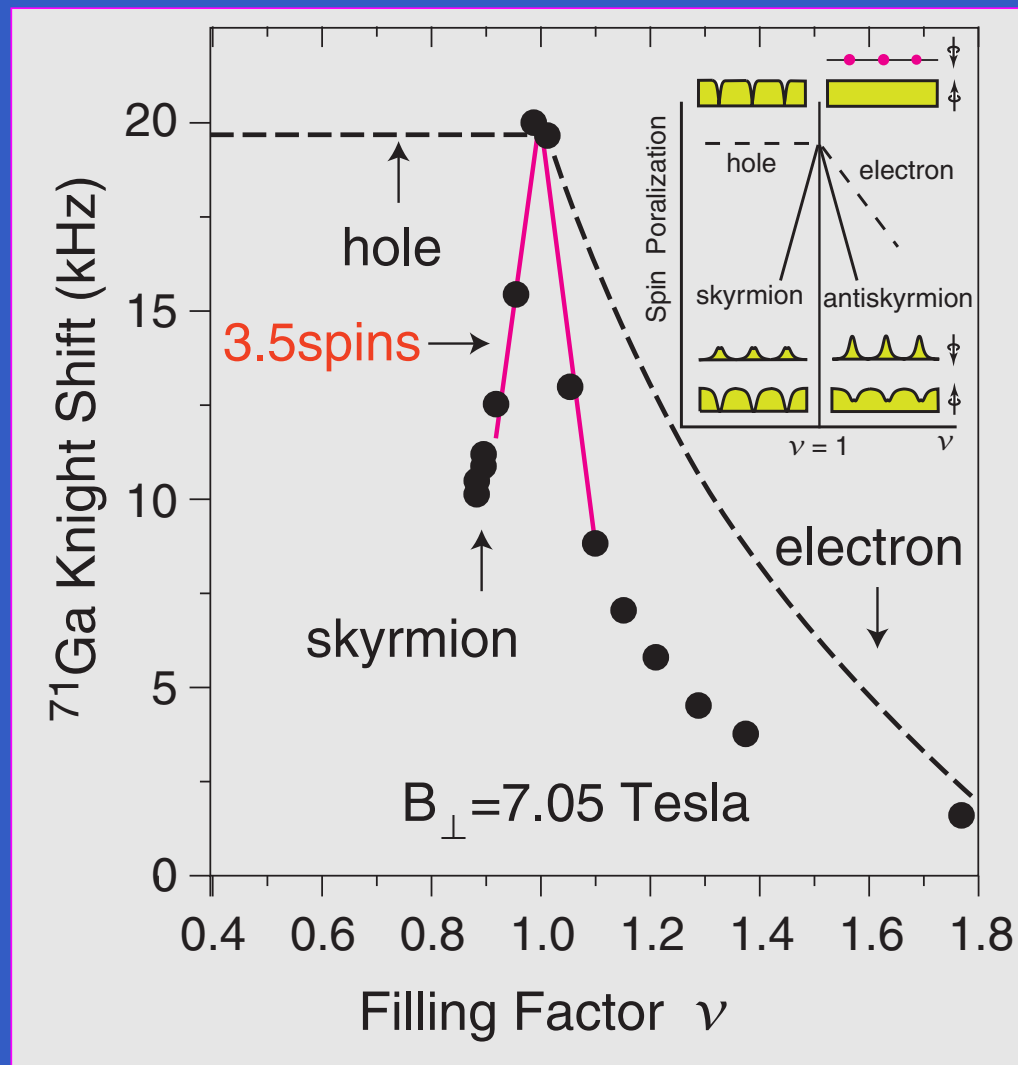
- Excitations around $\nu = 1$
 - $N = N_\Phi$ at $\nu = 1$
 - $N < N_\Phi$ at $\nu < 1 \Rightarrow$ holes excited (no spin excitation)
 - $N > N_\Phi$ at $\nu > 1 \Rightarrow$ electrons excited (spin excitation)
- Hall plateau is generated when quasiparticles are trapped by impurities



Spin Texture identified with Skyrmion

(experiment)

Experimentally 3.5 flipped spins are observed around $\nu = 1$



Barrett et al, PRL74(1995)5112

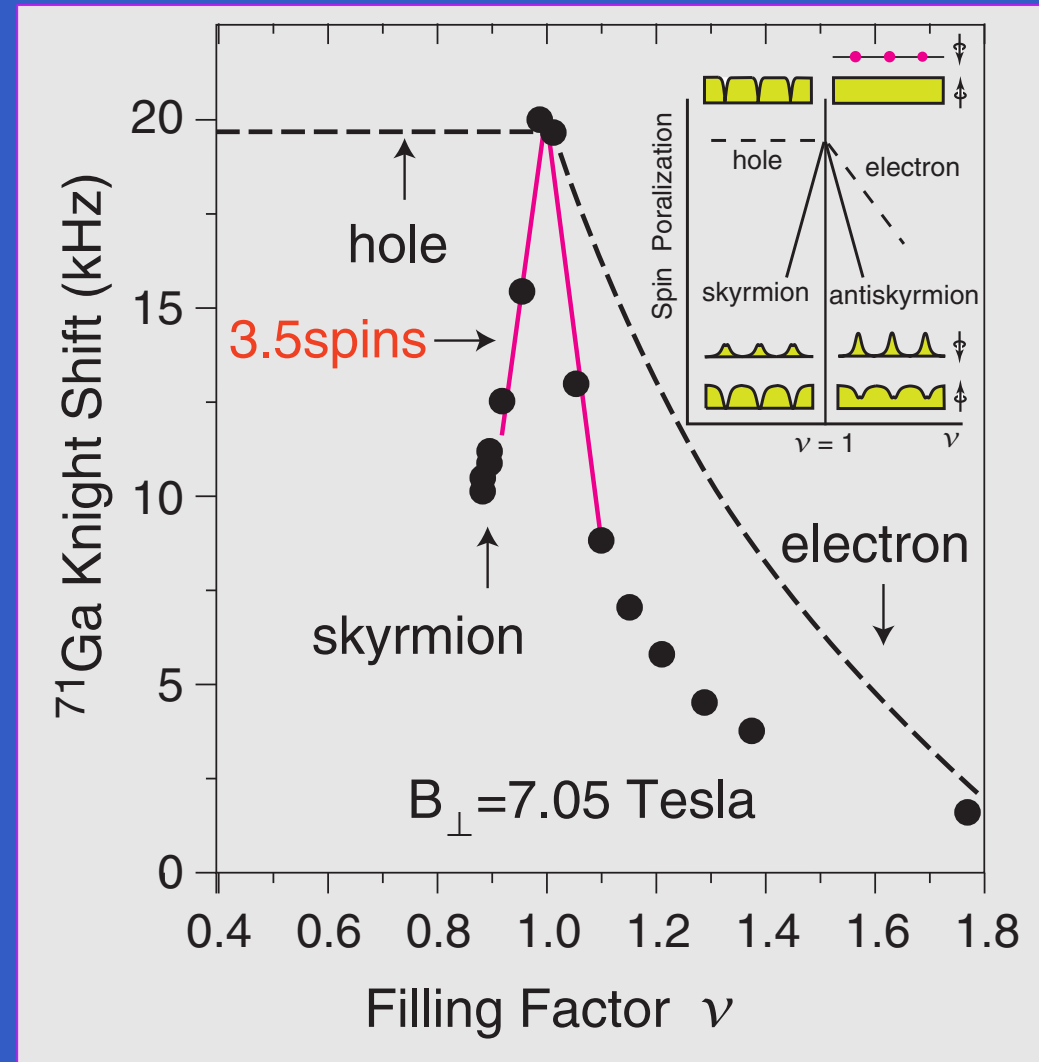
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Skyrmions must be excited !

A skyrmion and an antiskyrmion flips the same number of spins

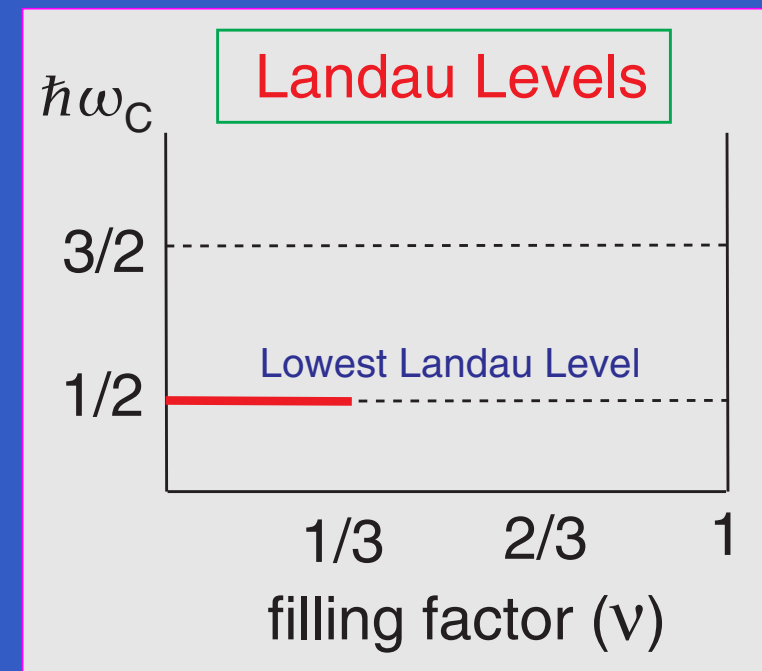


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Lowest-Landau-Level Projection

- Kinetic Hamiltonian generates Landau levels

$$H_K = \frac{P^2}{2M} = \frac{1}{2M}(P_x - iP_y)(P_x + iP_y) + \frac{1}{2}\hbar\omega_c \quad \text{with} \quad P_k = -i\hbar\partial_k + eA_x^{\text{ext}}$$



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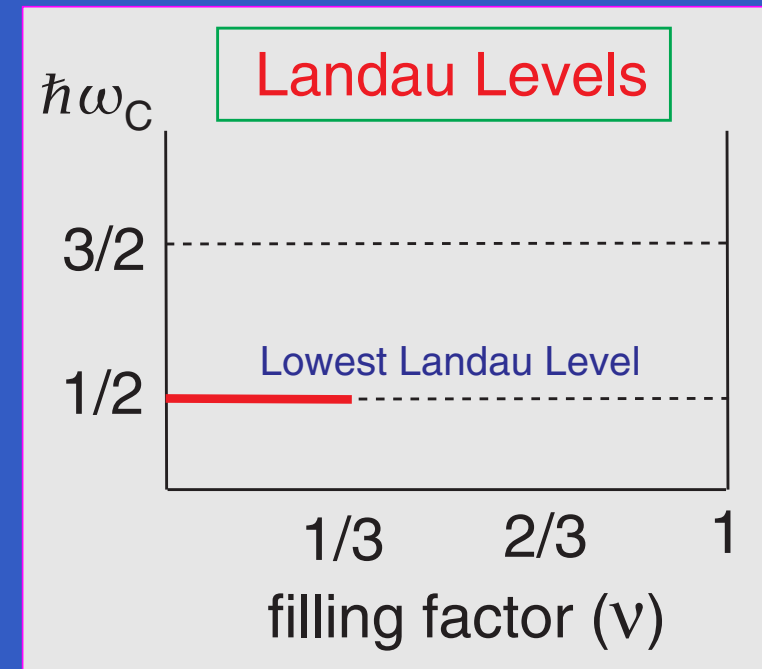
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- Electron coordinate $\mathbf{x} \Rightarrow$ guiding center X and relative coordinate R

$$\mathbf{x} = X + R \quad \text{with} \quad R = \left(-\frac{1}{eB_\perp}P_y, \frac{1}{eB_\perp}P_x \right)$$

$$[X, Y] = -i\ell_B^2, \quad [P_x, P_y] = i\hbar^2/\ell_B^2,$$

$$[X, P_x] = [X, P_y] = [Y, P_x] = [Y, P_y] = 0$$



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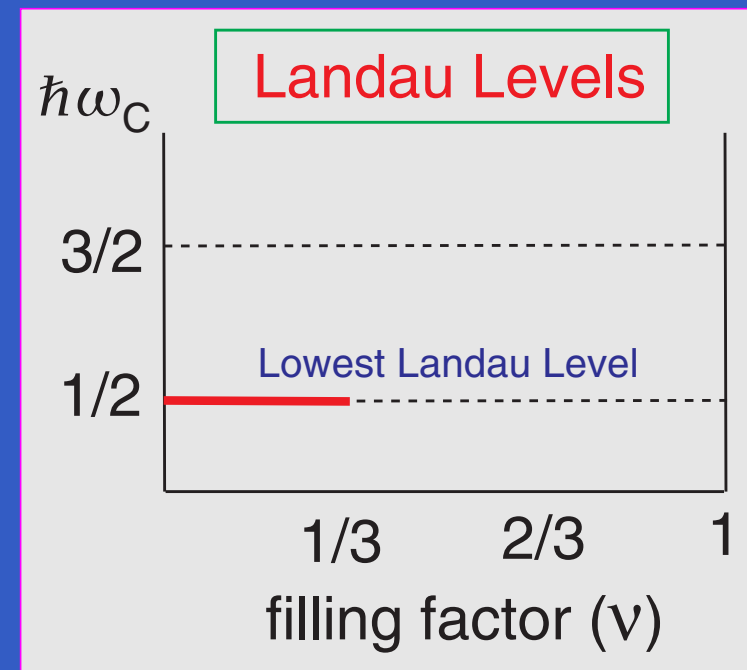
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- Electrons are confined to LLL if $\hbar\omega_c \rightarrow \infty$

$$(\mathbf{x}, y) \Rightarrow X = (X, Y) \quad \text{with} \quad [X, Y] = -i\ell_B^2$$



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with

$$b = \frac{1}{\sqrt{2}\ell_B} (X - iY), \quad b^\dagger = \frac{1}{\sqrt{2}\ell_B} (X + iY)$$

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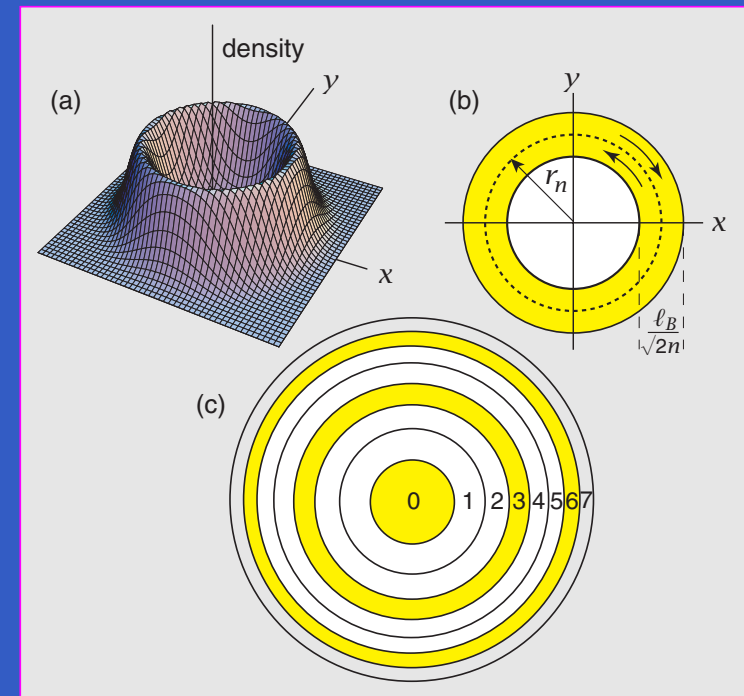
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- The Fock space \mathbb{H}_W is made of the states

$$|n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle, \quad n = 0, 1, 2, \dots$$



Landau site

QH system is governed by noncommutative geometry

Weyl Ordering

- Weyl-ordering of a classical quantity $f(\mathbf{r})$

$$W[f] = \frac{1}{(2\pi)^2} \int d^2q d^2r e^{iq(\mathbf{r}-\mathbf{X})} f(\mathbf{r})$$

with $\mathbf{X} = (X, Y)$

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- We apply this to the electron density $f(\mathbf{r}) = \delta^2(\mathbf{x} - \mathbf{r}) \equiv \rho(\mathbf{x})$
Here, \mathbf{r} is the electron trajectory, while \mathbf{x} is the coordinate (parameter)

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- Field-theoretical density reads

$$\hat{\rho}(\mathbf{q}) \equiv \langle \Psi | W[\rho(\mathbf{q})] | \Psi \rangle = \frac{1}{2\pi} \sum_{mn} \langle m | e^{-i\mathbf{q}\mathbf{X}} | n \rangle c^\dagger(m) c(n)$$

with electron field $\Psi(\mathbf{x})$ confined within the LLL

$$\Psi(\mathbf{x}) = \langle \mathbf{x} | \Psi \rangle = \sum_n \langle \mathbf{x} | n \rangle c(n)$$

Density Operators with Spin (Monolayer)

- The observables in the monolayer system are the densities

$$\hat{\rho}(\mathbf{q}) = \frac{1}{2\pi} \sum_{mn} \langle m | e^{-i\mathbf{q}X} | n \rangle c_{\alpha}^{\dagger}(m) c_{\alpha}(n)$$

$$\hat{S}_a(\mathbf{q}) = \frac{1}{2\pi} \sum_{mn} \langle m | e^{-i\mathbf{q}X} | n \rangle c_{\alpha}^{\dagger}(m) \frac{1}{2} (\tau_a)_{\alpha\beta} c_{\beta}(n)$$

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- The density algebra $W_{\infty}(2)$ is the $SU(2)$ extension of W_{∞}

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Spin rotation \Rightarrow U(1) density modulation \Rightarrow Increase of Coulomb energy

Landau-site Hamiltonian

Coulomb Hamiltonian \Rightarrow LLL projection \Rightarrow Landau-site Hamiltonian

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• LLL-projection of Coulomb Hamiltonian

$$H_C = \pi \int d^2q V(\mathbf{q}) \rho(-\mathbf{q}) \rho(\mathbf{q}) \Rightarrow \hat{H}_C = \sum_{mni j} V_{mni j} \sum_{\sigma, \tau} c_{\sigma}^{\dagger}(m) c_{\sigma}(n) c_{\tau}^{\dagger}(i) c_{\tau}(j)$$

$$V_{mni j} = \frac{1}{4\pi} \int d^2k V_D(\mathbf{k}) \langle m | e^{iX\mathbf{k}} | n \rangle \langle i | e^{-iX\mathbf{k}} | j \rangle, \quad V_D(\mathbf{q}) = \frac{e^2}{4\pi\epsilon|\mathbf{q}|} e^{-\ell_B^2 \mathbf{q}^2 / 2}$$

• V_{nnjj} yields the direct interaction, while V_{njjn} the exchange interaction

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• Homogeneous eigen states \Rightarrow degenerate ground states

$$\hat{H}_C |g\rangle = -\epsilon_X N_{\Phi} |g\rangle, \quad \epsilon_X = \frac{1}{4\pi} \int d^2k V_D(\mathbf{k}) = \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{e^2}{4\pi\epsilon\ell_B}$$

$$|g\rangle = \prod_n \left[\sin \theta c_{\uparrow}^{\dagger}(n) |0\rangle + \cos \theta c_{\downarrow}^{\dagger}(n) |0\rangle \right]$$

Exchange Interaction

- Exchange interaction is extracted by using the algebraic identity for SU(N)

$$\delta_{\sigma\beta}\delta_{\tau\alpha} = \frac{1}{2} \sum_A^{N^2-1} \lambda_{\sigma\tau}^A \lambda_{\alpha\beta}^A + \frac{1}{N} \delta_{\sigma\tau} \delta_{\alpha\beta}$$

- Projected Coulomb Hamiltonian

$$\hat{H}_C = \sum_{mni j} V_{mni j} \sum_{\sigma, \tau} c_{\sigma}^{\dagger}(m) c_{\sigma}(n) c_{\tau}^{\dagger}(i) c_{\tau}(j) = -\pi \int d^2 p V_D(\mathbf{p}) \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p})$$

$$\Rightarrow \hat{H}_X = -2 \sum_{mni j} V_{mni j} [S_a(m, j) S_a(i, n) + \frac{1}{2N} \rho(m, j) \rho(i, n)]$$

$$\hat{H}_X = -\pi \int d^2 p V_X(\mathbf{p}) \left[\hat{\mathbf{S}}(-\mathbf{p}) \hat{\mathbf{S}}(\mathbf{p}) + \frac{1}{4} \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p}) \right]$$

$$\text{with } V_X(\mathbf{p}) = \frac{\ell_B^2}{\pi} \int d^2 k e^{-i\ell_B^2 \mathbf{p} \wedge \mathbf{k}} e^{-\ell_B^2 \mathbf{p}^2 / 2} V(\mathbf{k}) = \frac{\sqrt{2\pi} e^2 \ell_B}{4\pi \epsilon} I_0(\ell_B^2 \mathbf{p}^2 / 4) e^{-\ell_B^2 \mathbf{p}^2 / 4}$$

Effective Field Theory

$$\hat{H}_\chi = -\pi \int d^2p V_\chi(\mathbf{p}) \left[\hat{\mathbf{S}}(-\mathbf{p})\hat{\mathbf{S}}(\mathbf{p}) + \frac{1}{4}\hat{\rho}(-\mathbf{p})\hat{\rho}(\mathbf{p}) \right]$$

Effective Field Theory

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Landau-site Hamiltonian \Rightarrow Derivative Expansion \Rightarrow Effective Field Theory

Effective Field Theory

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Landau-site Hamiltonian \Rightarrow Derivative Expansion \Rightarrow Effective Field Theory

- For sufficient smooth configuration we make the derivative expansion

$$V_X(\mathbf{p}) = \frac{\sqrt{2\pi}e^2\ell_B}{4\pi\epsilon} I_0(\ell_B^2\mathbf{p}^2/4)e^{-\ell_B^2\mathbf{p}^2/4} = V_X(0) - \frac{2J_s}{\pi\rho_\Phi^2}\mathbf{p}^2 + O(\mathbf{p}^4)$$

with $\rho_\Phi = \frac{\rho_0}{\nu} = \frac{1}{2\pi\ell_B^2}, \quad V_X(0) = 4\ell_B^2\epsilon_X, \quad J_s = \frac{1}{8\pi}\epsilon_X$

Effective Field Theory

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- For sufficient smooth configuration we make the derivative expansion

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with $\rho_\Phi = \frac{\rho_0}{\nu} = \frac{1}{2\pi \ell_B^2}$, $V_X(0) = 4\ell_B^2 \epsilon_X$, $J_s = \frac{1}{8\pi} \epsilon_X$

- Exchange interaction leads to a nonlinear sigma model as an effective theory

$$\mathcal{H}_X^{\text{spin}} = 2J_s \partial_k \mathbf{S}(\mathbf{x}) \partial_k \mathbf{S}(\mathbf{x}) \quad \text{with} \quad \mathbf{S}(\mathbf{x}) = \frac{\rho_0}{\nu} \mathbf{S}(\mathbf{x})$$

Nonlinear Sigma Model

$$H_{\text{eff}} = 2J_s \int d^2x \partial_k \mathcal{S}^{\text{sky}}(\mathbf{x}) \partial_k \mathcal{S}^{\text{sky}}(\mathbf{x})$$

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- Topological solitons are **Skyrmions** associated with $\pi_2(CP^1) = \mathbb{Z}$

$$Q^{\text{sky}}(\mathbf{x}) = \frac{1}{\pi} \varepsilon_{abc} \varepsilon^{ij} \mathcal{S}_a \partial_i \mathcal{S}_b \partial_j \mathcal{S}_c = \frac{1}{\pi} \frac{4(\kappa \ell_B)^2}{[r^2 + 4(\kappa \ell_B)^2]^2}$$

- Skyrmion configuration (its size is fixed to minimize the energy)

$$\mathcal{S}_x^{\text{sky}} = \sqrt{1 - \sigma_{\text{sky}}^2} \cos \theta, \quad \mathcal{S}_y^{\text{sky}} = -\sqrt{1 - \sigma_{\text{sky}}^2} \sin \theta, \quad \mathcal{S}_z^{\text{sky}} = \sigma_{\text{sky}}$$

$$\text{with } \sigma_{\text{sky}}(\mathbf{x}) = \frac{r^2 - 4(\kappa \ell_B)^2}{r^2 + 4(\kappa \ell_B)^2}$$

- The skyrmion scale κ is arbitral in 2-dimensional nonlinear sigma model

Spin Wave (Goldstone Mode)

- Effective Hamiltonian in the presence of the Zeeman effect

$$\mathcal{H}_{\text{eff}} = 2J_s \partial_k \mathcal{S}(\mathbf{x}) \partial_k \mathcal{S}(\mathbf{x}) - \rho_0 \Delta_Z \mathcal{S}_Z(\mathbf{x})$$

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- For small fluctuation, i.e., **pseudo-Goldstone mode**

$$\mathcal{H}_{\text{eff}} = \frac{1}{2} J_s (\partial_k \sigma)^2 + \frac{1}{2} J_s (\partial_k \vartheta)^2 + \frac{\rho_0 \Delta_Z}{4} (\sigma^2 + \vartheta^2)$$

with

$$\mathcal{S}_x(\mathbf{x}) = \frac{1}{2} \sigma(\mathbf{x}), \quad \mathcal{S}_y(\mathbf{x}) = \frac{1}{2} \sqrt{1 - \sigma^2(\mathbf{x})} \sin \vartheta(\mathbf{x}), \quad \mathcal{S}_z(\mathbf{x}) = \frac{1}{2} \sqrt{1 - \sigma^2(\mathbf{x})} \cos \vartheta(\mathbf{x})$$

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- Quantum coherence develops if coherence length \gg magnetic length

$$\xi_{\text{spin}} = \sqrt{\frac{2\rho_s}{\rho_0 \Delta_Z}} = \frac{7.33}{\sqrt{B_{\perp}}} \ell_B$$

CP¹ Skyrmion

- Due to the $W_\infty(2)$ algebra the spin rotation induces a density modulation

$$[\hat{S}_a(\mathbf{p}), \hat{\rho}(\mathbf{q})] = \frac{i}{\pi} \hat{S}_a(\mathbf{p} + \mathbf{q}) \sin\left(\frac{\ell_B^2}{2} \mathbf{p} \wedge \mathbf{q}\right)$$

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- A Skyrmion excitation induces a density modulation

$$\rho^{\text{sky}}(\mathbf{x}) \equiv \langle \mathfrak{S}_{\text{sky}} | \hat{\rho}(\mathbf{x}) | \mathfrak{S}_{\text{sky}} \rangle \simeq Q^{\text{sky}}(\mathbf{x}) - \rho_0$$

with the topological charge

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- Effective Hamiltonian for nonperturbative excitations

$$H_{\text{eff}} = 2J_s \int d^2x \partial_k \mathcal{S}^{\text{sky}}(\mathbf{x}) \partial_k \mathcal{S}^{\text{sky}}(\mathbf{x}) - \rho_0 \Delta_Z \int d^2x \mathcal{S}_Z^{\text{sky}}(\mathbf{x}) \\ + \frac{1}{2} \int d^2x d^2y \rho^{\text{sky}}(\mathbf{x}) V(\mathbf{x} - \mathbf{y}) \rho^{\text{sky}}(\mathbf{y})$$

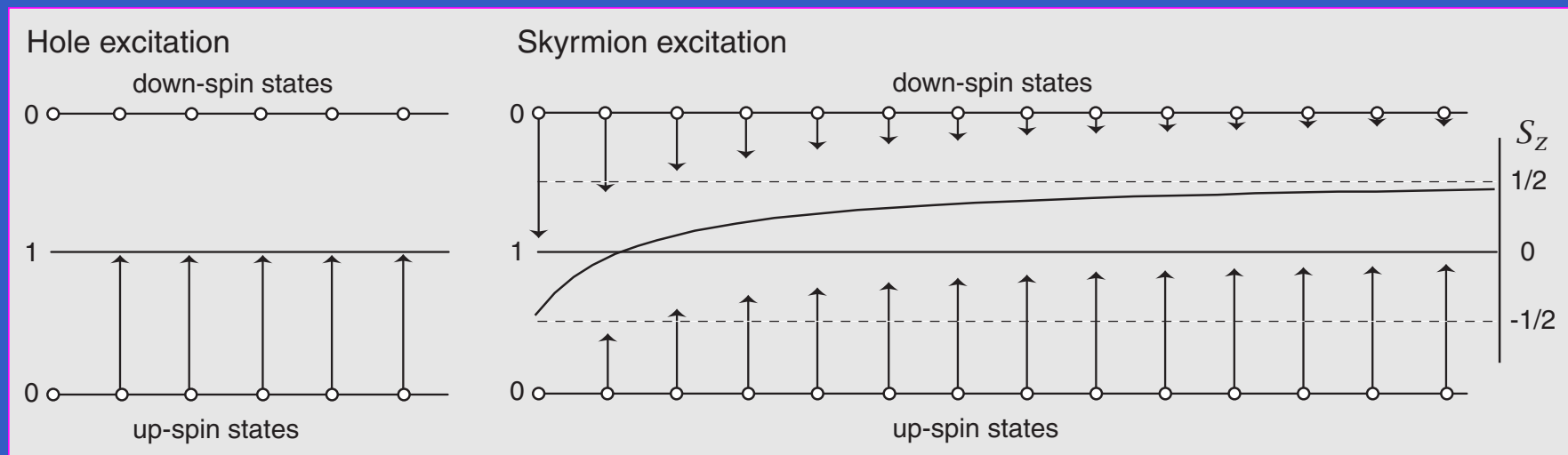
CP¹ Skyrmion in Microscopic Theory

- A microscopic Skyrmion state

$$|\mathfrak{S}_{\text{sky}}\rangle = \prod_{n=0}^{\infty} \left[u(n)c_{\downarrow}^{\dagger}(n) + v(n)c_{\uparrow}^{\dagger}(n+1) \right] |0\rangle, \quad u^2(n) + v^2(n) = 1$$

- A hole excitation is a special one: $u(n) = 0, v(n) = 1$ for all n

$$|\mathfrak{S}_{\text{sky}}\rangle = \prod_{n=1}^{\infty} c_{\uparrow}^{\dagger}(n) |0\rangle$$



CP¹ Skyrmion in Microscopic Theory (continued)

• Density and spin modulations

$$\rho^{\text{sky}}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int d^2q \sum_{mn} \langle m | e^{-i\mathbf{q}(\mathbf{x}-\mathbf{X})} | n \rangle \langle \mathfrak{E}_{\text{sky}} | c^\dagger(m) c(n) | \mathfrak{E}_{\text{sky}} \rangle$$

$$S_a^{\text{sky}}(\mathbf{x}) = \frac{1}{(2\pi)^2} \int d^2q \sum_{mn} \langle m | e^{-i\mathbf{q}(\mathbf{x}-\mathbf{X})} | n \rangle \langle \mathfrak{E}_{\text{sky}} | c^\dagger(m) \frac{\tau_a}{2} c(n) | \mathfrak{E}_{\text{sky}} \rangle$$

⇒

$$\rho^{\text{sky}}(\mathbf{x}) = \frac{1}{2\pi} - \frac{1}{2\pi} \frac{1}{\omega^2 + 1} e^{-\frac{1}{2}r^2} M(\omega^2; \omega^2 + 2; r^2/2)$$

$$S_z^{\text{sky}}(\mathbf{x}) = \frac{1}{4\pi} - \frac{1}{4\pi} e^{-\frac{1}{2}r^2} M(\omega^2; \omega^2 + 1; r^2/2) - \frac{1}{4\pi} \frac{\omega^2}{\omega^2 + 1} e^{-\frac{1}{2}r^2} M(\omega^2 + 1; \omega^2 + 2; r^2/2)$$

$$S_x^{\text{sky}}(\mathbf{x}) = \frac{1}{4\pi} \frac{\sqrt{2}\omega x}{\omega^2 + 1} e^{-\frac{1}{2}r^2} M(\omega^2 + 1; \omega^2 + 2; r^2/2)$$

$$S_y^{\text{sky}}(\mathbf{x}) = \frac{1}{4\pi} \frac{-\sqrt{2}\omega y}{\omega^2 + 1} e^{-\frac{1}{2}r^2} M(\omega^2 + 1; \omega^2 + 2; r^2/2)$$

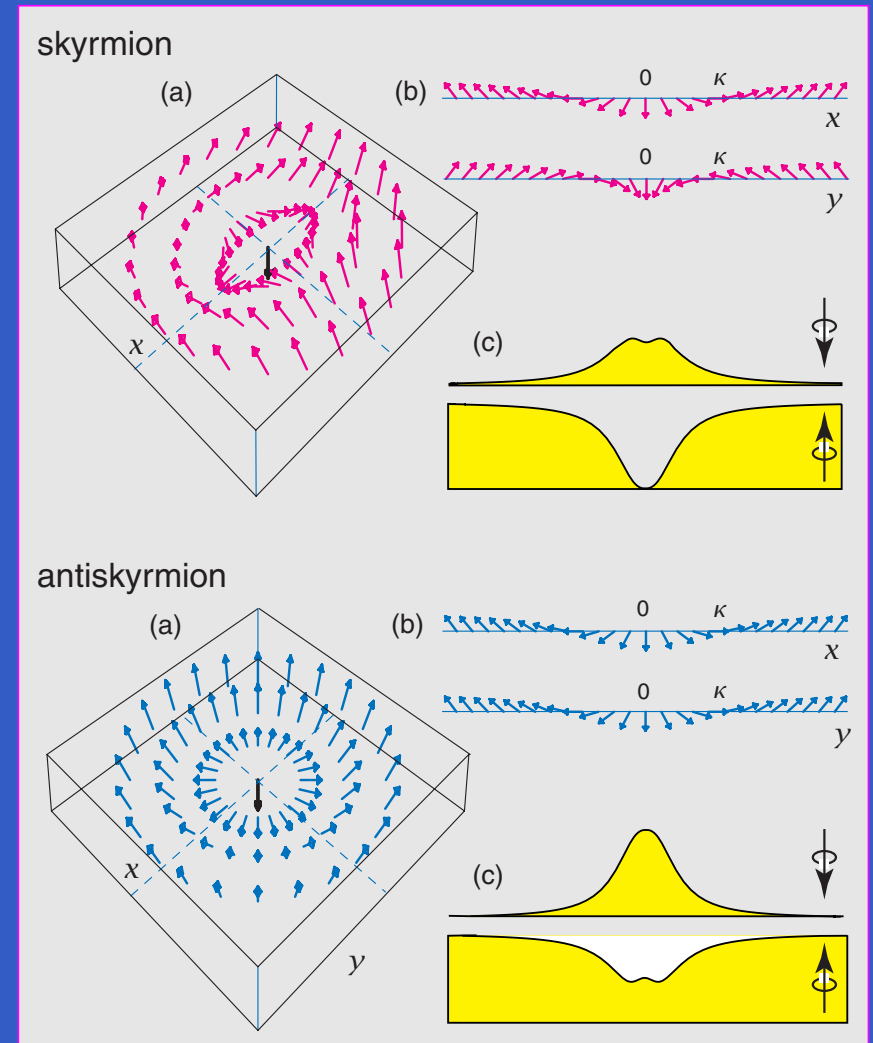
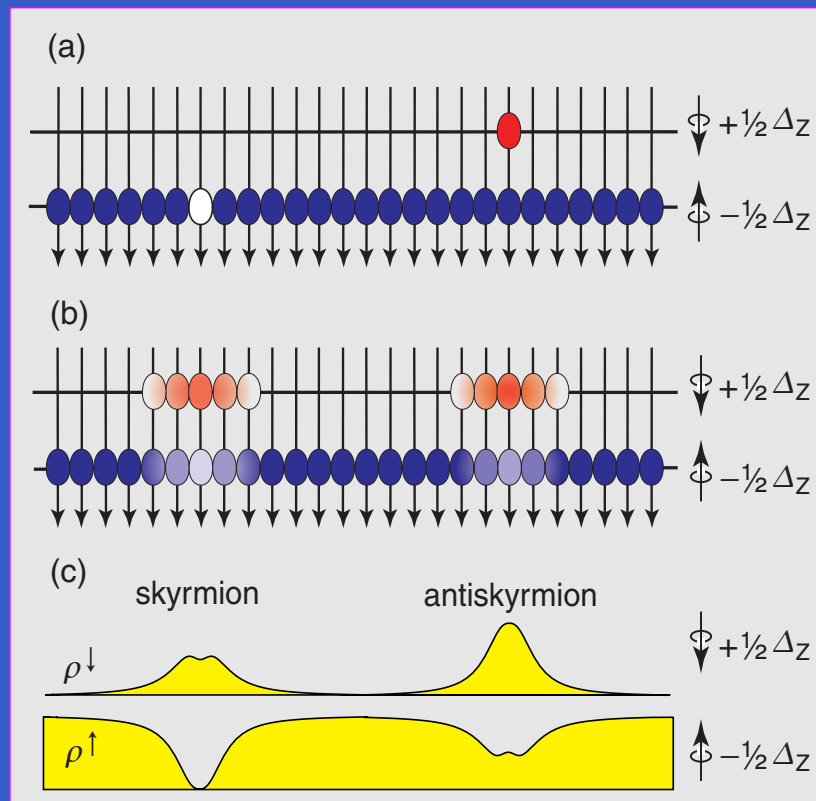
M is the Kummer function

$$\text{anzats: } u^2(n) = \frac{\omega^2}{n + 1 + \omega^2}, \quad v^2(n) = \frac{n + 1}{n + 1 + \omega^2}, \quad \omega = \sqrt{2}\kappa\ell_B \quad (\kappa: \text{Skyrmion scale})$$

CP¹ Skyrmion and Spin Flip

- Skyrmion has a fixed scale to optimize the Coulomb and Zeeman energies
- Skyrmion flips many spins coherently

$$N^{\text{spin}} = -\frac{1}{2} \int d^2x \left[2S_z^{\text{sky}}(\mathbf{x}) - \rho_0 \right]$$



● Skyrmion excitation energy ($E_C^0 = e^2 / 4\pi\epsilon\ell_B$)

$$E_{\text{sky}} \simeq \left\{ \frac{1}{4} \sqrt{\frac{\pi}{2}} + \frac{0.26}{2\kappa} + 2\tilde{g}\kappa^2 \ln \left(\frac{\sqrt{2\pi}}{32\tilde{g}} + 1 \right) \right\} E_C^0$$

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$$\kappa \simeq \frac{1}{2} \left\{ \tilde{g} \ln \left(\frac{\sqrt{2\pi}}{32\tilde{g}} + 1 \right) \right\}^{-1/3} \simeq 1$$

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$$N^{\text{spin}} \simeq 3.5 \text{ at } B = 7 \text{ Tesla}$$

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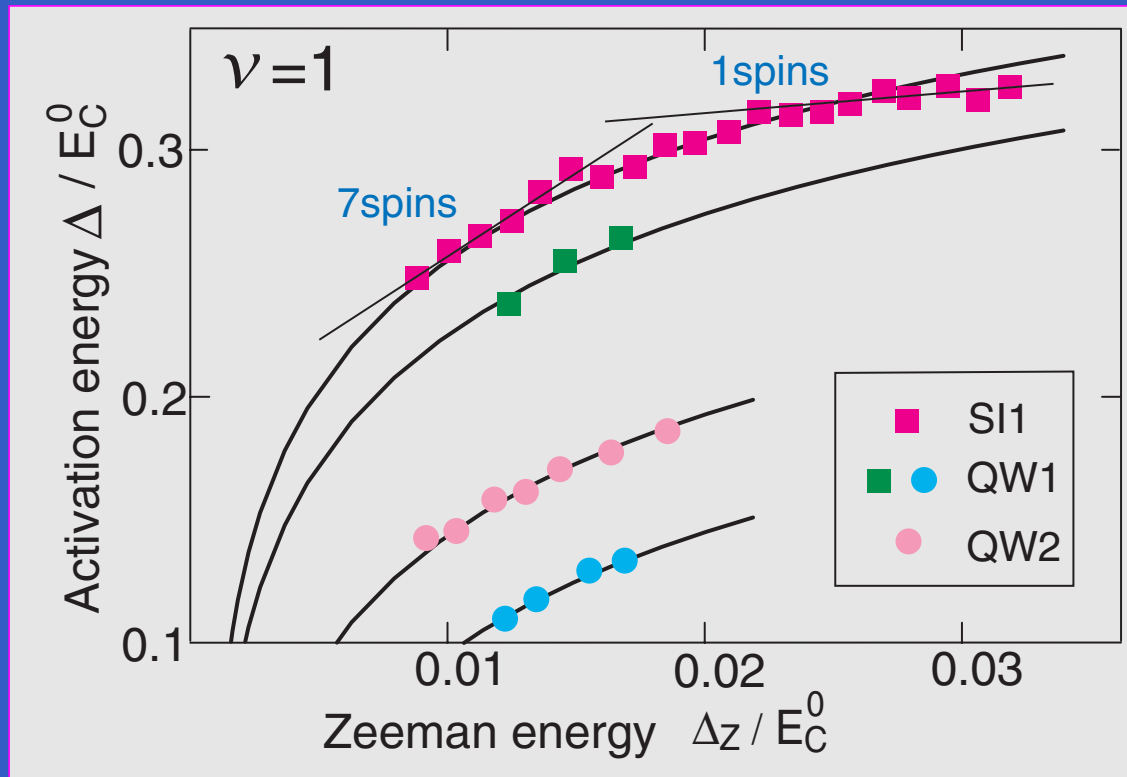
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Schmeller et al, PRL75(1995)4290

Monolayer QH System with SU(N) Symmetry

- Kinetic Hamiltonian for N-component electrons with SU(N) symmetry

$$H_K = \frac{1}{2M} \Psi^\dagger (P_x - iP_y)(P_x + iP_y) \Psi + \frac{1}{2} \hbar \omega_c \quad \text{with} \quad P_k = -i\hbar \partial_k + eA_x^{\text{ext}}$$

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- There are N^2 densities: $\rho(\mathbf{x}) = \Psi^\dagger(\mathbf{x})\Psi(\mathbf{x})$, $I_A(\mathbf{x}) = \frac{1}{2} \Psi^\dagger(\mathbf{x}) \lambda_A \Psi(\mathbf{x})$

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$$\hat{H}_C = -\pi \int d^2p V_D(\mathbf{p}) \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p})$$

$$\Rightarrow \hat{H}_X = -\pi \int d^2p V_X(\mathbf{p}) \left[\hat{I}(-\mathbf{p}) \hat{I}(\mathbf{p}) + \frac{1}{4} \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p}) \right]$$

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- Low-energy effective Hamiltonian is SU(N) nonlinear sigma model

$$H_X^{\text{eff}} = 2J_s \int d^2x \partial_k \mathcal{J}(\mathbf{x}) \partial_k \mathcal{J}(\mathbf{x}) \quad \text{with} \quad \hat{I}(\mathbf{x}) = \rho_\Phi \mathcal{J}(\mathbf{x})$$

- One Landau site can accommodate k electrons ($k \leq N$)

Grassmannian Field

- SU(N) nonlinear sigma model \Rightarrow with spontaneous symmetry breakdown

$$H_X^{\text{eff}} = 2J_s \int d^2x \partial_k \mathcal{J}(\mathbf{x}) \partial_k \mathcal{J}(\mathbf{x})$$

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$$G^{N,k} = \frac{U(N)}{U(k) \otimes U(N-k)} = \frac{SU(N)}{U(1) \otimes SU(k) \otimes SU(N-k)},$$

$$G^{N,1} \equiv CP^{N-1}$$

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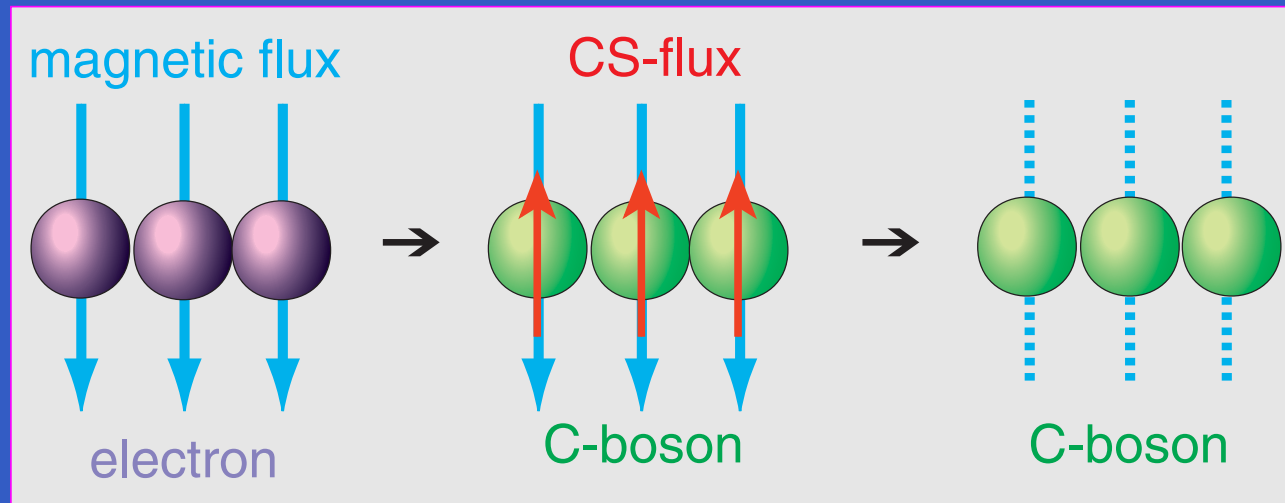
- Dimension is $2k(N-k) \Rightarrow 2k(N-k)$ Goldstone modes
- Topological solitons are **$G^{N,k}$ Skyrmions** at $\nu = k$

according to $\pi_2(G^{N,k}) = \mathbb{Z}$

Charge-Isospin Separation

- Spinless electrons are bosonized by attaching a flux \Rightarrow **composite bosons**

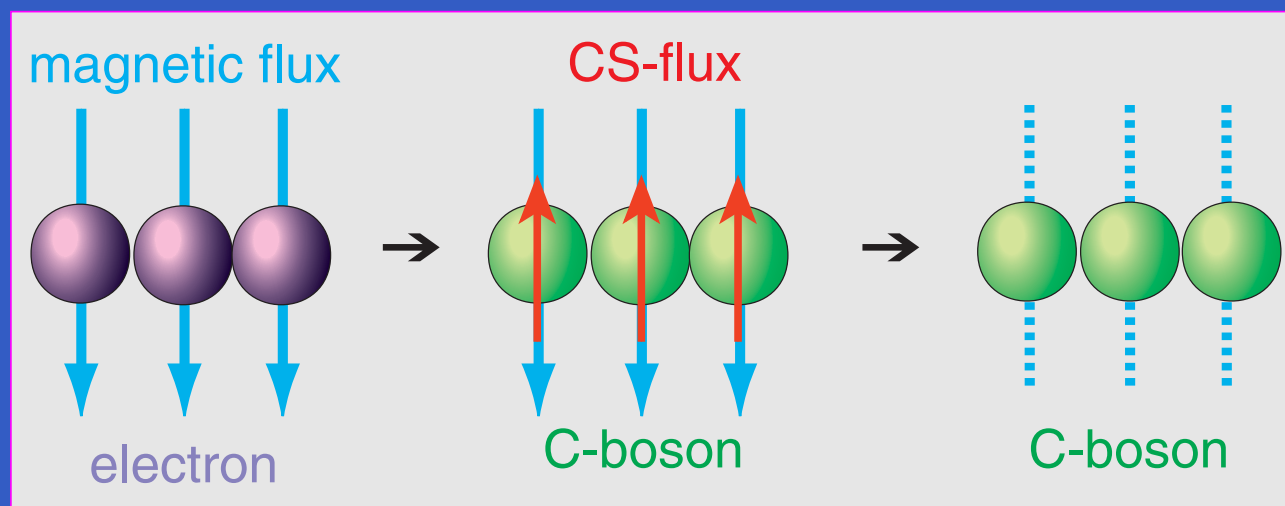
$$\psi(\mathbf{x}) = e^{-i\Theta(\mathbf{x})} \phi(\mathbf{x}) \quad \text{with} \quad \varepsilon_{jk} \partial_j \partial_k \Theta(\mathbf{x}) = 2\pi \rho(\mathbf{x})$$



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- $U(N)$ electrons are also bosonized

$$\psi_\alpha(\mathbf{x}) = e^{-i\Theta(\mathbf{x})} \phi(\mathbf{x}) n_\alpha(\mathbf{x}) \quad \text{with} \quad \sum n_\alpha^\dagger(\mathbf{x}) n_\alpha(\mathbf{x}) = 1$$

- There are $N - 1$ complex degree of freedom in $n_\alpha \Rightarrow$ **CP^{N-1} field**

Effective Field Theory with CP^{N-1} Field at $\nu = 1$

• Relation between isospin field and CP^{N-1} field:

$$J_A(\mathbf{x}) = \mathbf{n}^\dagger(\mathbf{x}) \frac{\lambda_A}{2} \mathbf{n}(\mathbf{x})$$

Effective Field Theory with $\mathbb{C}P^{N-1}$ Field at $\nu = 1$

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$$\mathcal{J}_A(\mathbf{x}) = \mathbf{n}^\dagger(\mathbf{x}) \frac{\lambda_A}{2} \mathbf{n}(\mathbf{x})$$

- Equivalence between **the SU(N) sigma model** and **the $\mathbb{C}P^{N-1}$ model**

$$\mathcal{H}_X = 2J_s \sum_{A=1}^{N^2-1} [\partial_k \mathcal{J}_A]^2 = 2J_s \sum_{\alpha=1}^N \left(\partial_j n^{\alpha\dagger} + iK_j n^{\alpha\dagger} \right) \left(\partial^j n^\alpha + iK^j n^\alpha \right)$$

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- There are N -fold degeneracy in the ground state

- One ground state is chosen spontaneously with a **$\mathbb{C}P^{N-1}$ Skyrmion** on it

$$\mathbf{n}_g(\mathbf{x}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \Rightarrow \mathbf{n}_{\text{sky}}(\mathbf{x}) = \begin{pmatrix} K_{N-1} \\ \vdots \\ K_1 \\ Z \end{pmatrix}$$

Effective Field Theory with $G^{N,k}$ Field at $\nu = k$

- At $\nu = k$ there are N electrons in one Landau site
- We need k CP^{N-1} fields to describe such a system: [Grassmannian field](#)

Grassmannian field: $Z(\mathbf{x}) = (\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_k)$

Isospin field: $J_A(\mathbf{x}) = \text{Tr} \left[Z^\dagger(\mathbf{x}) \frac{\lambda_A}{2} Z(\mathbf{x}) \right] = \frac{1}{2} \sum_{i=1}^k \mathbf{n}_i^\dagger(\mathbf{x}) \lambda_A \mathbf{n}_i(\mathbf{x})$

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$$\mathcal{H}_X = 2J_s \sum_{A=1}^{N^2-1} [\partial_k J_A]^2 = 2J_s \text{Tr} \left[(\partial_j Z - iK_j Z)^\dagger (\partial_j Z - iK_j Z) \right]$$

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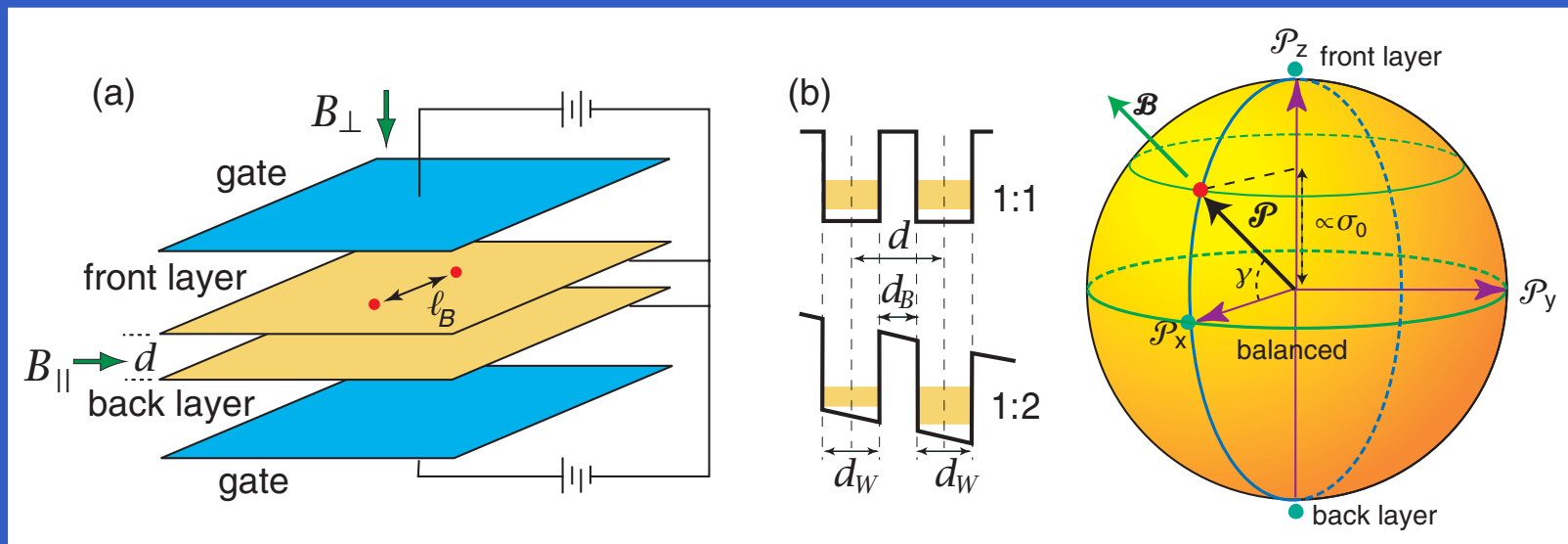
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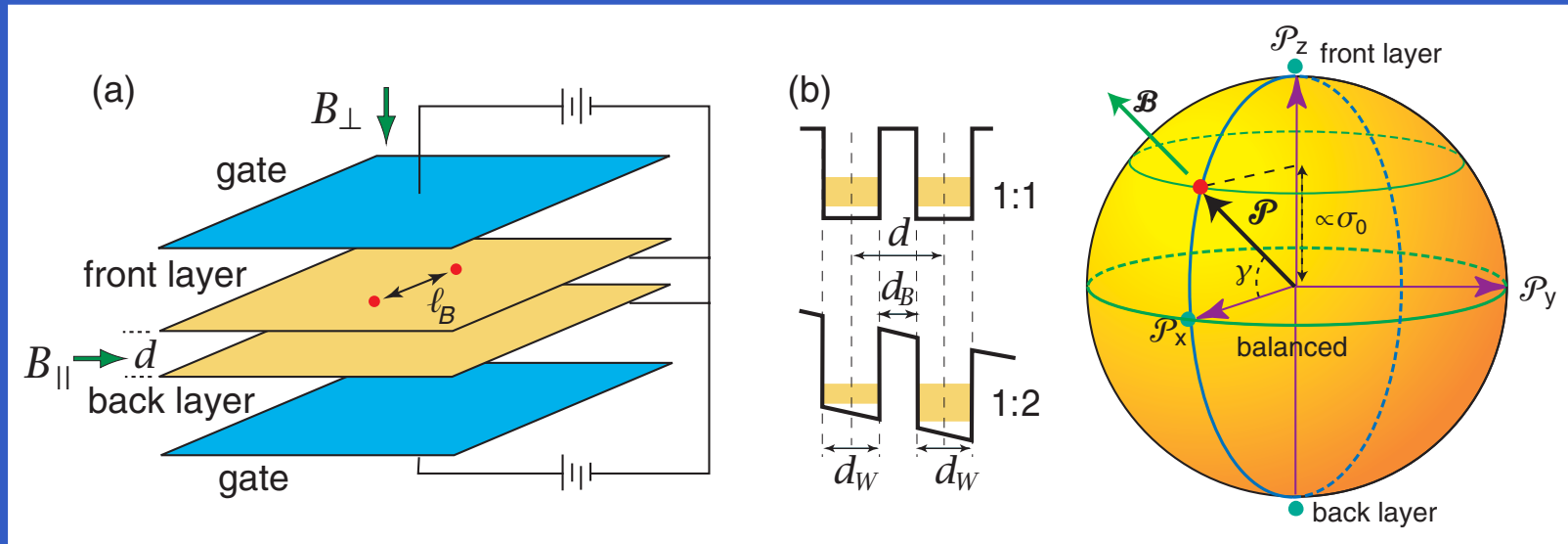
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Pseudospins in Bilayer System



Pseudospins in Bilayer System



● Pseudospin (ppin) \mathcal{P} is a powerful tool to elucidate bilayer system

$$H_{pZ} = \sum_i [-\Delta_{SAS} P_x(i, i) - eV_{bias} P_z(i, i)]$$

$$\langle P_z \rangle = \frac{1}{2} \sigma_0, \quad \langle P_z \rangle = \frac{1}{2} \sqrt{1 - \sigma_0^2}$$

● Imbalance parameter P_z is controlled experimentally

$$\sigma_0 \equiv \frac{\rho^{front} - \rho^{back}}{\rho^{front} + \rho^{back}} = 2P_z, \quad eV_{bias} = \frac{\sigma_0}{\sqrt{1 - \sigma_0^2}} \Delta_{SAS}$$

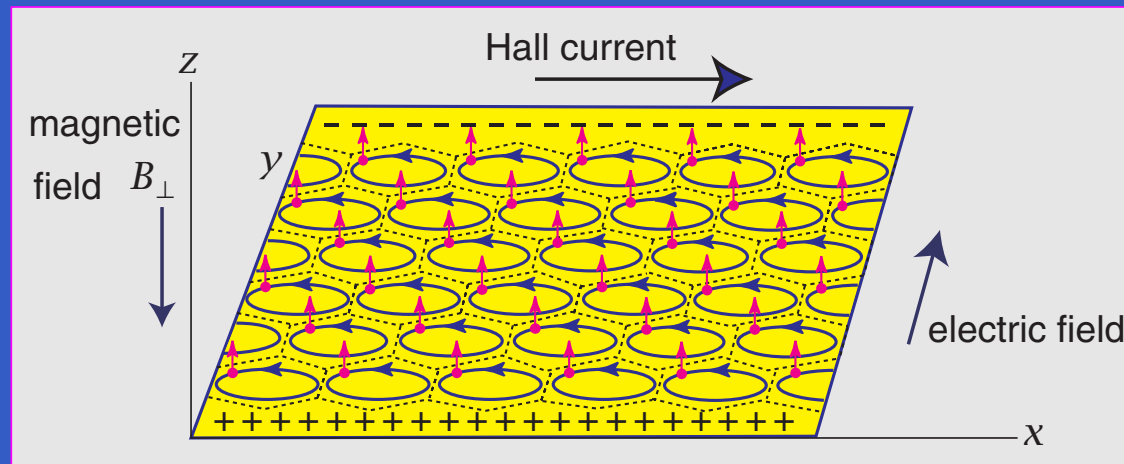
Coulomb Hamiltonian \Rightarrow LLL-projection \Rightarrow Landau-site Hamiltonian

Coulomb interactions induce **spin coherence**

$$H_C = \pi \int d^2q \rho(-\mathbf{p}) V(\mathbf{p}) \rho(\mathbf{p}) \quad \Rightarrow$$

$$H_X^{\text{spin}} = -\pi \int d^2p V_X(\mathbf{p}) \left[\hat{S}(-\mathbf{p}) \hat{S}(\mathbf{p}) + \frac{1}{4} \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p}) \right]$$

$$V_X(\mathbf{p}) = \frac{\ell_B^2}{\pi} \int d^2k e^{-i\ell_B^2 \mathbf{p} \wedge \mathbf{k}} e^{-\ell_B^2 \mathbf{p}^2 / 2} V(\mathbf{k}), \quad V(\mathbf{k}) = \frac{e^2}{4\pi\epsilon|\mathbf{k}|}$$



Quantum Hall Ferromagnets: Spinless Bilayer with SU(2)

Coulomb Hamiltonian \Rightarrow LLL-projection \Rightarrow Landau-site Hamiltonian

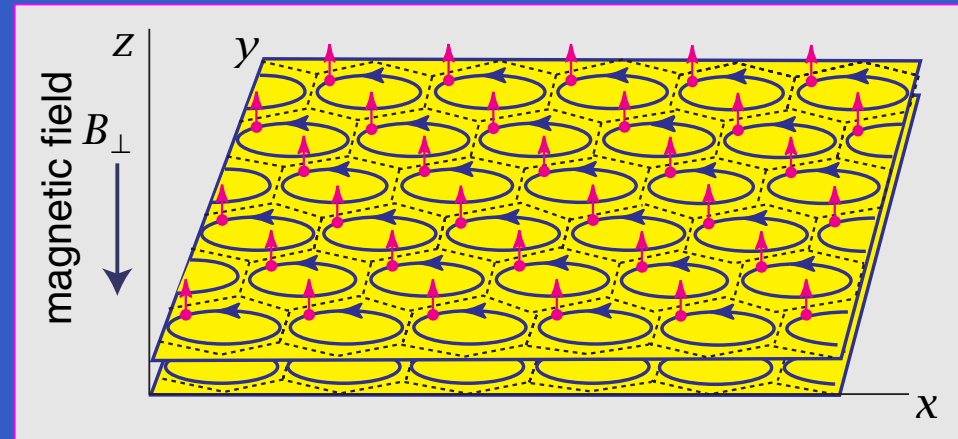
Coulomb interactions induce **interlayer coherence** in bilayer system

$$H_C = \pi \int d^2q [\rho(-\mathbf{p})V^+(\mathbf{p})\rho(\mathbf{p}) + 2P_z(-\mathbf{p})V^-(\mathbf{p})P_z(\mathbf{p})] \Rightarrow$$

$$H_X^{\text{ppin}} = -\pi \int d^2p [V_X^d(\mathbf{p}) \sum_{a=xy} \hat{P}_a(-\mathbf{p})\hat{P}_a(\mathbf{p}) + 2V_X^-(\mathbf{p})\hat{P}_z(-\mathbf{p})\hat{P}_z(\mathbf{p}) + \frac{1}{4}V_X(\mathbf{p})\hat{\rho}(-\mathbf{p})\hat{\rho}(\mathbf{p})]$$

$$V_X^\pm(\mathbf{p}) = \frac{\ell_B^2}{\pi} \int d^2k e^{-i\ell_B^2 \mathbf{p} \wedge \mathbf{k}} e^{-\ell_B^2 \mathbf{p}^2 / 2} V^\pm(\mathbf{k}), \quad V^\pm(\mathbf{k}) = \frac{e^2}{8\pi\epsilon|\mathbf{k}|} (1 \pm e^{-d/\ell_B})$$

$$V_X^\pm(\mathbf{p}) = \frac{1}{2} [V_X(\mathbf{p}) \pm V_X^d(\mathbf{p})]$$



Effective Hamiltonian in Spinless Bilayer with SU(2)

$$H_X^{\text{ppin}} = -\pi \int d^2p \left[V_X^d(\mathbf{p}) \sum_{a=xy} \hat{P}_a(-\mathbf{p}) \hat{P}_a(\mathbf{p}) + 2V_X^-(\mathbf{p}) \hat{P}_z(-\mathbf{p}) \hat{P}_z(\mathbf{p}) + \frac{1}{4} V_X(\mathbf{p}) \hat{\rho}(-\mathbf{p}) \hat{\rho}(\mathbf{p}) \right]$$

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Landau-site Hamiltonian \Rightarrow Derivative Expansion \Rightarrow Effective Field Theory

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Landau-site Hamiltonian \Rightarrow Derivative Expansion \Rightarrow Effective Field Theory

• Coulomb interactions induce **interlayer coherence** in bilayer system

$$H_C^{\text{ppin}} \simeq 2 \int d^2x \left(\sum_{a=x,y} J_s^d [\partial_k \mathcal{P}_a(\mathbf{x})]^2 + J_s [\partial_k \mathcal{P}_z(\mathbf{x})]^2 \right)$$

$$\frac{J_s^d}{J_s} = -\sqrt{\frac{2}{\pi}} \frac{d}{\ell_B} + \left(1 + \frac{d^2}{\ell_B^2} \right) e^{d^2/2\ell_B^2} \text{erfc}(d/\sqrt{2}\ell_B), \quad J_s = \frac{1}{16\pi} \sqrt{\frac{\pi}{2}} \frac{e^2}{4\pi\epsilon\ell_B}$$

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- Topological solitons **ppin-Skyrmions**: $\pi_2(\text{CP}^1) = \mathbb{Z}$
 \Rightarrow flipping many pseudospins coherently

Bilayer Quantum Hall Ferromagnets with SU(4)

Physical Variables

$$\begin{aligned}\rho(\mathbf{x}) &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), & S_a(\mathbf{x}) &= \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{spin}}\psi(\mathbf{x}) \\ P_a(\mathbf{x}) &= \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{ppin}}\psi(\mathbf{x}), & R_{ab}(\mathbf{x}) &= \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\psi(\mathbf{x})\end{aligned}$$

with

$$\begin{aligned}\tau_x^{\text{spin}} &= \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_x \end{pmatrix} & \tau_y^{\text{spin}} &= \begin{pmatrix} \tau_y & 0 \\ 0 & \tau_y \end{pmatrix} & \tau_z^{\text{spin}} &= \begin{pmatrix} \tau_z & 0 \\ 0 & \tau_z \end{pmatrix} \\ \tau_x^{\text{ppin}} &= \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} & \tau_y^{\text{ppin}} &= \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix} & \tau_z^{\text{ppin}} &= \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\end{aligned}$$

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Effective Hamiltonian describes SU(4) coherence with **SU(4) Skyrmions**

$$\begin{aligned}\mathcal{H}_X &= J_s^d \left(\sum [\partial_k \mathcal{S}_a]^2 + [\partial_k \mathcal{P}_a]^2 + [\partial_k \mathcal{R}_{ab}]^2 \right) \\ &\quad + 2J_s^- \left(\sum [\partial_k \mathcal{S}_a]^2 + [\partial_k \mathcal{P}_z]^2 + [\partial_k \mathcal{R}_{az}]^2 \right)\end{aligned}$$

CP^3 Skyrmions in Bilayer Quantum Hall Ferromagnets

- Charge-isospin separation $\Rightarrow CP^3$ field \mathbf{n}

$$\begin{aligned}\rho(\mathbf{x}) &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), & \mathcal{S}_a(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{spin}}\mathbf{n}(\mathbf{x}) \\ \mathcal{P}_a(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{ppin}}\mathbf{n}(\mathbf{x}), & \mathcal{R}_{ab}(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\mathbf{n}(\mathbf{x})\end{aligned}$$

CP³ Skyrmions in Bilayer Quantum Hall Ferromagnets

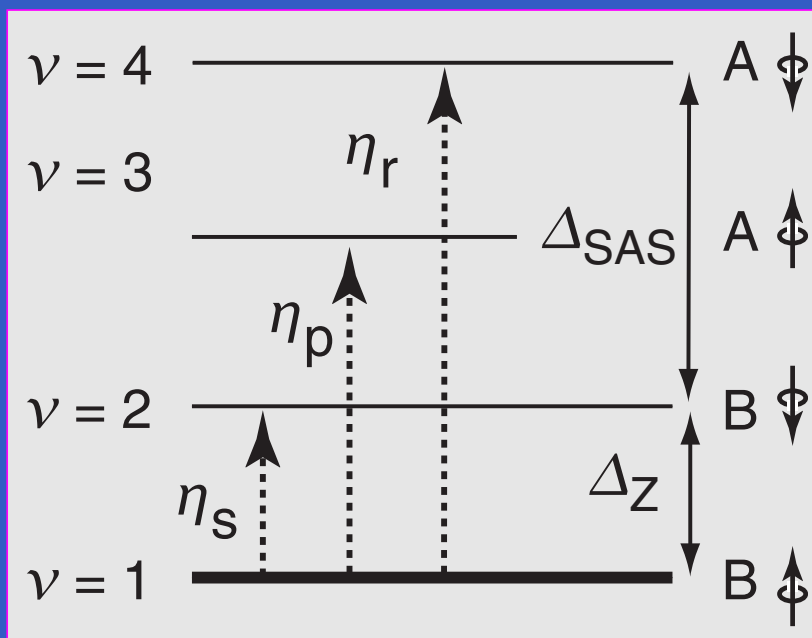
- Charge-isospin separation \Rightarrow CP³ field \mathbf{n}

$$\begin{aligned} \rho(\mathbf{x}) &= \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), & \mathcal{S}_a(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{spin}}\mathbf{n}(\mathbf{x}) \\ \mathcal{P}_a(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{ppin}}\mathbf{n}(\mathbf{x}), & \mathcal{R}_{ab}(\mathbf{x}) &= \frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\tau_a^{\text{spin}}\tau_b^{\text{ppin}}\mathbf{n}(\mathbf{x}) \end{aligned}$$

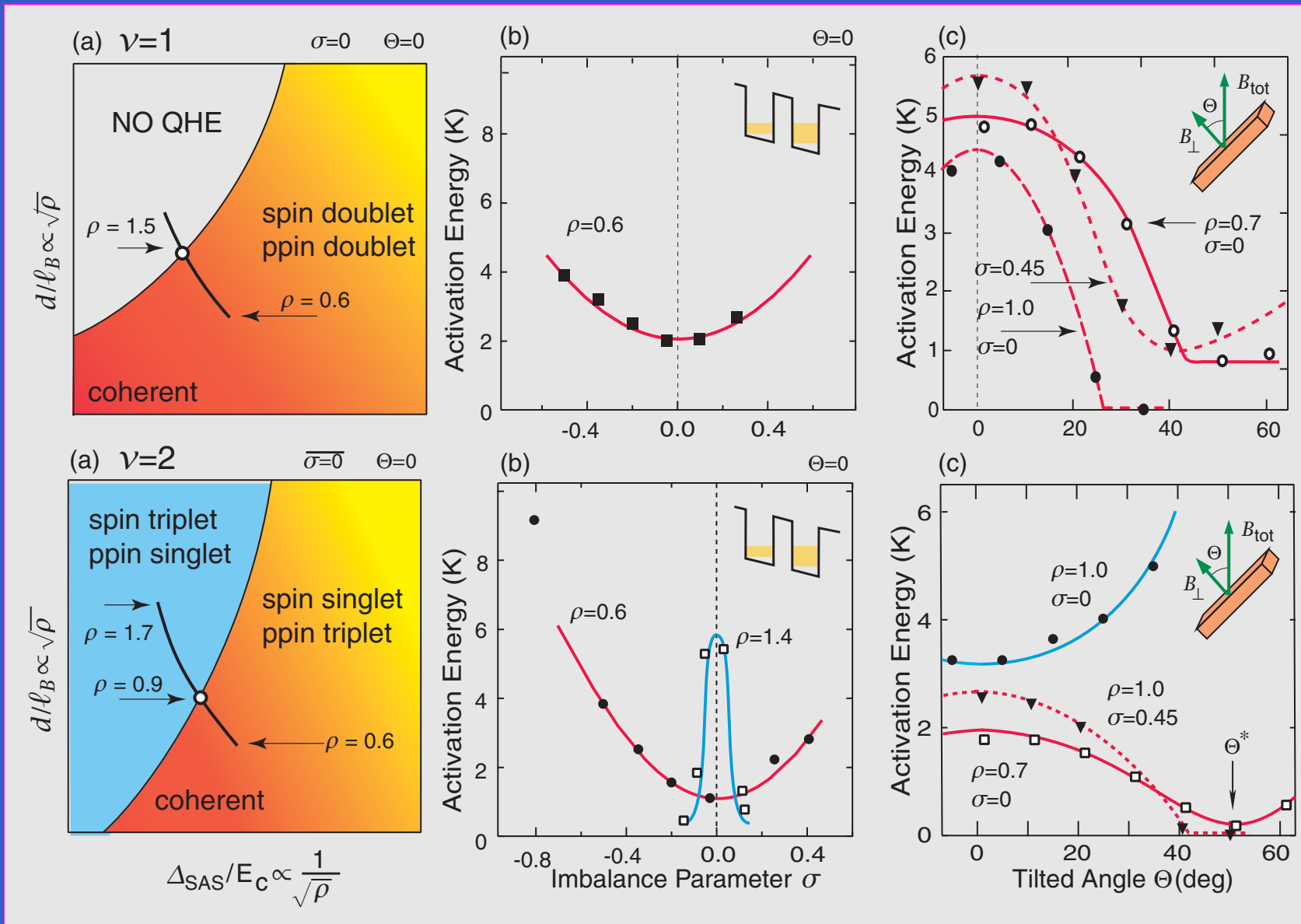
- CP³ Skyrmions

$$\begin{pmatrix} n^{A\downarrow} \\ n^{A\uparrow} \\ n^{B\downarrow} \\ n^{B\uparrow} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_g \Rightarrow \begin{pmatrix} K_r \\ K_p \\ K_s \\ Z \end{pmatrix}_{\text{sky}}$$

$$\begin{pmatrix} n^{f\uparrow}(\mathbf{x}) \\ n^{f\downarrow}(\mathbf{x}) \\ n^{b\uparrow}(\mathbf{x}) \\ n^{b\downarrow}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} Z\sqrt{1+\sigma_0} + K_p\sqrt{1-\sigma_0} \\ K_s\sqrt{1+\sigma_0} + K_r\sqrt{1-\sigma_0} \\ Z\sqrt{1-\sigma_0} - K_p\sqrt{1+\sigma_0} \\ K_s\sqrt{1-\sigma_0} - K_r\sqrt{1+\sigma_0} \end{pmatrix}$$



Experiments in Bilayer QH States ($\nu = 1 \& 2$)



✓ Sawada et al, PRL80(1998)4534

Experimental Results I ($\nu = 1$)

• Bilayer QH system

⇒ **Isospin SU(4)**

✓ Ezawa, PRL82(1999)3512

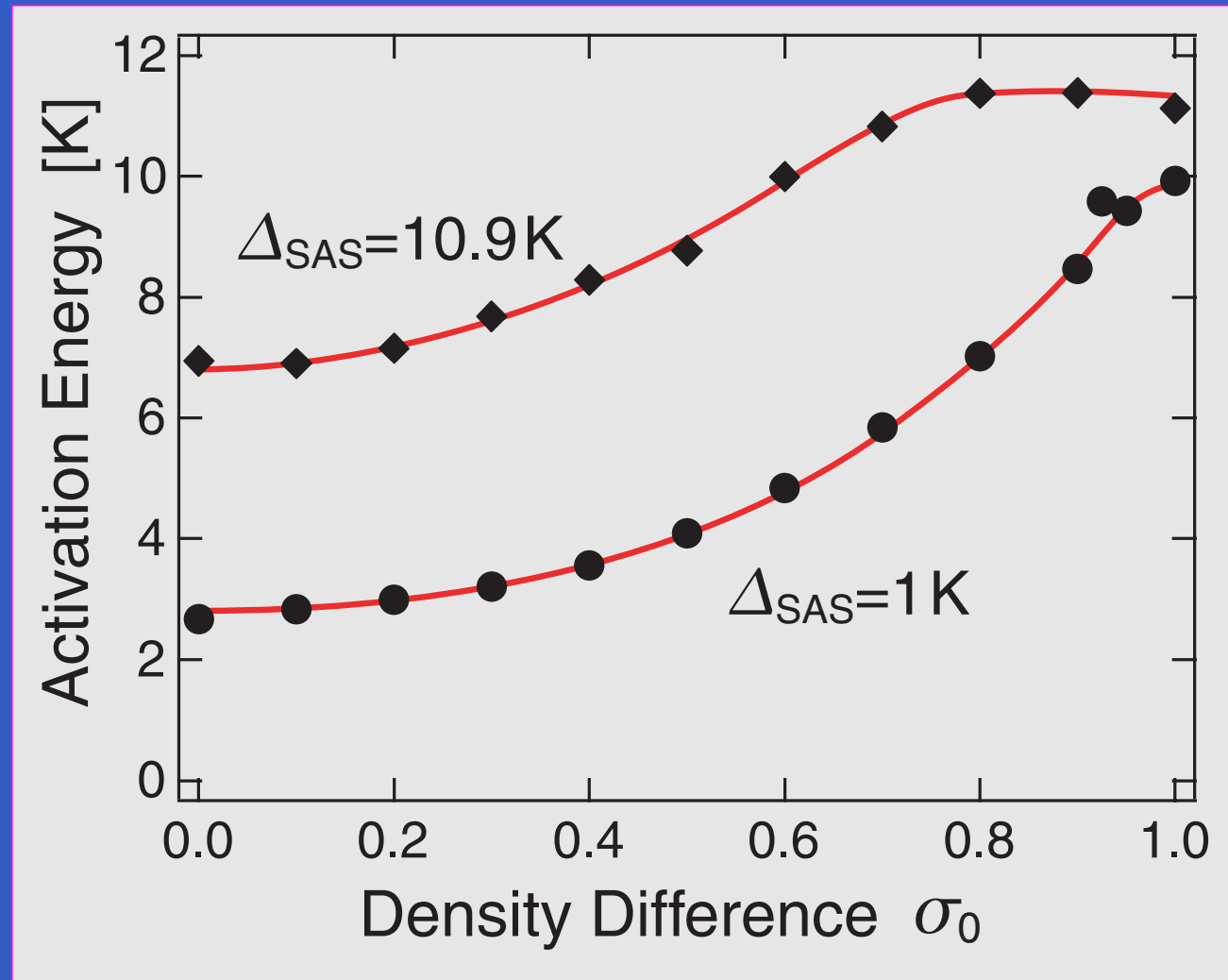
• Zeeman effect

⇒ **Spin S_z**

• Bias voltage

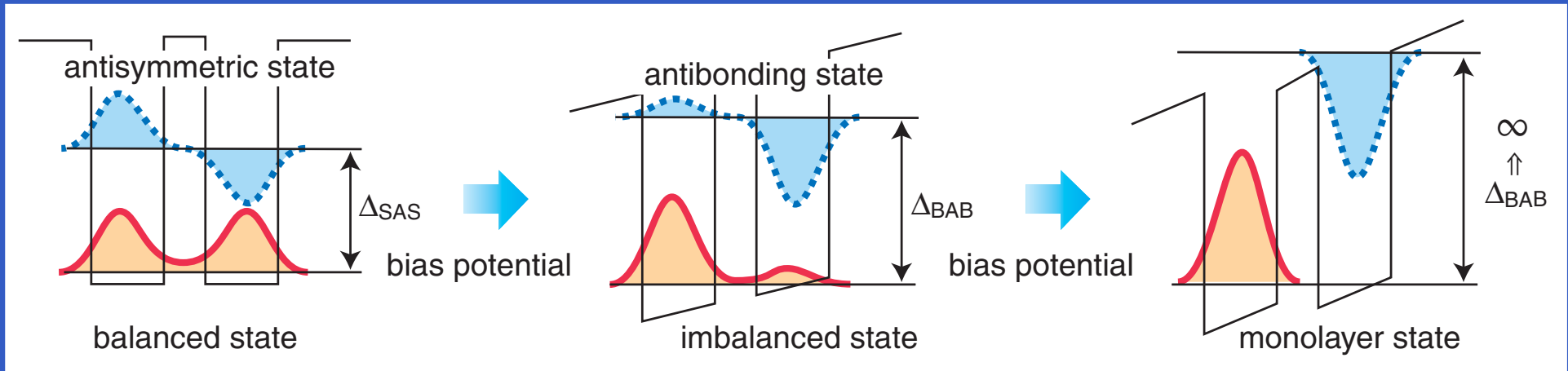
⇒ **Pseudospin \mathcal{P}_z**

Data by changing the density balance



✓ Terasawa et al. Physica E (2004)52

Bilayer to Monolayer QH Systems ($\nu = 1$)



System is controlled continuously from **balanced point** to **monolayer limit**

Pseudospin textures in bilayer \Rightarrow **Spin textures** in monolayer

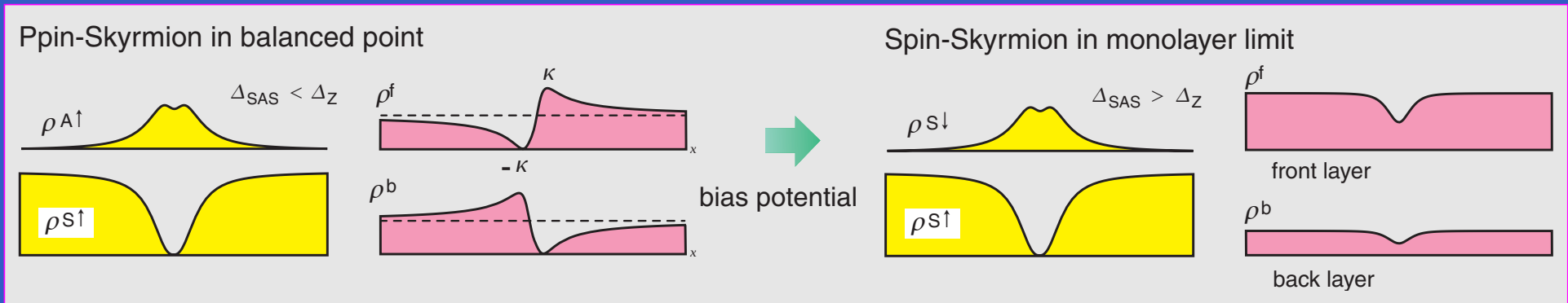
Pseudospin SU(2) \Rightarrow SU(4) \Rightarrow Spin SU(2)

Ppin CP¹ Skyrmion \Rightarrow **CP³ Skyrmion** \Rightarrow **Spin CP¹ Skyrmion**

Topological charge is the same: $\pi_2(\text{CP}^1) = \pi_2(\text{CP}^3) = \mathbb{Z}$

imbalance parameter: $\sigma_0 \equiv \frac{\rho^{\text{front}} - \rho^{\text{back}}}{\rho^{\text{front}} + \rho^{\text{back}}} = 2\mathcal{P}_z$

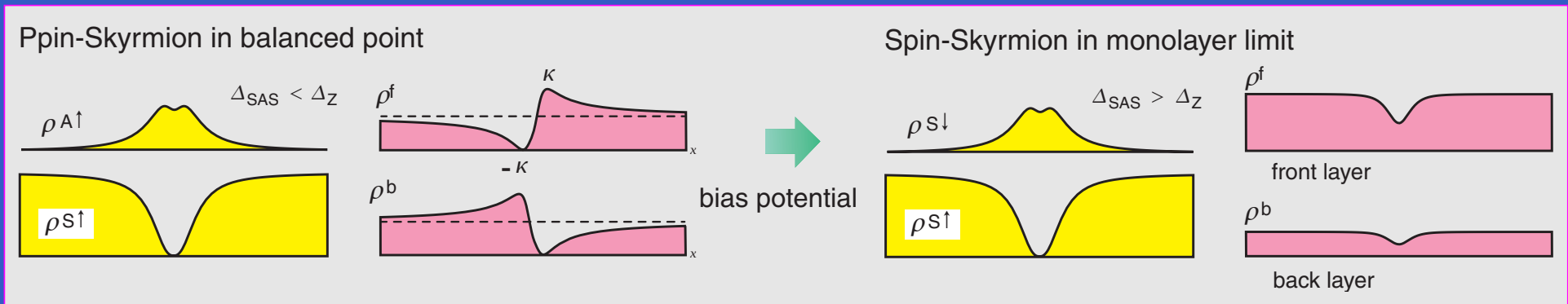
CP³ Skyrmion vs Imbalance Parameter ($\nu = 1$)



CP³ Skyrmion interpolates ppin Skyrmion to spin Skyrmion continuously

$$n_{\text{sky}}(\mathbf{x}) = \begin{pmatrix} n^{A\downarrow} \\ n^{A\uparrow} \\ n^{B\downarrow} \\ n^{B\uparrow} \end{pmatrix} = \begin{pmatrix} K_r \\ K_p \\ K_s \\ Z \end{pmatrix}, \quad \begin{pmatrix} n^{f\uparrow}(\mathbf{x}) \\ n^{f\downarrow}(\mathbf{x}) \\ n^{b\uparrow}(\mathbf{x}) \\ n^{b\downarrow}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} Z\sqrt{1+\sigma_0} + K_p\sqrt{1-\sigma_0} \\ K_s\sqrt{1+\sigma_0} + K_r\sqrt{1-\sigma_0} \\ Z\sqrt{1-\sigma_0} - K_p\sqrt{1+\sigma_0} \\ K_s\sqrt{1-\sigma_0} - K_r\sqrt{1+\sigma_0} \end{pmatrix}$$

CP³ Skyrmion vs Imbalance Parameter ($\nu = 1$)

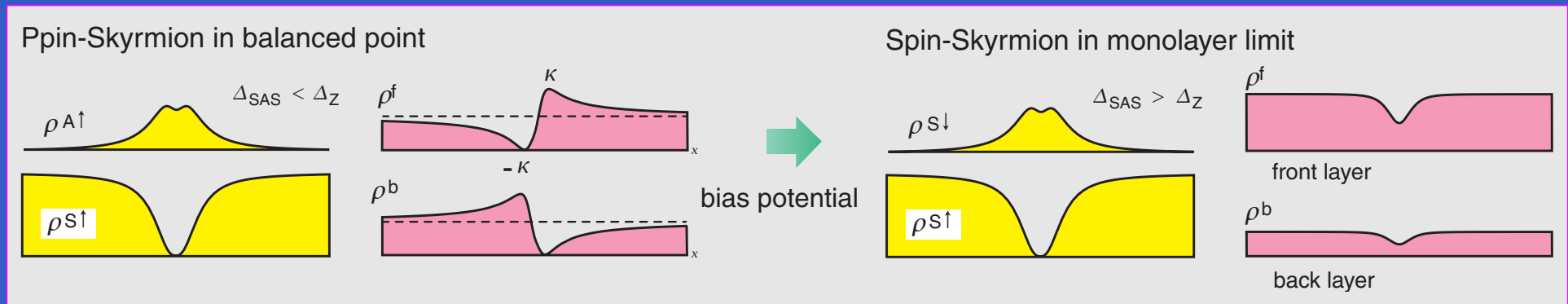


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ppin CP¹ Skyrmion ($K_{\text{spin}} = K_{\text{res}} = 0; K_{\text{ppin}} \neq 0$) is excited at balanced point

CP³ Skyrmion vs Imbalance Parameter ($\nu = 1$)



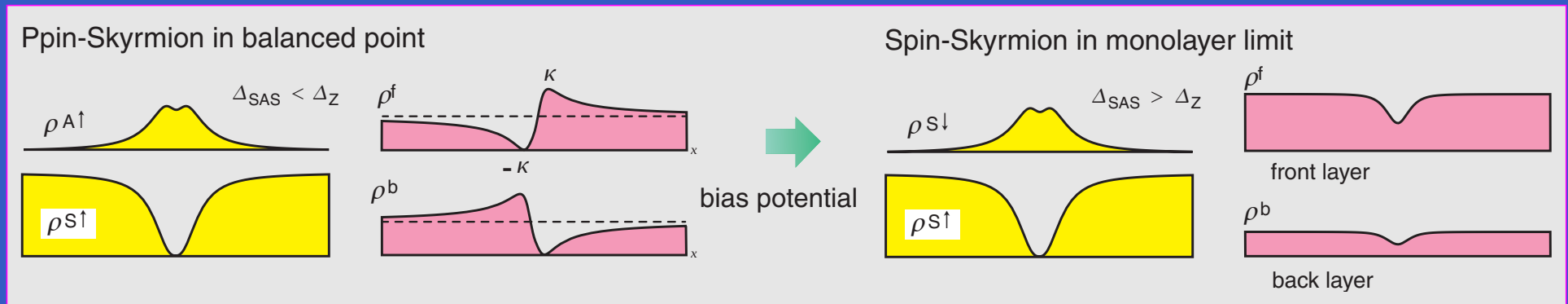
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isospin CP³ Skyrmion ($K_{\text{spin}} \neq 0; K_{\text{ppin}} \neq 0; K_{\text{res}} \simeq 0$) is excited in general

CP³ Skyrmion vs Imbalance Parameter ($\nu = 1$)



CP³ Skyrmion interpolates ppin Skyrmion to spin Skyrmion continuously

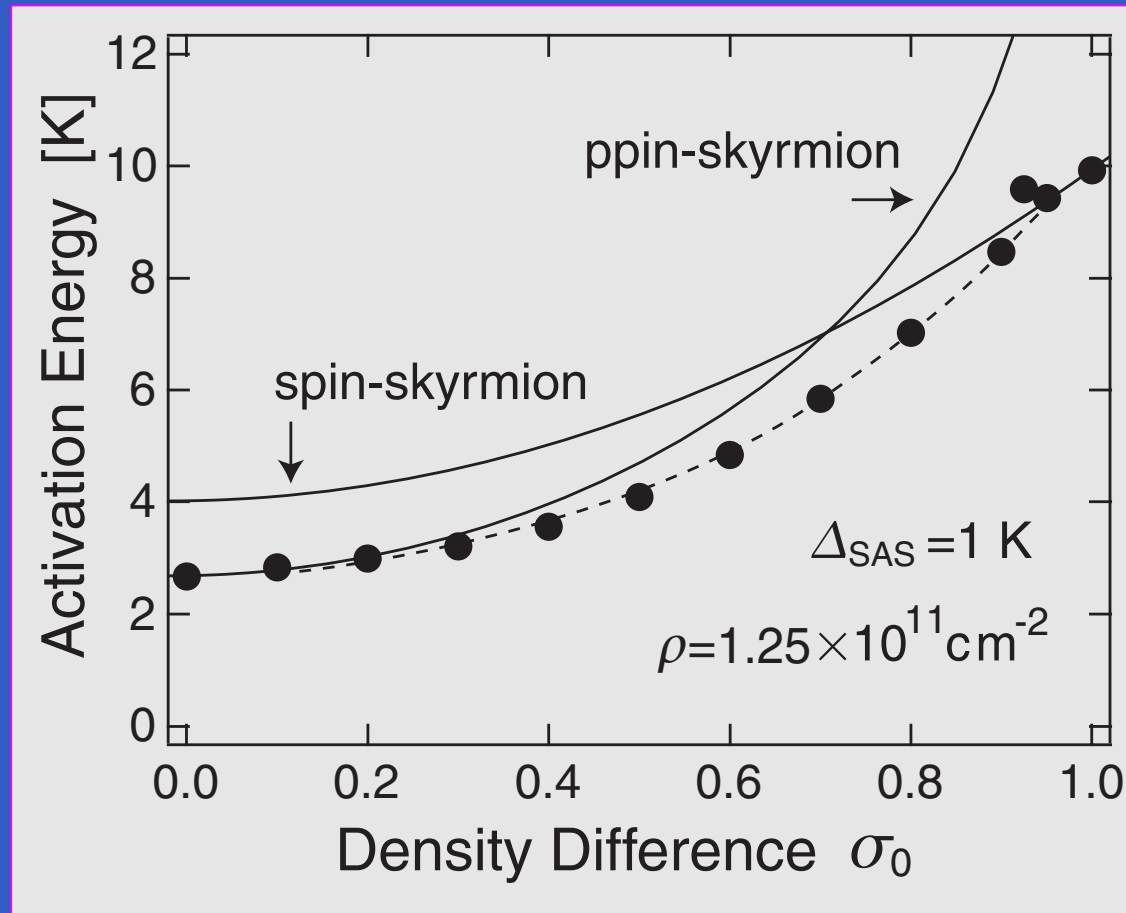
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isospin CP³ Skyrmion ($K_{\text{spin}} \neq 0; K_{\text{ppin}} \neq 0; K_{\text{res}} \simeq 0$) is excited in general

spin CP¹ Skyrmion ($K_{\text{ppin}} = K_{\text{res}} = 0; K_{\text{spin}} \neq 0$) is excited in monolayer limit

Activation Energy vis Density Difference ($\nu = 1$)



- ppin CP^1 Skyrmion ($k_{\text{spin}} = k_{\text{res}} = 0; k_{\text{ppin}} \neq 0$) is excited at balanced point
- isospin CP^3 Skyrmion ($k_{\text{spin}} \neq 0; k_{\text{ppin}} \neq 0; k_{\text{res}} \simeq 0$) is excited in general
- spin CP^1 Skyrmion ($k_{\text{ppin}} = k_{\text{res}} = 0; k_{\text{spin}} \neq 0$) is excited in monolayer limit

Experimental Results II ($\nu = 1$)

● Bilayer QH system

⇒ **Isospin SU(4)**

✓ Ezawa, PRL82(1999)3512

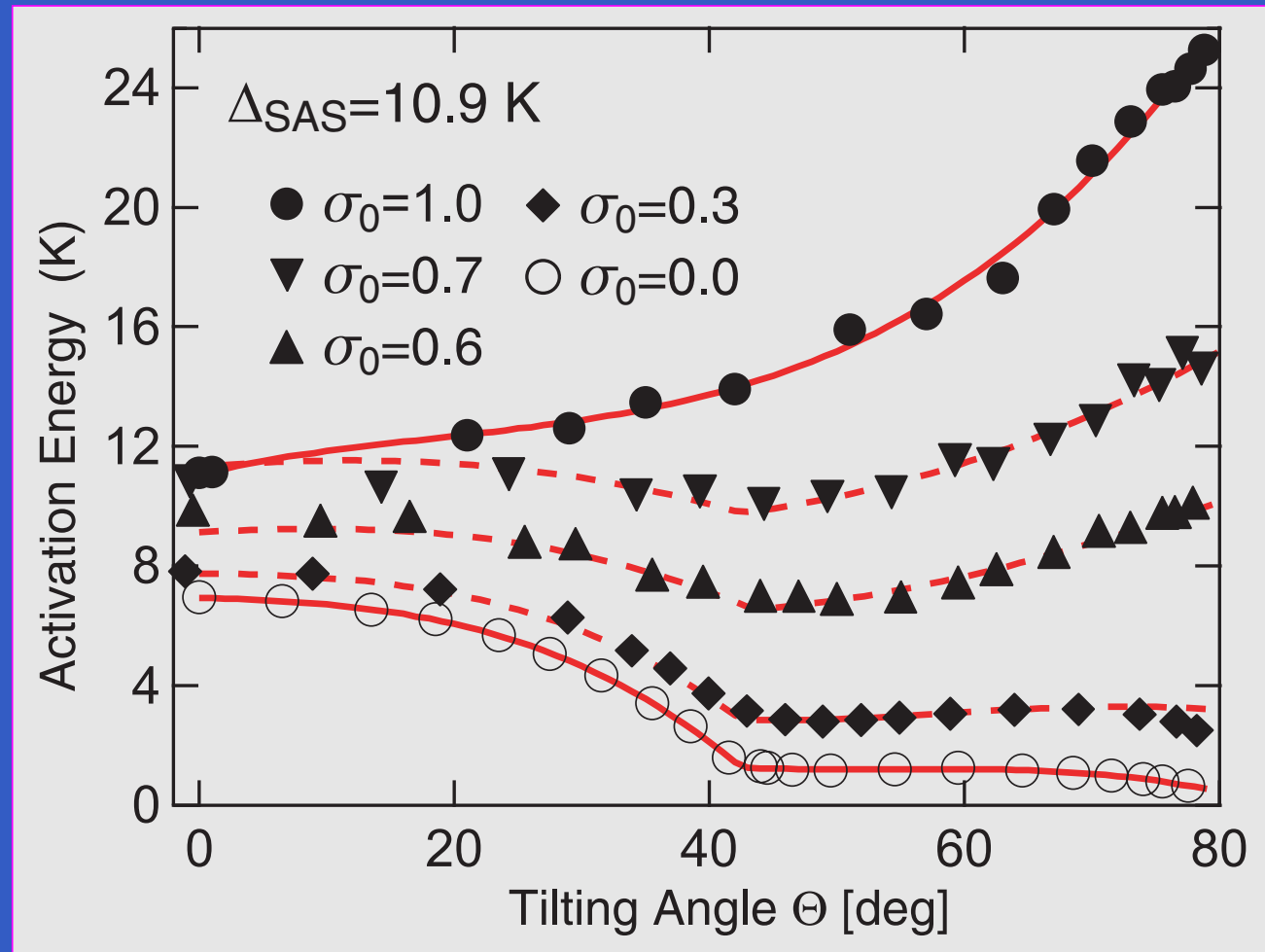
● Zeeman effect

⇒ **Spin S_z**

● Bias voltage

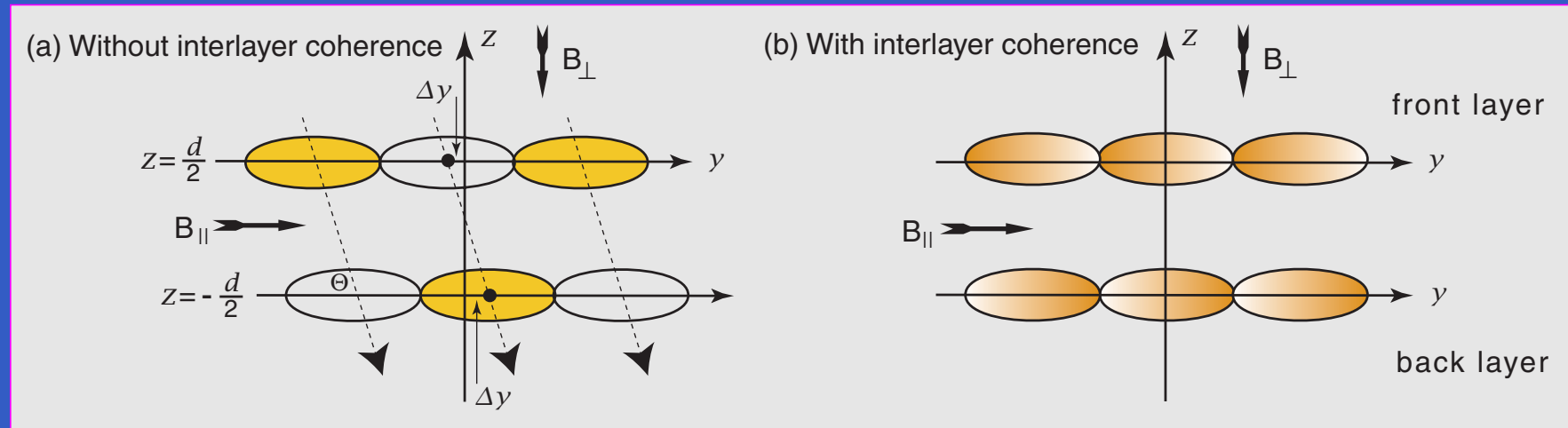
⇒ **Pseudospin \mathcal{P}_z**

Data by applying the parallel field



✓ Terasawa et al. Physica E (2004)52

Parallel Magnetic Field in Bilayer System



• The effect of parallel magnetic field

$$\psi^\alpha(\mathbf{x}; B_{\parallel}) = \exp \left[\mp (y - \bar{y}_k^0) \delta_m - \frac{1}{8} \delta_m^2 \ell_B^2 \right] \psi^\alpha(\mathbf{x}) \quad \text{without coherence}$$

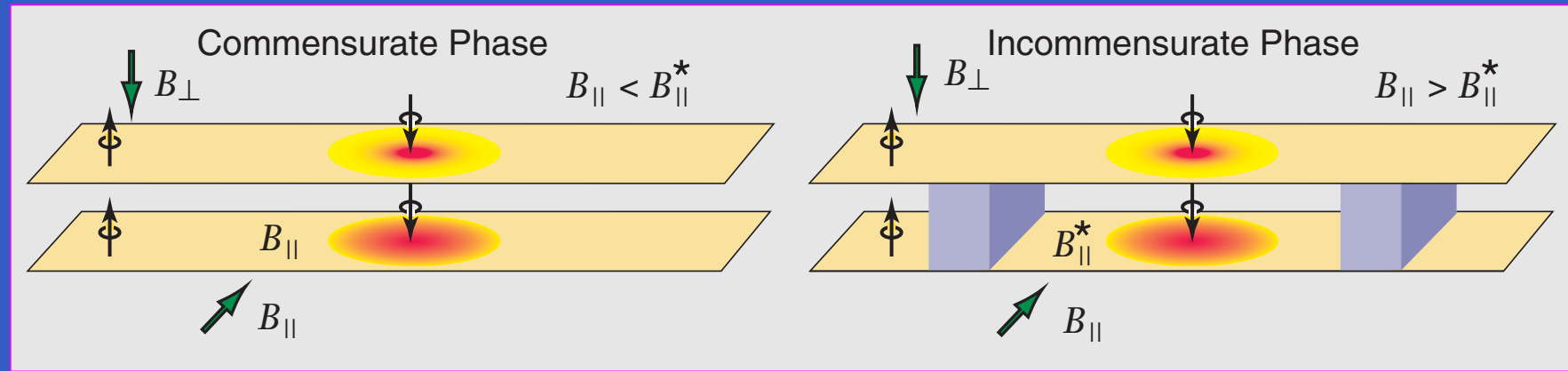
$$\psi^\alpha(\mathbf{x}; B_{\parallel}) = \exp \left(\mp \frac{i}{2} \delta_m x \right) \psi^\alpha(\mathbf{x}) \quad \text{with coherence}$$

• **CP³ Skyrmion** is modified only by phase in coherent phase

$$n_{\text{sky}}(\mathbf{x}; B_{\parallel}) = \exp \left(\mp \frac{i}{2} \delta_m x \right) n_{\text{sky}}(\mathbf{x}),$$

$$\delta_m = \frac{edB_{\parallel}}{\hbar}$$

Activation Energy of CP³ Skyrmions ($\nu = 1$)



Excitations require the exchange, Coulomb, Zeeman and tunneling energies

$$E_{\text{sky}} = E_X^{\Theta=0} + E_{\text{self}}^+(\kappa) + \frac{1}{2} \varepsilon_{\text{cap}} (1 - \sigma_0^2) N_{\text{ppin}}(\kappa_p) + N_{\text{spin}}(\kappa_s) \Delta_Z^{\Theta} + N_{\text{ppin}}(\kappa_p) \Delta_{\text{SAS}}^{\Theta}$$

$$\Delta_Z^{\Theta} = g^* \mu_B B_{\perp} \sqrt{1 + \tan^2 \Theta}, \quad N_{\text{spin}}(\kappa_s) \propto \kappa_s^2, \quad N_{\text{ppin}}(\kappa_p) \propto \kappa_p^2, \quad \kappa^2 \equiv \kappa_s^2 + \kappa_p^2$$

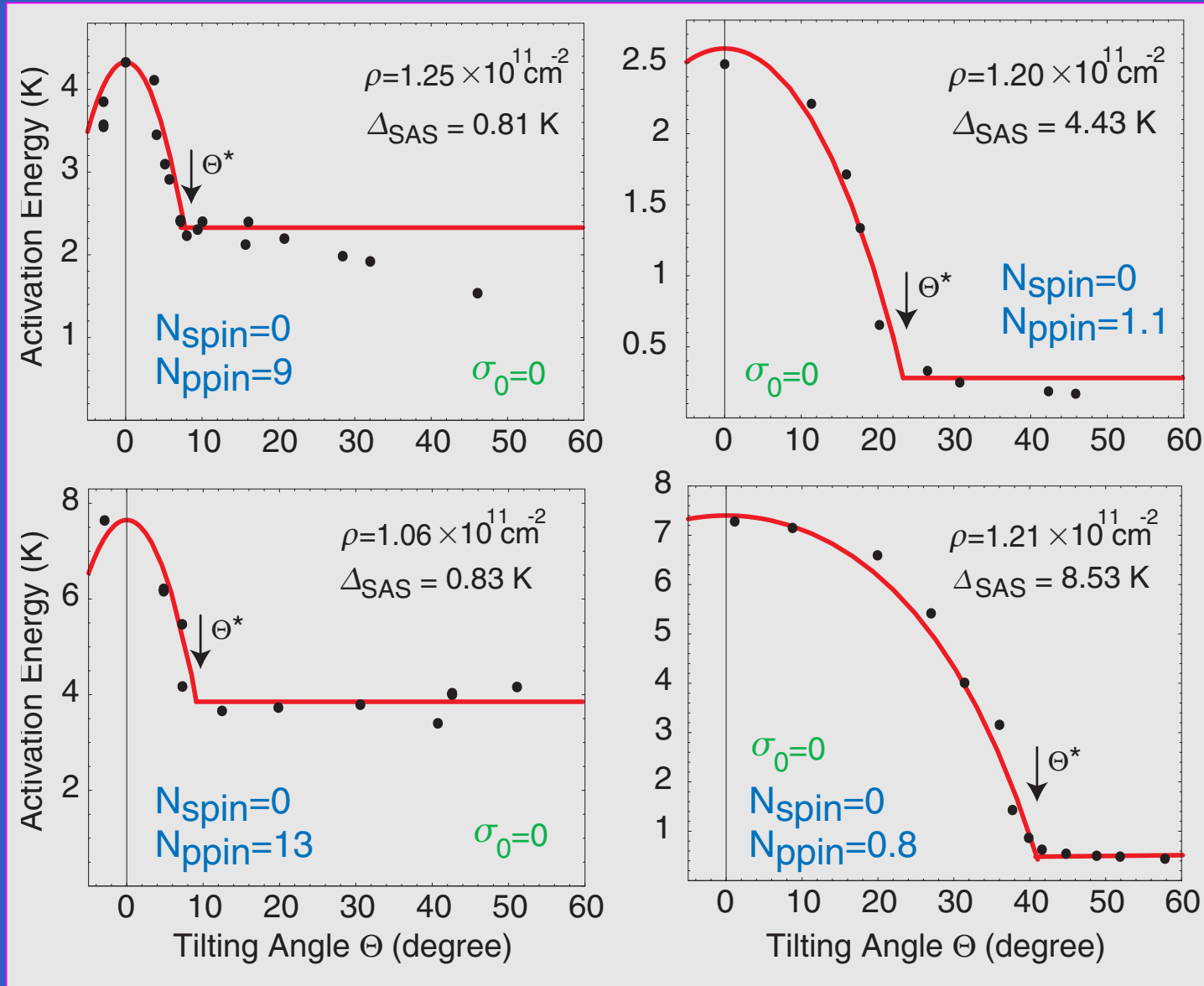
$$\Delta_{\text{SAS}}^{\Theta} = \begin{cases} \frac{1}{\sqrt{1-\sigma_0^2}} \Delta_{\text{SAS}} - \frac{2\pi d^2 J_s^d}{\ell_B^2} (1 - \sigma_0^2) \tan^2 \Theta & \text{for } \Theta < \Theta^* \\ \frac{1}{\sqrt{1-\sigma_0^2}} \Delta_{\text{SAS}} - \frac{2\pi d^2 J_s^d}{\ell_B^2} (1 - \sigma_0^2) \tan^2 \Theta^* & \text{for } \Theta > \Theta^* \end{cases}$$

where Θ^* is commensurate-incommensurate transition point, $\tan \Theta^* = B_{\parallel}^* / B_{\perp}$

✓Ezawa et al. PRB70(2004)125304

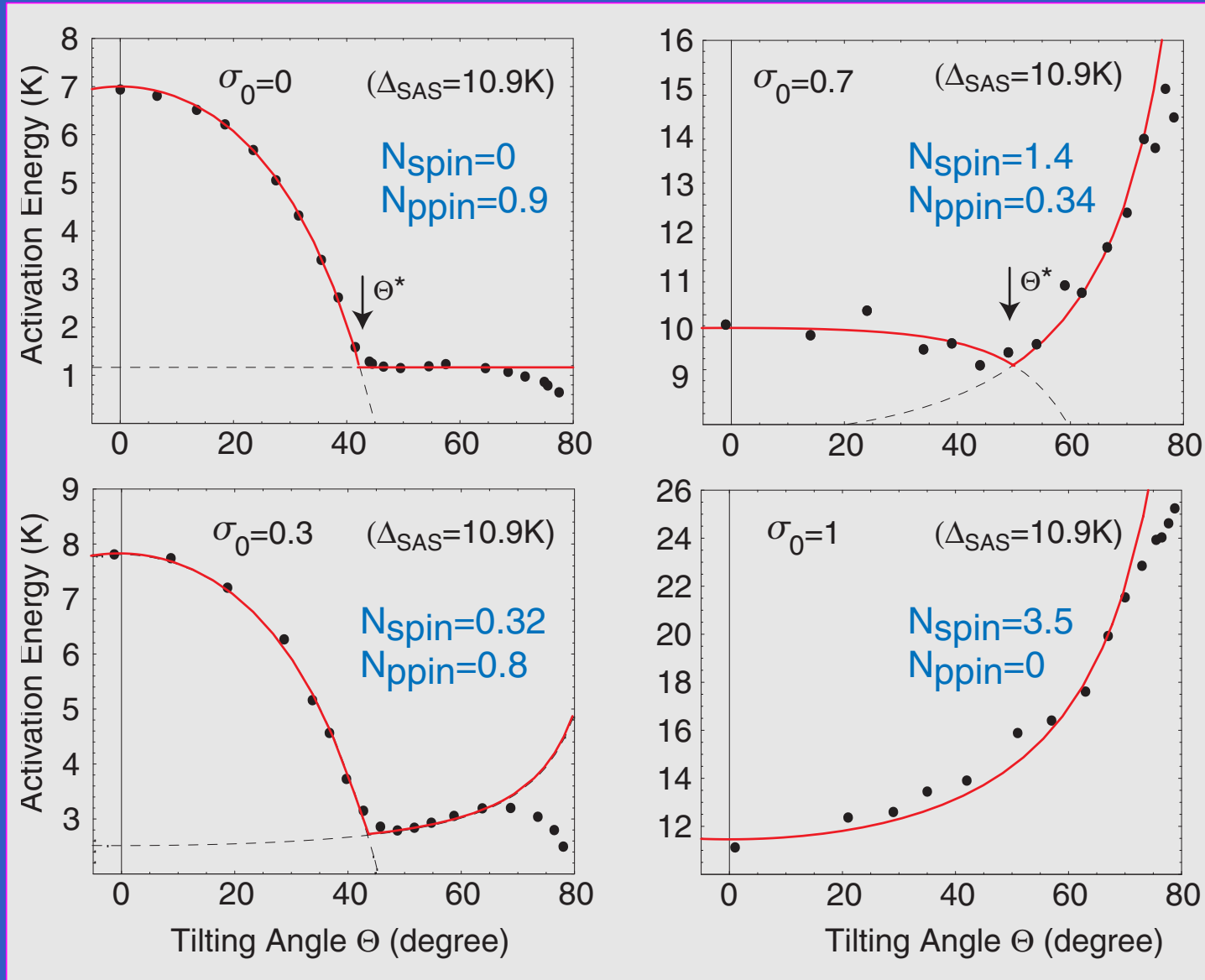
KIAS, Seoul (2004.10.25-27) – p.42/52

Excitation of Ppin CP¹ Skyrmions ($\nu = 1$)



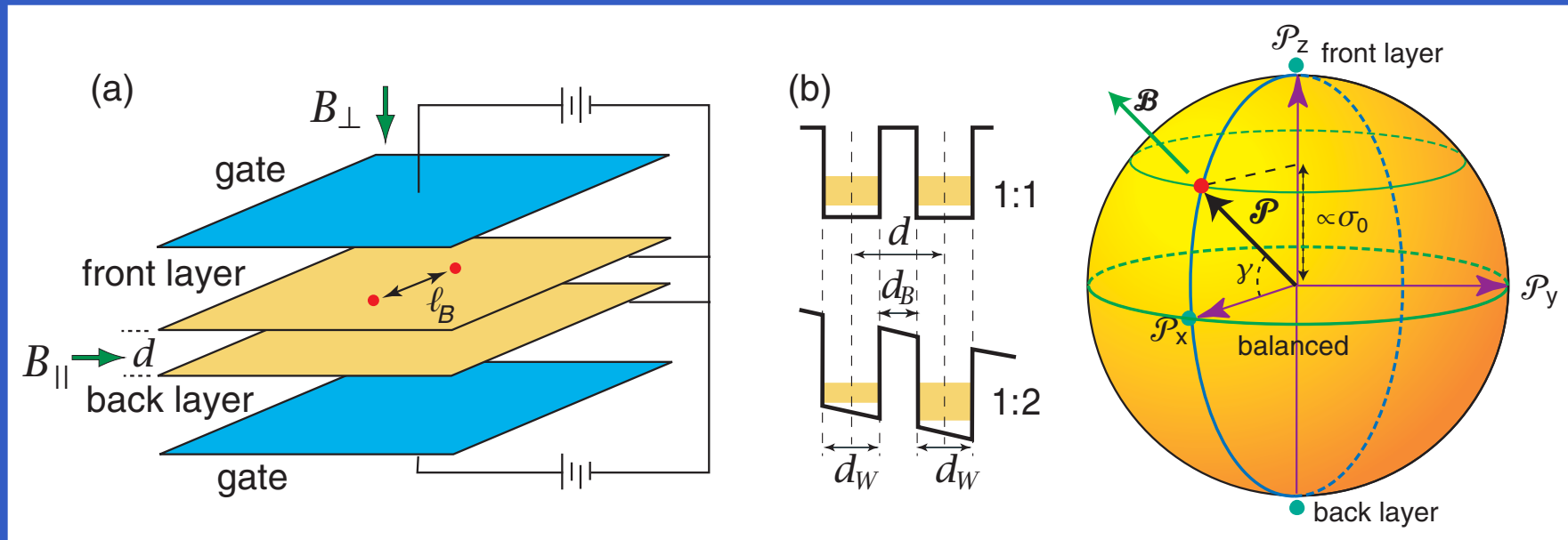
Murphy et al., PRL72(1994)728

Excitation of CP^3 Skyrmions ($\nu = 1$)

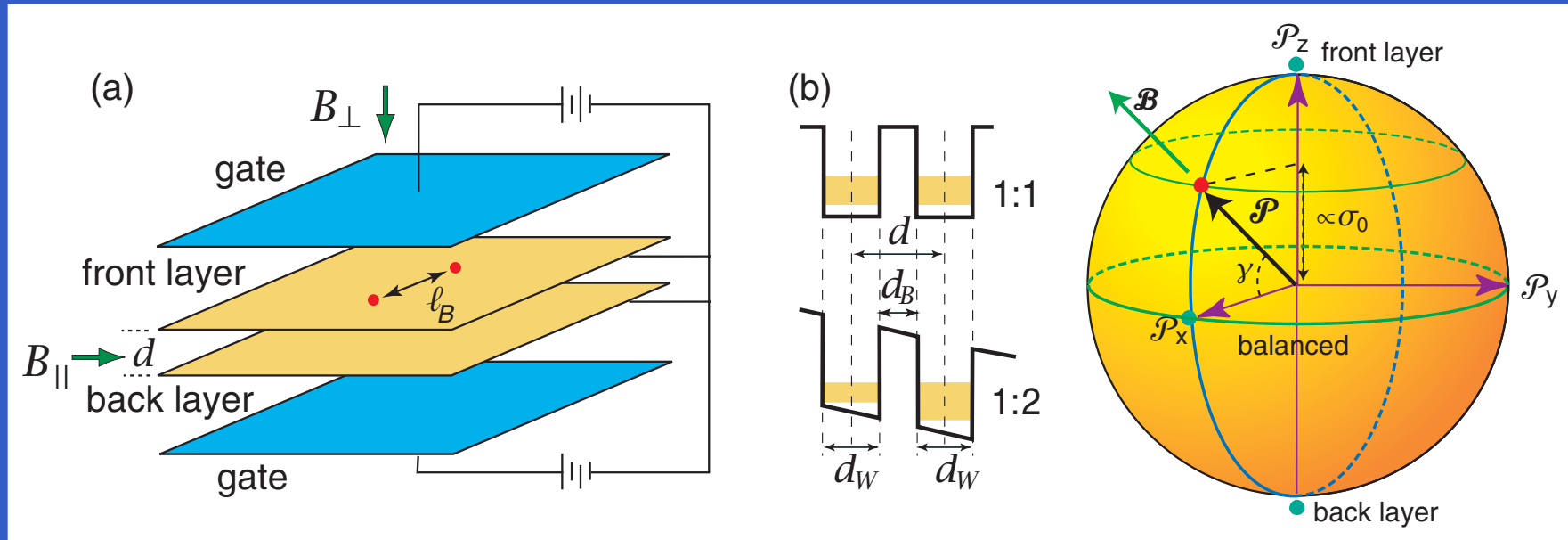


✓Terasawa et al. Physica E (2004)52

Bilayer QH States ($\nu = 2$)

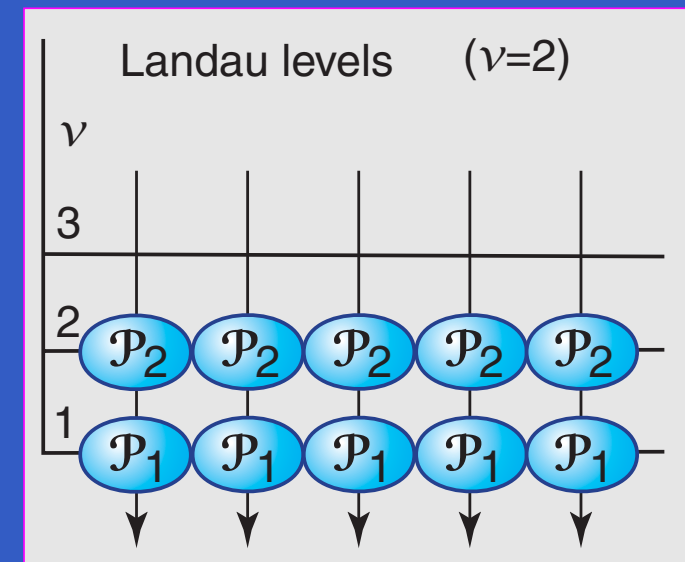


Bilayer QH States ($\nu = 2$)

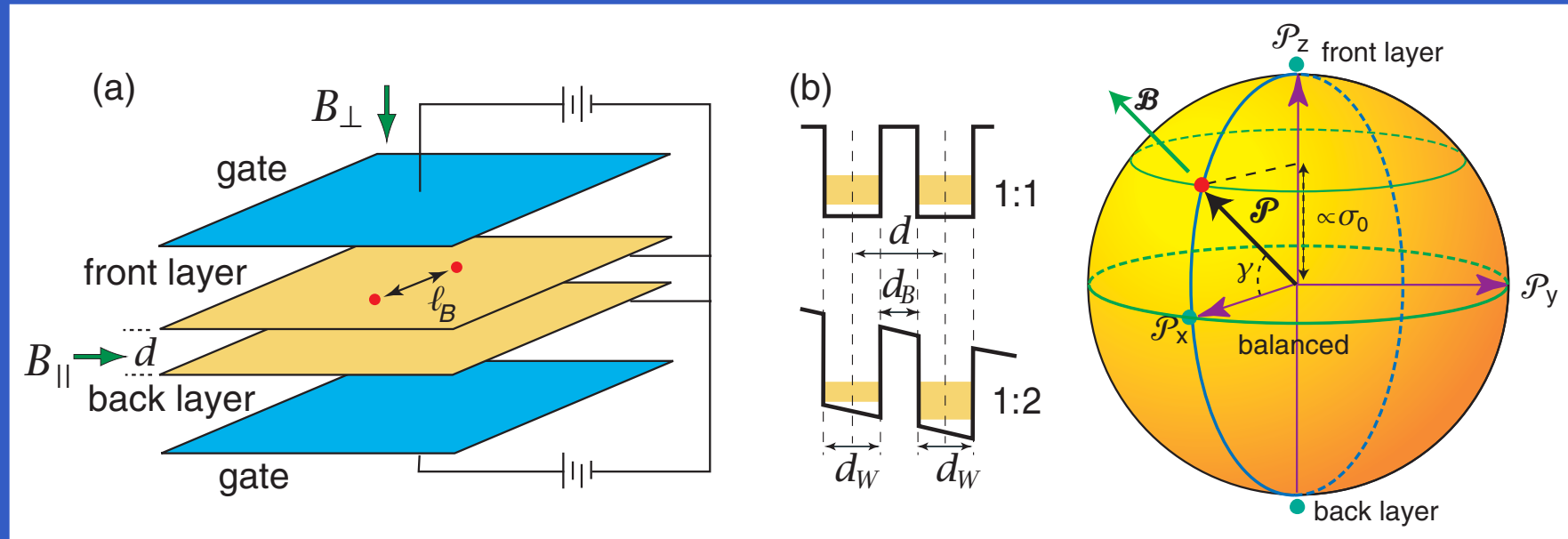


Composition of two spins (two pseudospins)

$$\nu = 2: \quad 2 \otimes 2 = 1 \oplus 3$$



Bilayer QH States ($\nu = 2$)

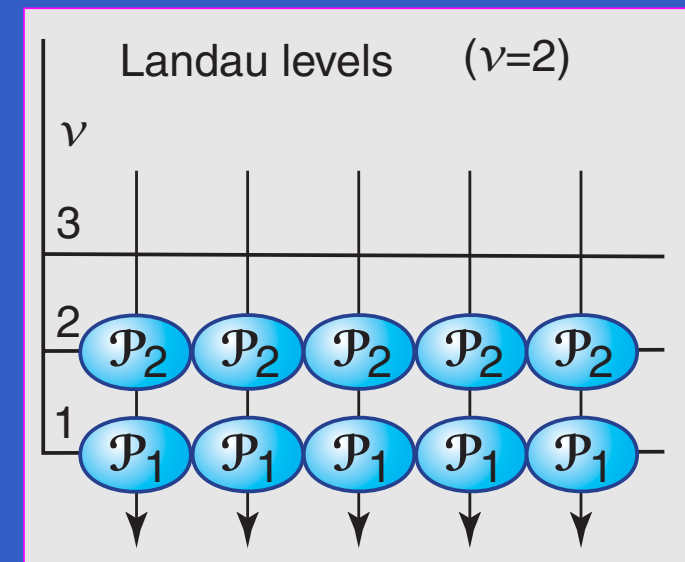


- Composition of two spins (two pseudospins)

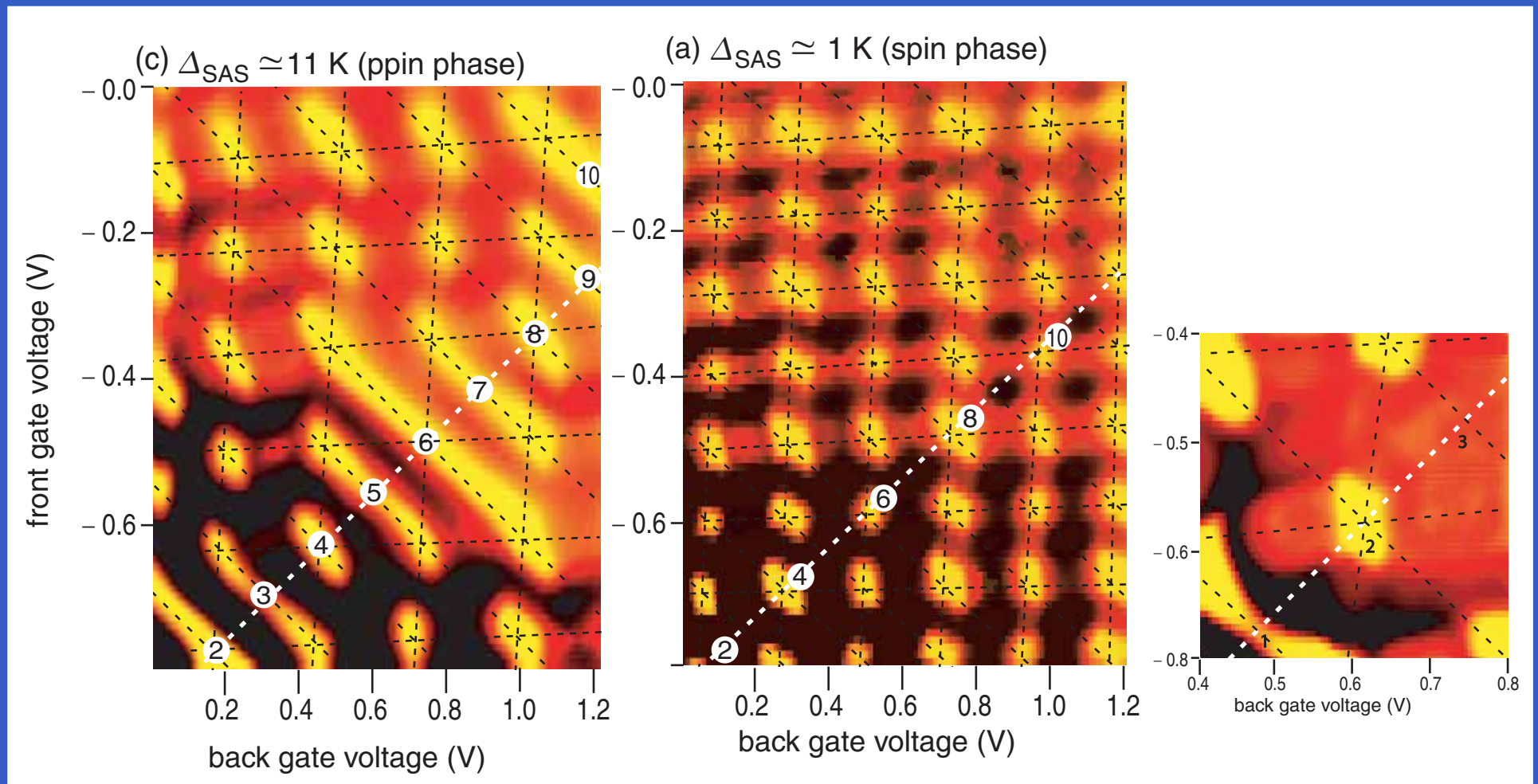
$$\nu = 2: \quad 2 \otimes 2 = 1 \oplus 3$$

- Two types of QH states

- spin-singlet and ppin-triplet **ppin-phase**
- spin-triplet and ppin-singlet **spin-phase**

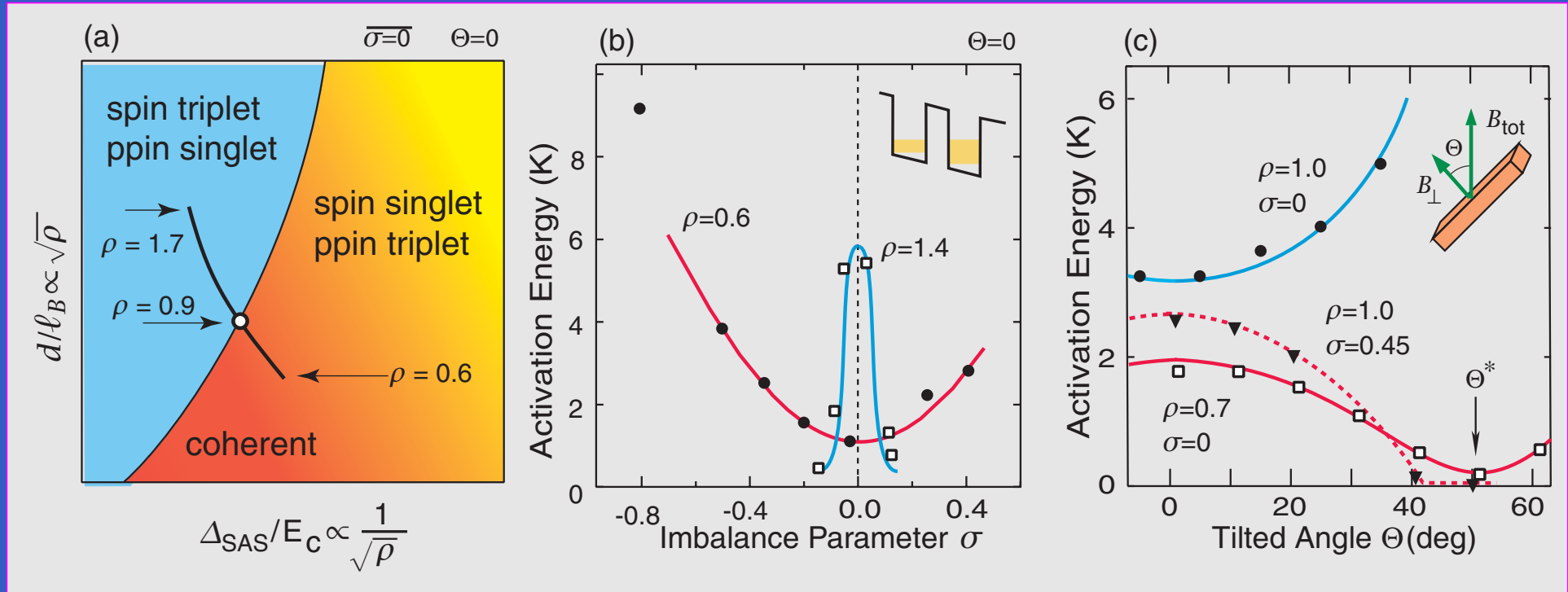


Bilayer QH States ($\nu = 2$)



✓ Muraki et al, SSC 112(1999)625

Bilayer QH States ($\nu = 2$)



Two types of QH states

spin-singlet and ppin-triplet **ppin-phase**

spin-triplet and ppin-singlet **spin-phase**

Is the spin-phase unique?

Is it an uncorrelated two-monolayer system or a genuine bilayer system

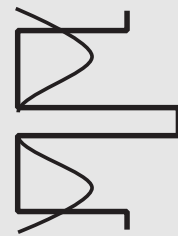
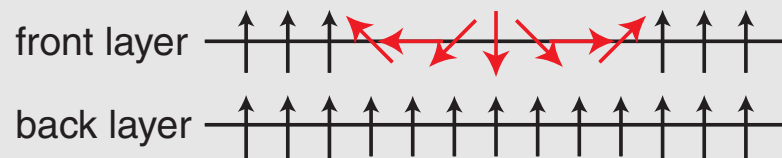
Grassmannian Soliton

- In zero tunnelling gap (two electrons distinguishable)
- Two layers behave independently \Rightarrow two CP^1 fields
- **one CP^1 Skyrmion with charge e** is excited

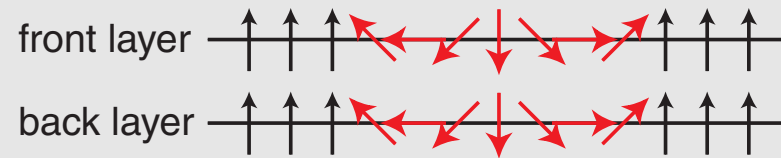
Grassmannian Soliton

- In zero tunnelling gap (two electrons distinguishable)
- Two layers behave independently \Rightarrow two CP^1 fields
- **one CP^1 Skyrmion with charge e** is excited

(a) $\Delta_{SAS}=1K$

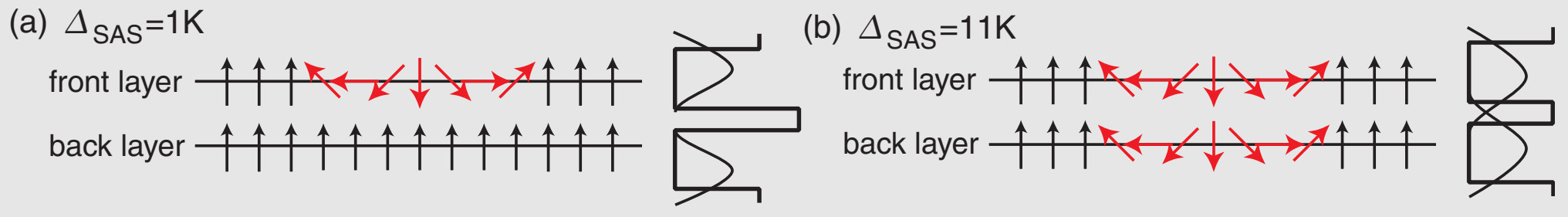


(b) $\Delta_{SAS}=11K$



Grassmannian Soliton

- In zero tunnelling gap (two electrons distinguishable)
- Two layers behave independently \Rightarrow two CP^1 fields
- one CP^1 Skyrmion with charge e** is excited

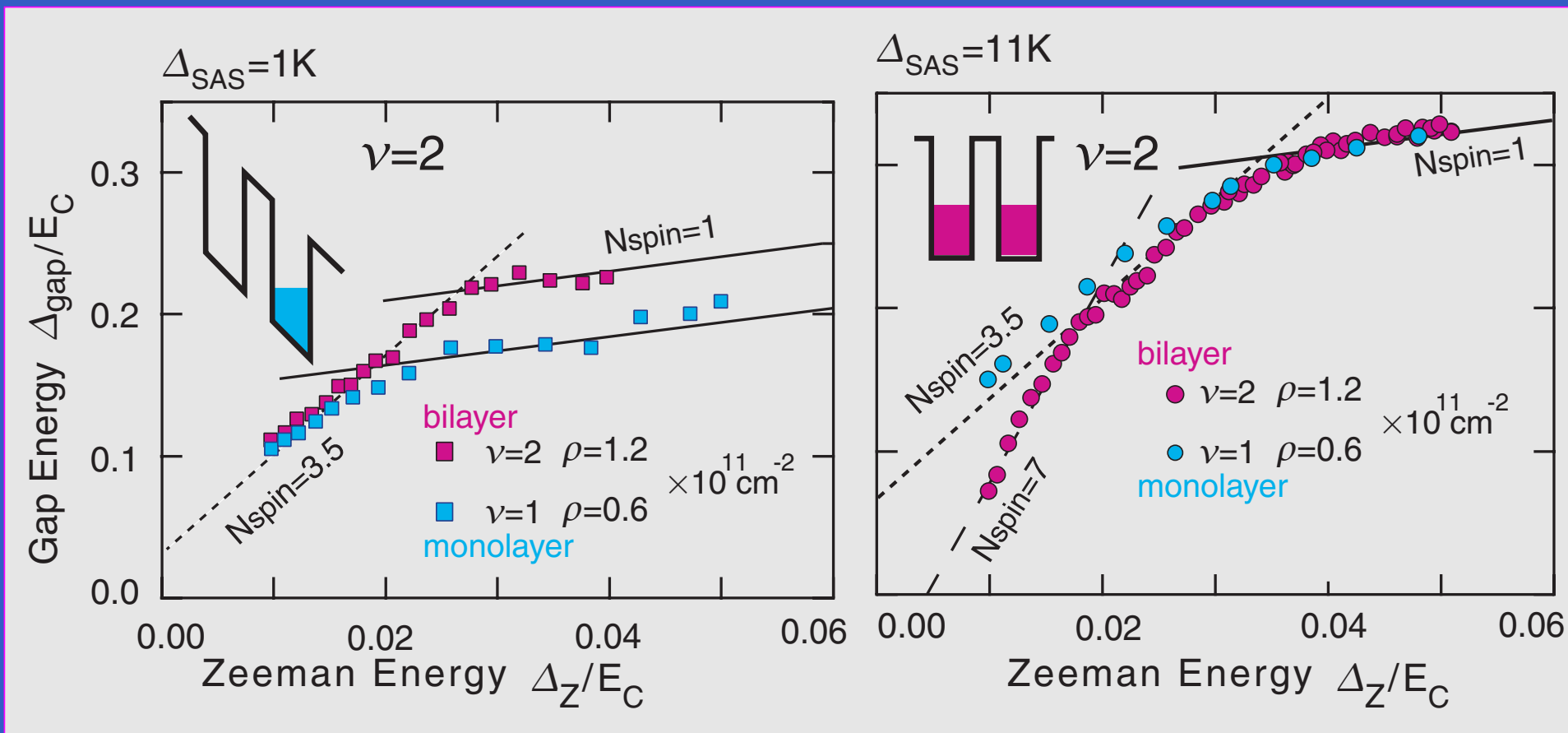


- In large tunnelling gap (two electrons indistinguishable)
- Two layers behave coherently \Rightarrow one Grassmannian $G^{4,2}$ field
- one $G^{4,2}$ Skyrmion with charge $2e$** is excited
- flipped spin number twice as much as that of CP^1 Skyrmion
- Grassmannian solitons arise based on

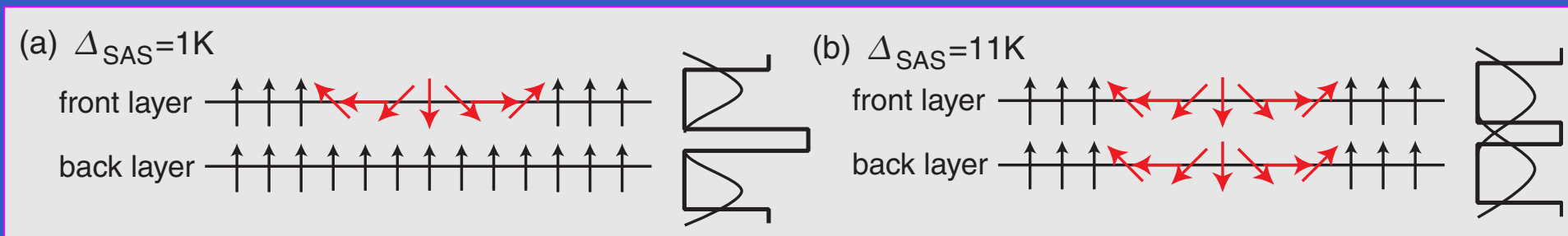
$$\pi_2(G^{N,k}) = \mathbb{Z}, \quad G^{N,k} = SU(N)/[U(1) \otimes SU(N) \otimes SU(N-k)]$$

CP¹ versus G^{4,2} Skyrmions

(experiment)

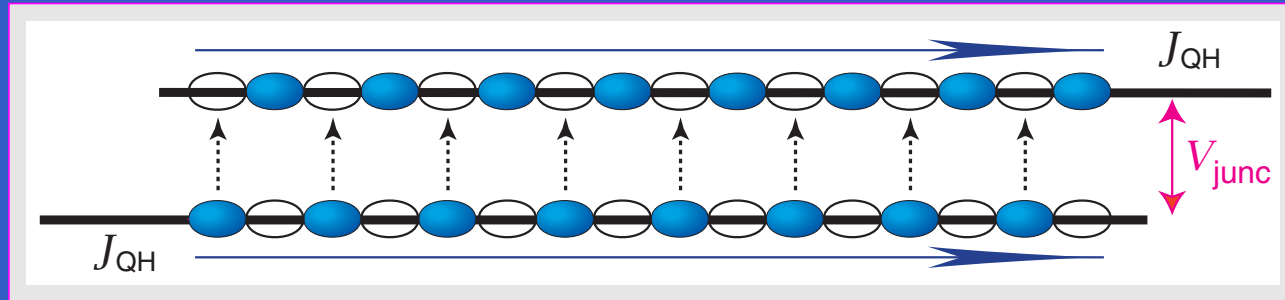


✓ Kumada et al, JPSJ69(2000)3178



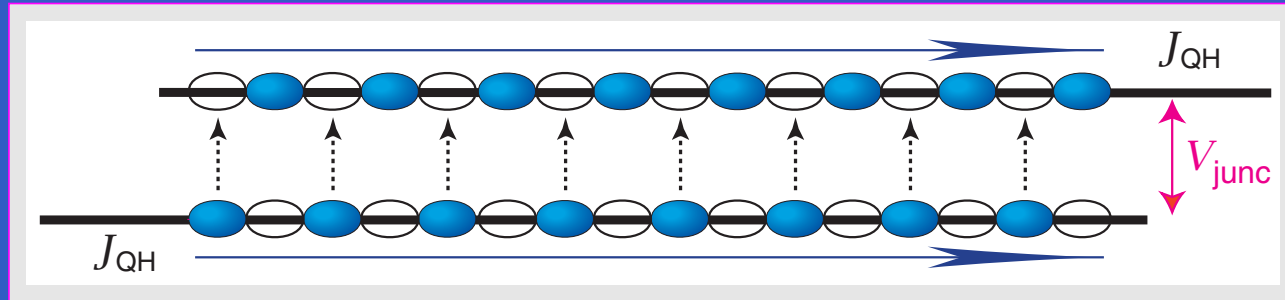
Josephson-like Effects

- **Josephson tunneling current** (predicted by Ezawa&Iwazaki, 1992)

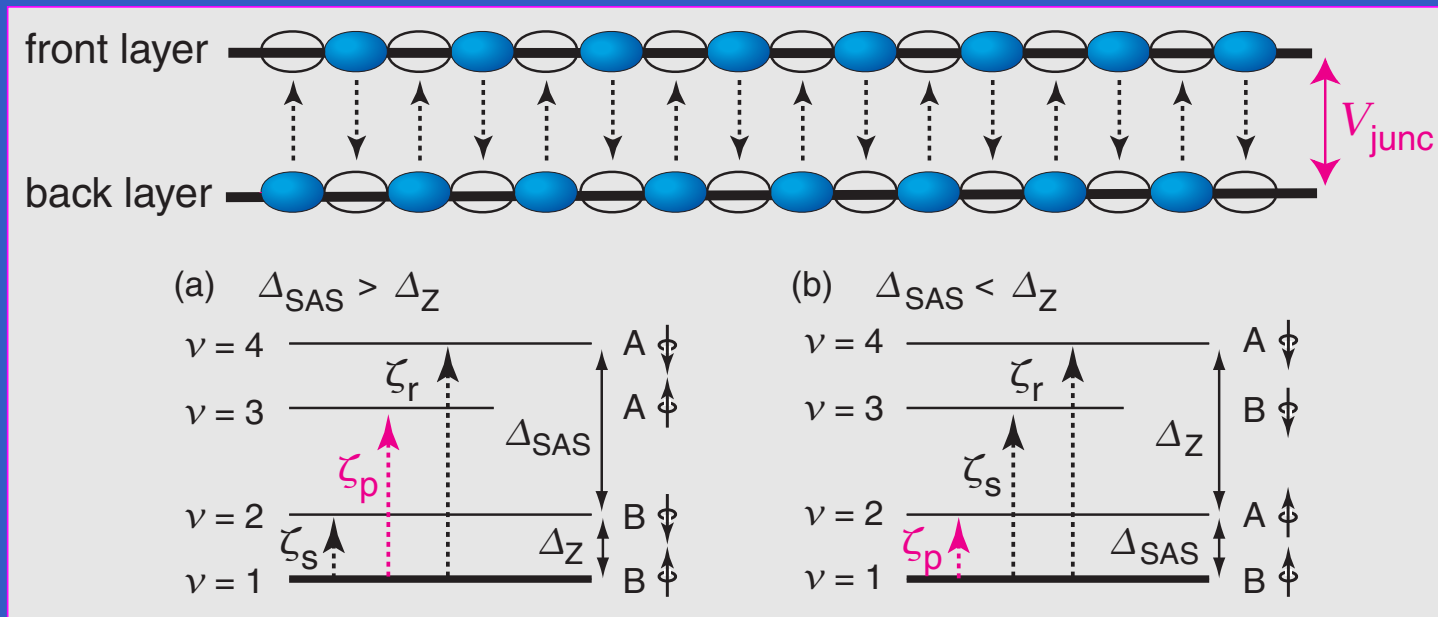


Josephson-like Effects

- Josephson tunneling current (predicted by Ezawa&Iwazaki, 1992)



- Plasmon excitations expected (detectable by microwave)



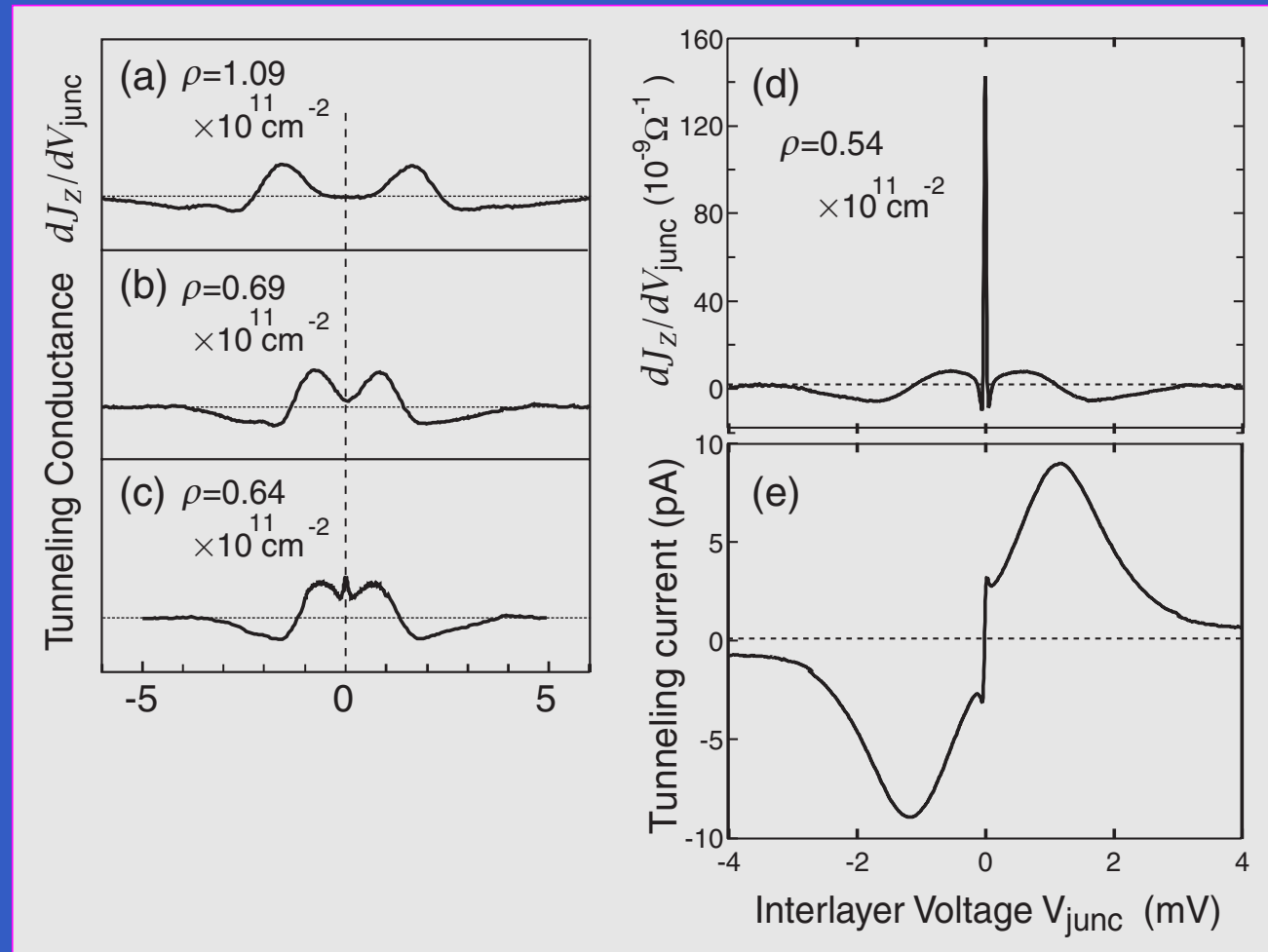
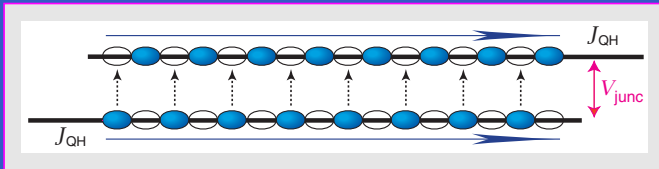
Josephson-like Tunneling (Experiments)

Layer coherence



Coherent tunneling

(Ezawa-Iwazaki, 1993)



Spielman, Eisenstein, PRL84(2000)5808;PRL87(2001)36803

Conclusions

- An ideal system realizing
 - ⇒ **noncommutative geometry**
- Noncommutative geometry
 - ⇒ **quantum coherence**
- SU(2) spin coherence
 - ⇒ **CP^1 Skyrmions**
- SU(4) isospin coherence
 - ⇒ **CP^3 and $G^{4,2}$ Skyrmions**
- Interlayer coherence
 - ⇒ **Josephson-like phenomena**
- **Condensation of single charge e**
 - ⇒ Statistical transformation

2nd Version Coming Soon !!

