Noncommutative Geometry, Effective Field Theory and Skyrmions in Quantum Hall Systems

Zyun F. Ezawa

Tohoku University

in collaboration with
A. Sawada (Kyoto University)
D. Terasawa (Tohoku University)
Y. Hirayama, N. Kumada, K. Muraki (NTT)
G. Tsitsishvili (Tbilisi Mathematical Institute)

Introduction

- Why **Quantum Hall Effects** are so interesting ?

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Noncommutative geometry $\Rrightarrow$ effective field theory $\Rrightarrow$ quantum coherence

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- Spin coherence develops spontaneously
  - Charged excitations are **CP^1 Skyrmions**

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- SU(4) quantum coherence develops in bilayer quantum Hall systems
  - Charged excitations are CP^3 at $\nu = 1$ and $G^{4,2}$ Skyrmions at $\nu = 2$
  - Interlayer coherence ⇛ Josephson phenomena
    ⇛ Single electron condensation

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- Fractional QH effects, Higher Landau levels, Edge phenomena, $\cdots\cdots$

Monolayer QH System

Two-Dimensional World

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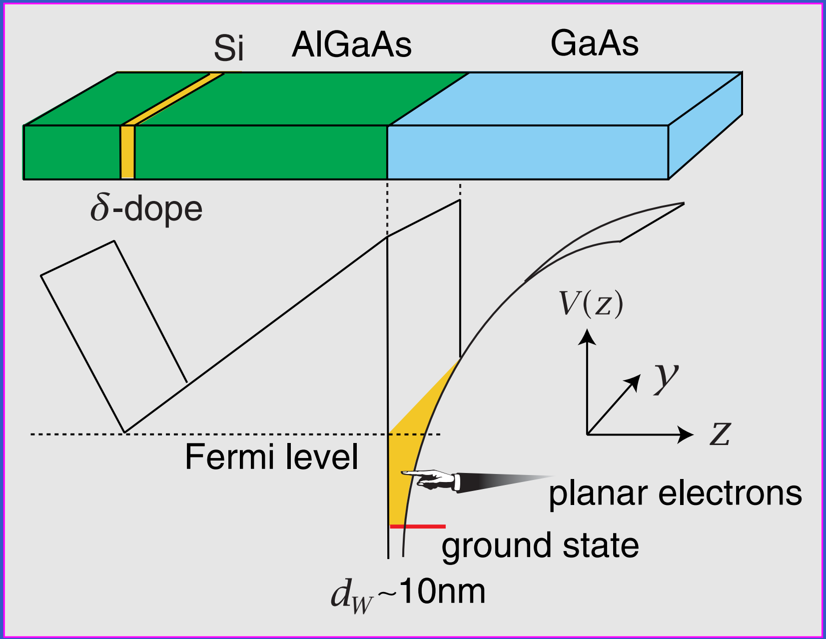
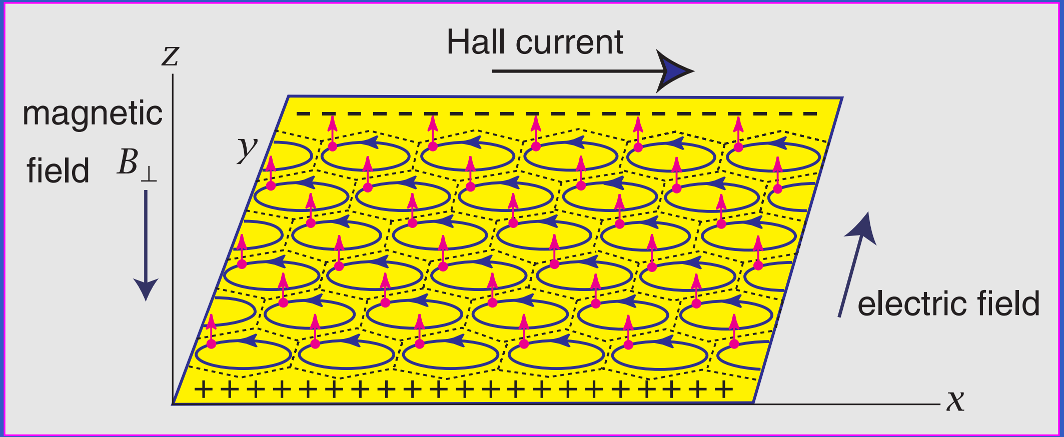
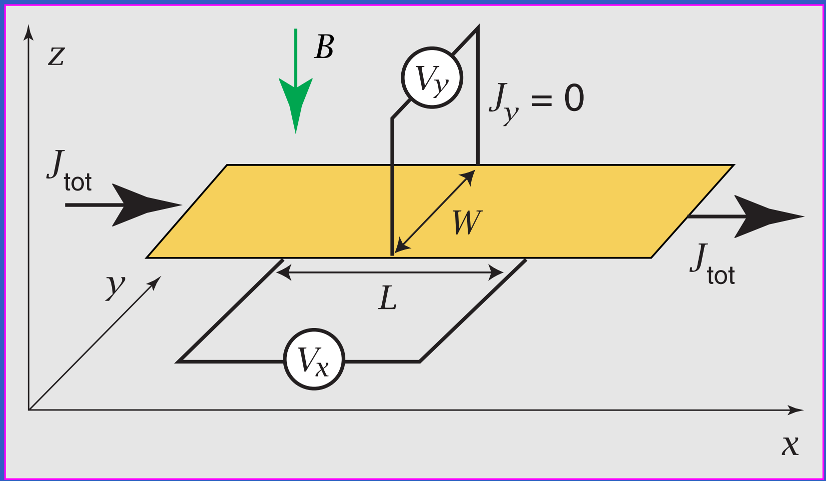
- A world of planar electrons emerges between two different semiconductors



- A electron makes cyclotron motion occupying an area $2\pi\ell_B^2$
  $\Rrightarrow$ Landau site
  - $\ell_B = \sqrt{\hbar/eB_\perp}$
  - Hall current by Lorentz force



Classical and Quantum Hall Effects



- Diagonal resistivity

$$R_{xx} = -\frac{W}{L}\frac{V_x}{J_{\text{total}}} = 0$$

- Hall resistivity

$$R_{xy} = \frac{V_y}{J_{\text{total}}} = \frac{1}{\nu}\frac{2\pi\hbar}{e^2} \propto B_\perp$$

with $\nu = \frac{2\pi\hbar\rho_0}{eB_\perp} \equiv \frac{N}{N_\Phi} \Rightarrow \frac{n}{m}$ (quantized: this is the QH effect)

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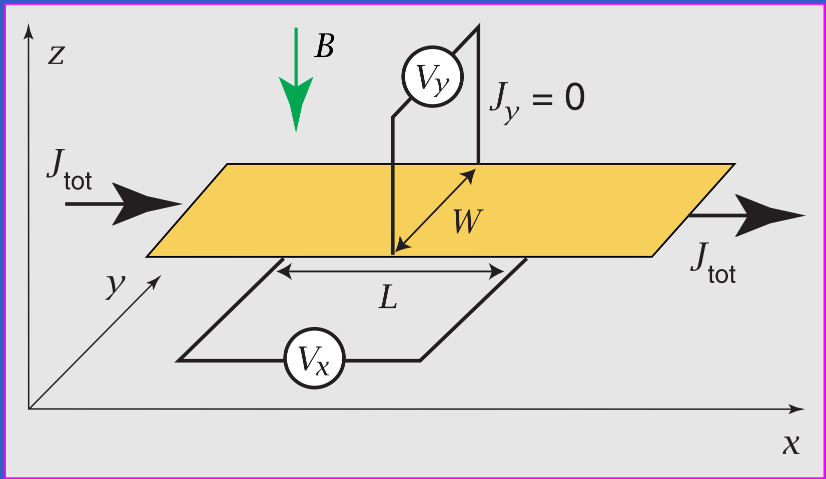

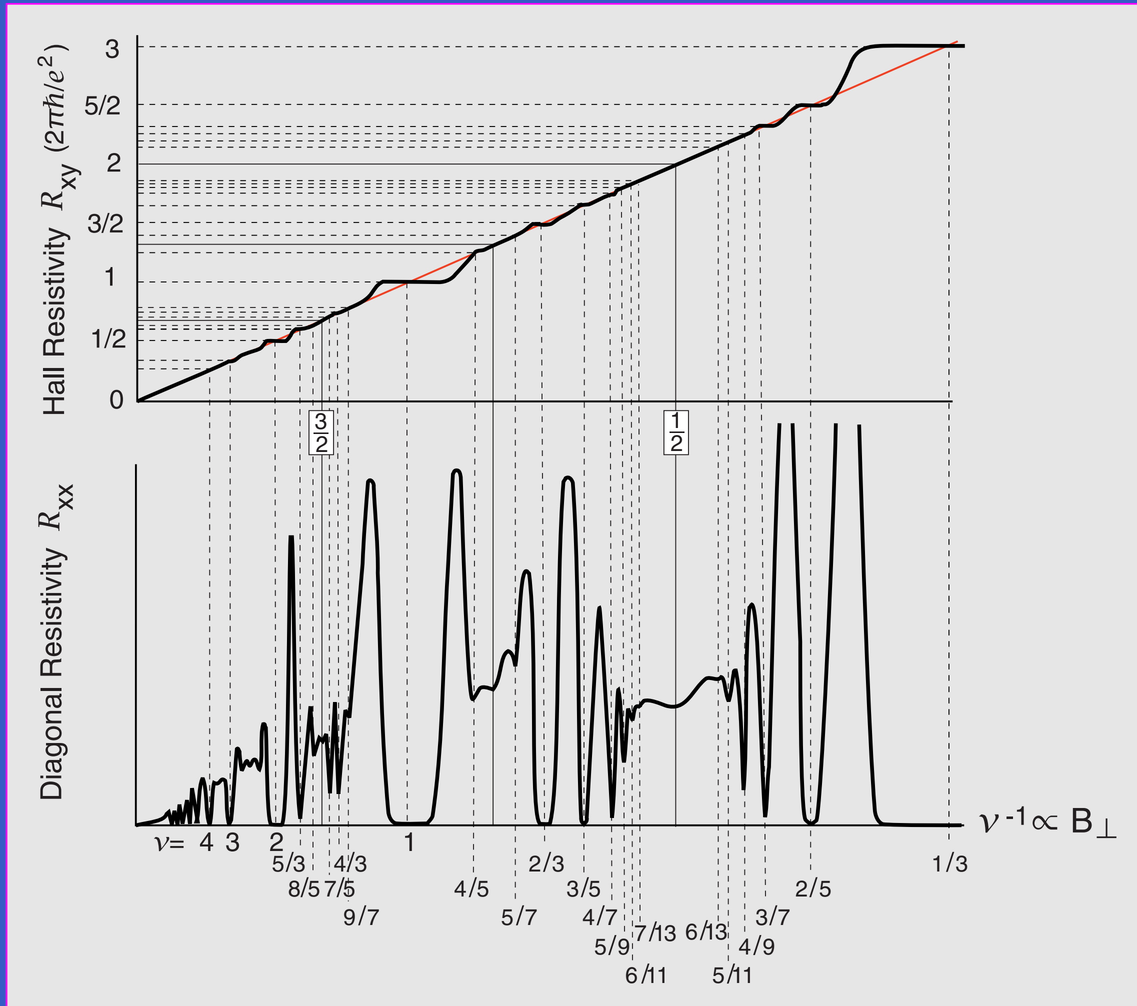

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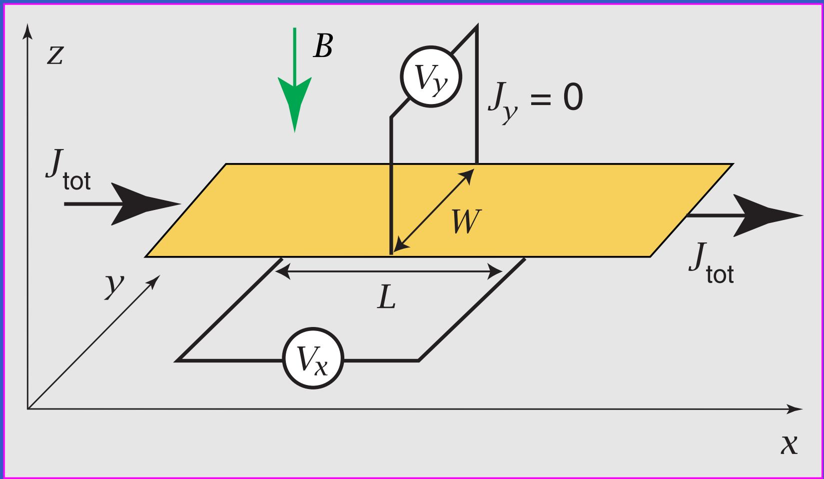

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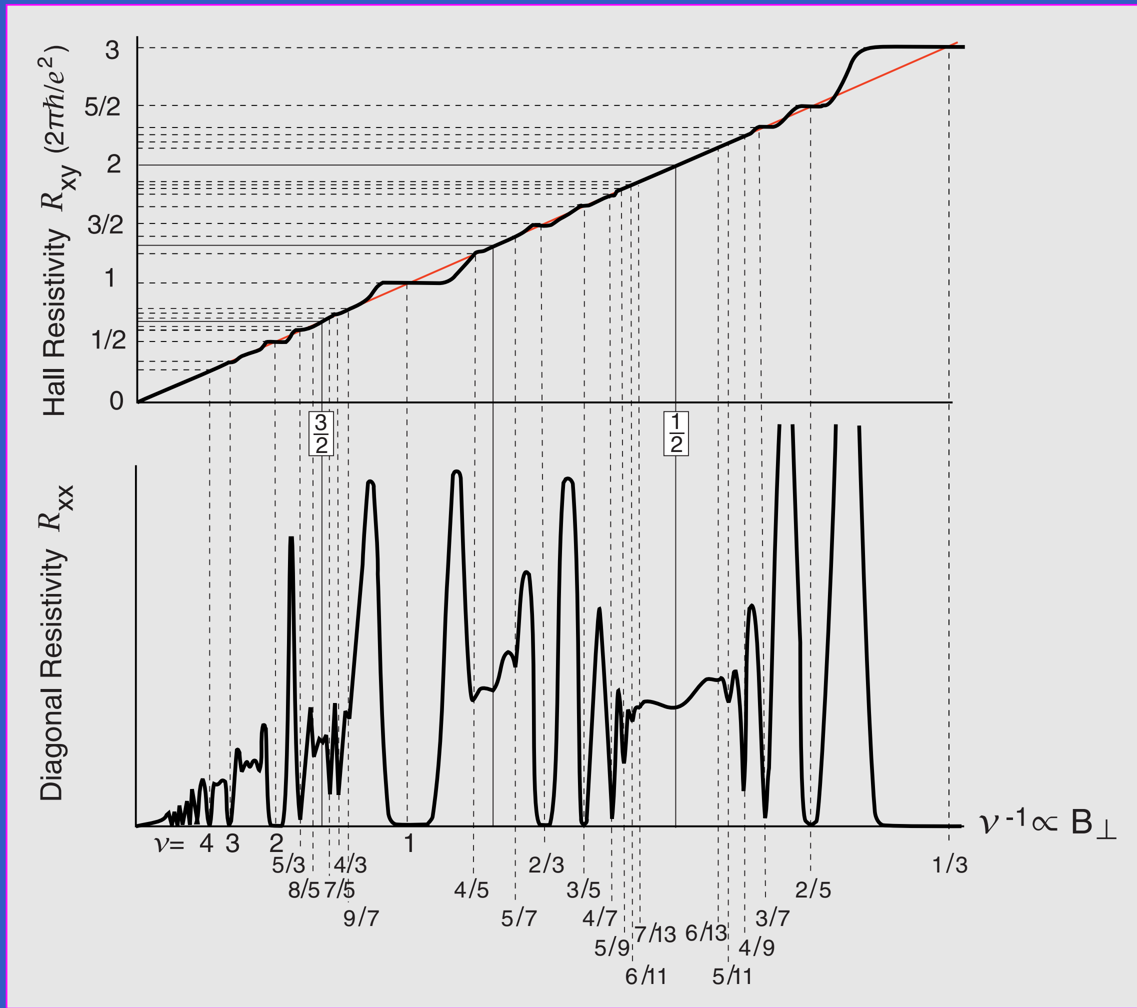
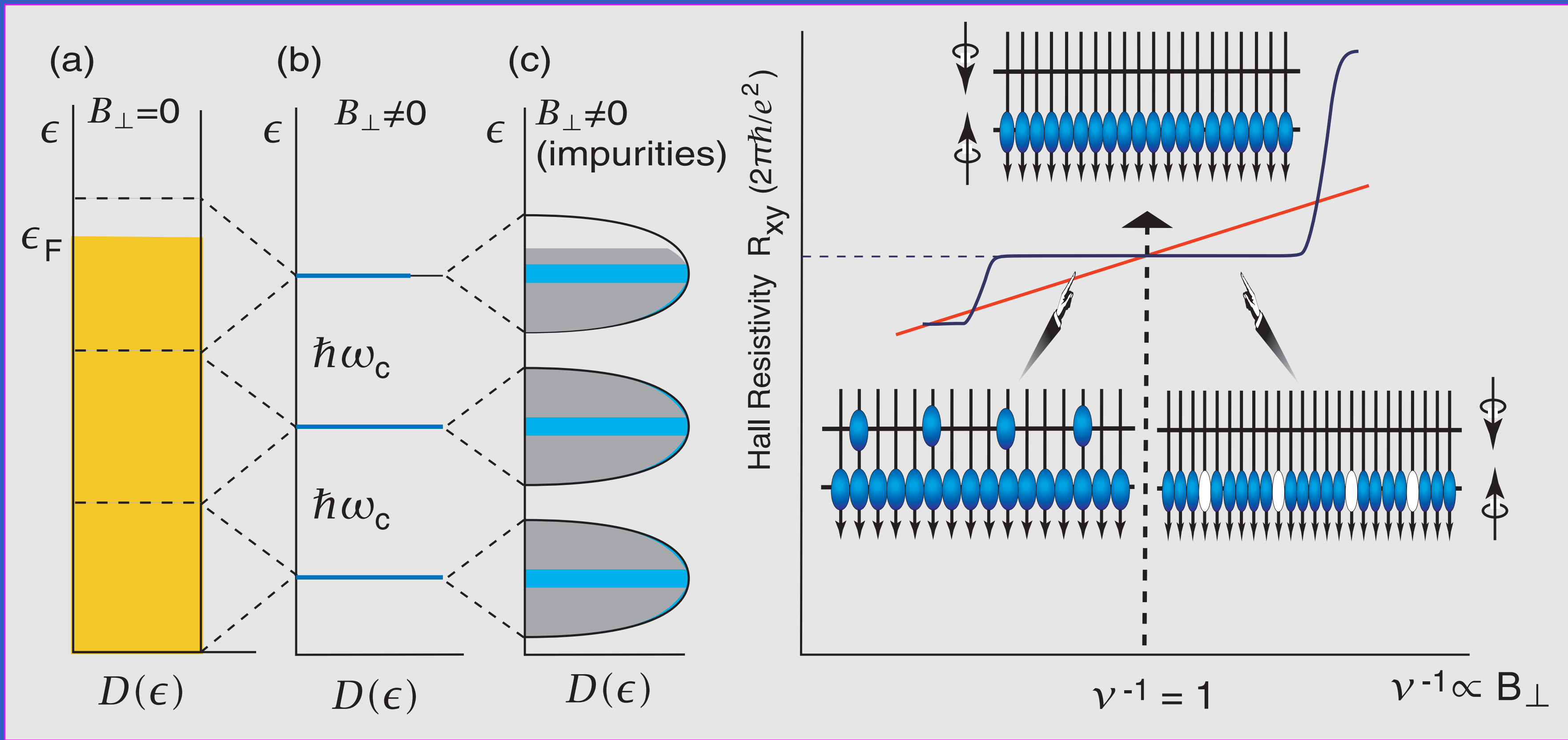


Noncommutative world develops in each plateau

with $\nu = \frac{2\pi\hbar\rho_0}{eB_\perp} \equiv \frac{N}{N_\Phi} \Rightarrow \frac{n}{m}$ (quantized: this is the QH effect)

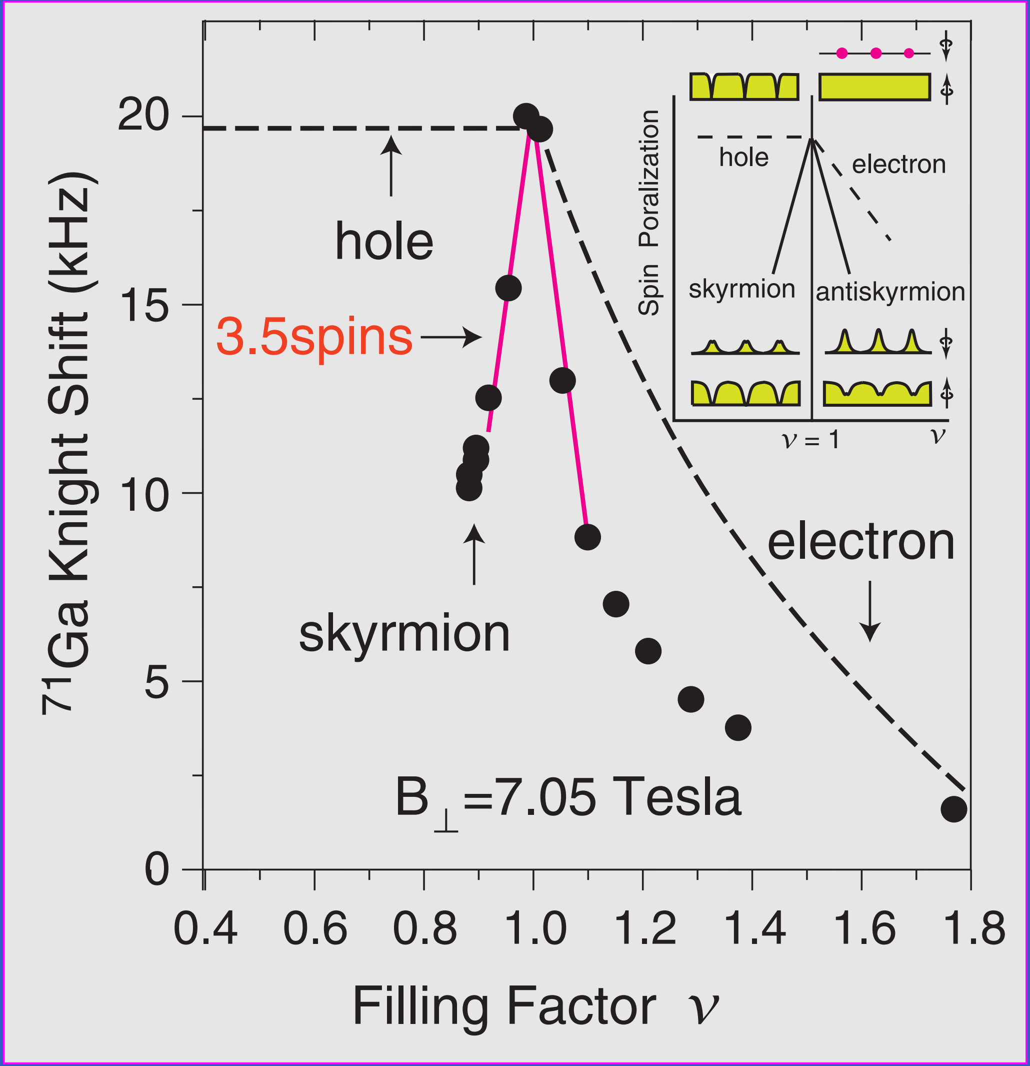
Hall Plateau

- Excitations around $\nu = 1$
  - $N = N_\Phi$ at $\nu = 1$
  - $N < N_\Phi$ at $\nu < 1 \Rightarrow$ holes excited (no spin excitation)
  - $N > N_\Phi$ at $\nu > 1 \Rightarrow$ electrons excited (spin excitation)
- Hall plateau is generated when quasiparticles are trapped by impurities



Spin Texture identified with Skyrmion (experiment)

- Experimentally 3.5 flipped spins are observed around $\nu = 1$



Barrett et al, PRL74(1995)5112

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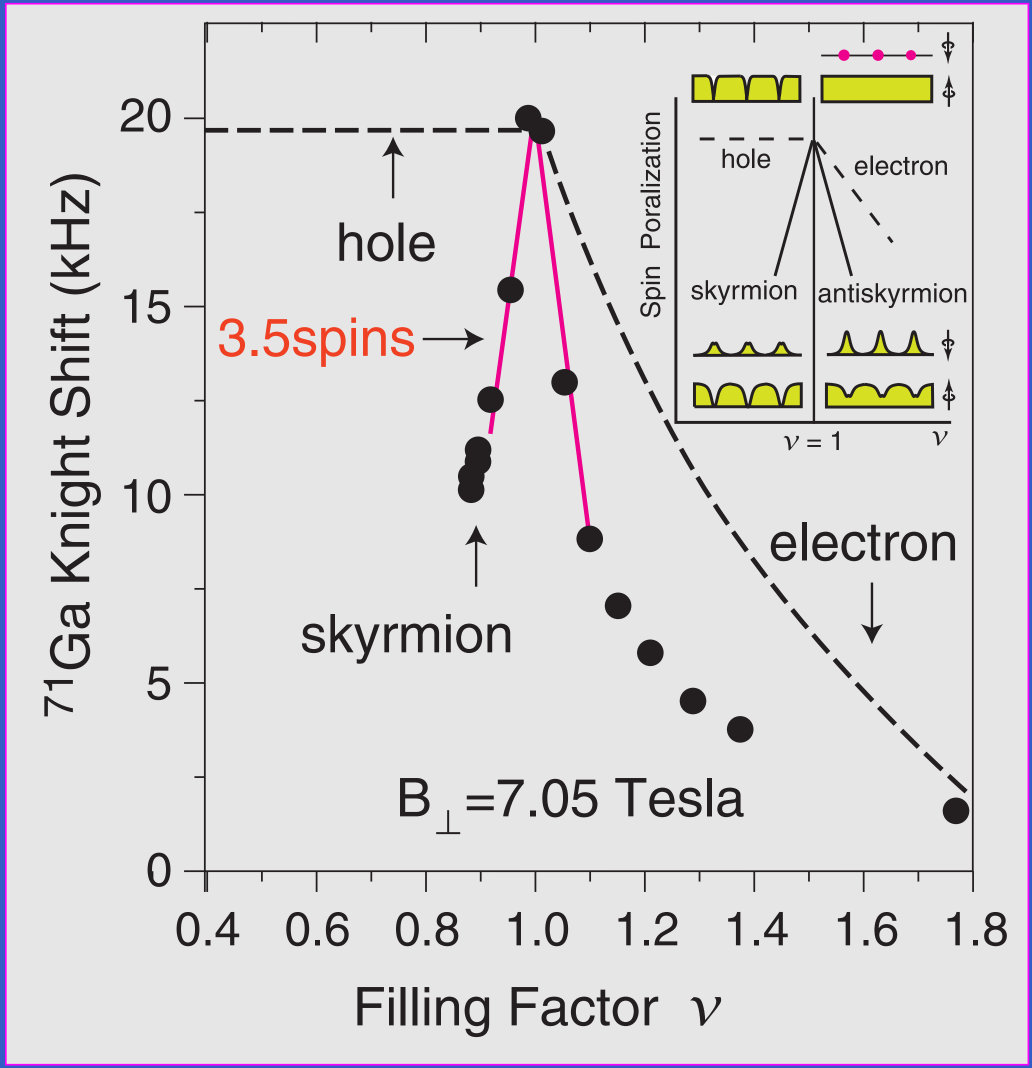
- Skrymions must be excited !
- A skyrmion and an antiskyrmion flips the same number of spins



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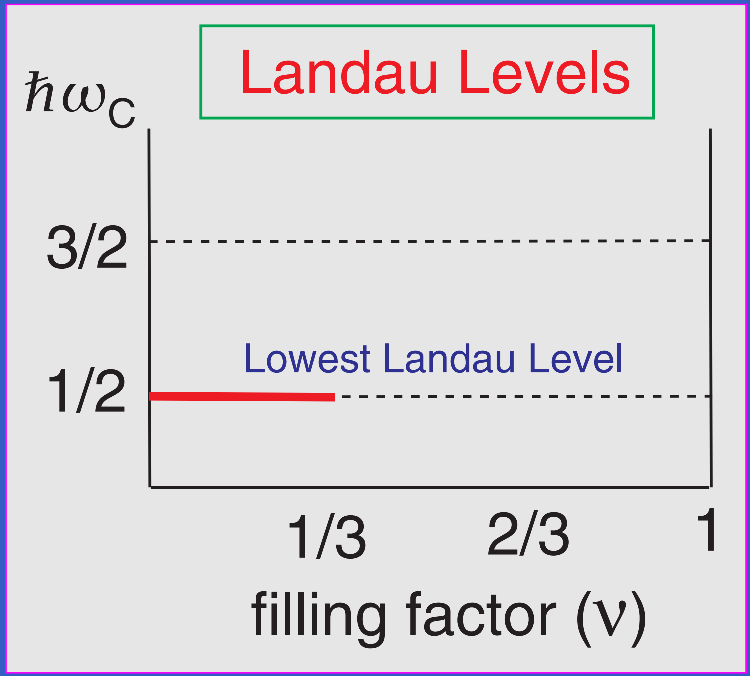
Lowest-Landau-Level Projection

- Kinetic Hamiltonian generates Landau levels

$$H_K = \frac{\boldsymbol{P}^2}{2M} = \frac{1}{2M}(P_x - iP_y)(P_x + iP_y) + \frac{1}{2}\hbar\omega_c \qquad \text{with} \quad P_k = -i\hbar\partial_k + eA_x^{\text{ext}}$$



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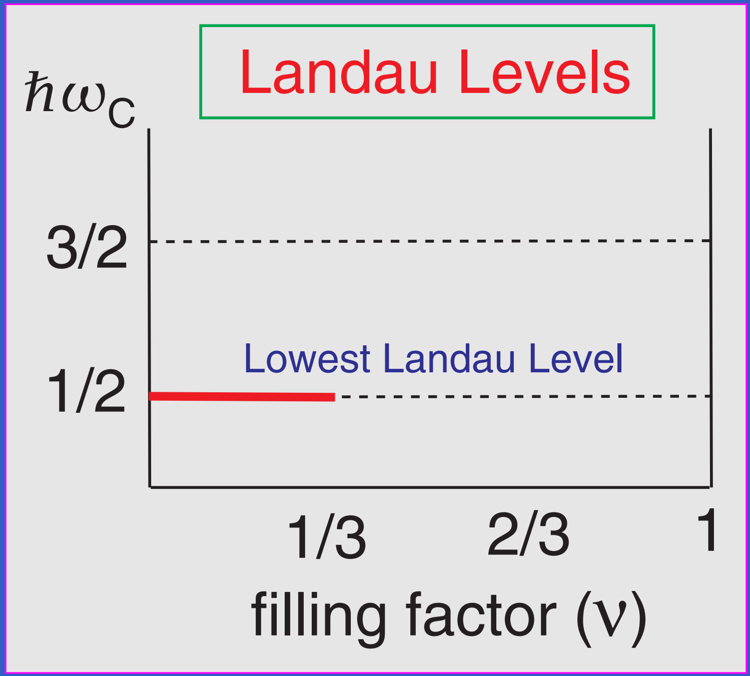
- Electron coordinate $\boldsymbol{x}$ $\Rrightarrow$ guiding center $\boldsymbol{X}$ and relative coordinate $\boldsymbol{R}$

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{R} \qquad \text{with} \quad \boldsymbol{R} = \left(-\frac{1}{eB_\perp}P_y, \frac{1}{eB_\perp}P_x\right)$$

$$[X, Y] = -i\ell_B^2, \quad [P_x, P_y] = i\hbar^2/\ell_B^2,$$

$$[X, P_x] = [X, P_y] = [Y, P_x] = [Y, P_y] = 0$$



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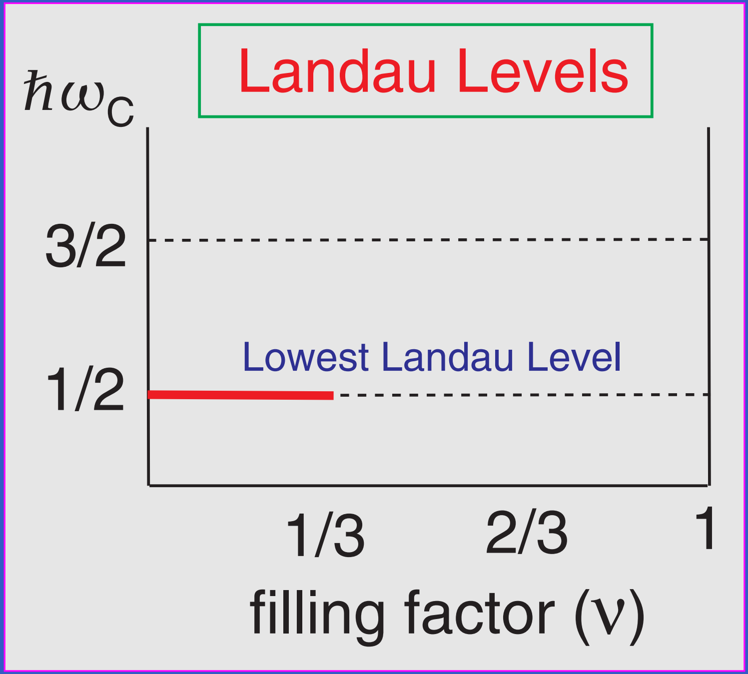

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- Electrons are confined to LLL if $\hbar\omega_c \to \infty$

$$(x, y) \Rrightarrow \boldsymbol{X} = (X, Y) \quad \text{with} \quad [X, Y] = -i\ell_B^2$$

Noncommutative Geometry

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$$[X, Y] = -i\ell_B^2 \quad \Rrightarrow \quad [b, b^\dagger] = 1$$

with

$$b = \frac{1}{\sqrt{2}\ell_B}(X - iY), \quad b^\dagger = \frac{1}{\sqrt{2}\ell_B}(X + iY)$$

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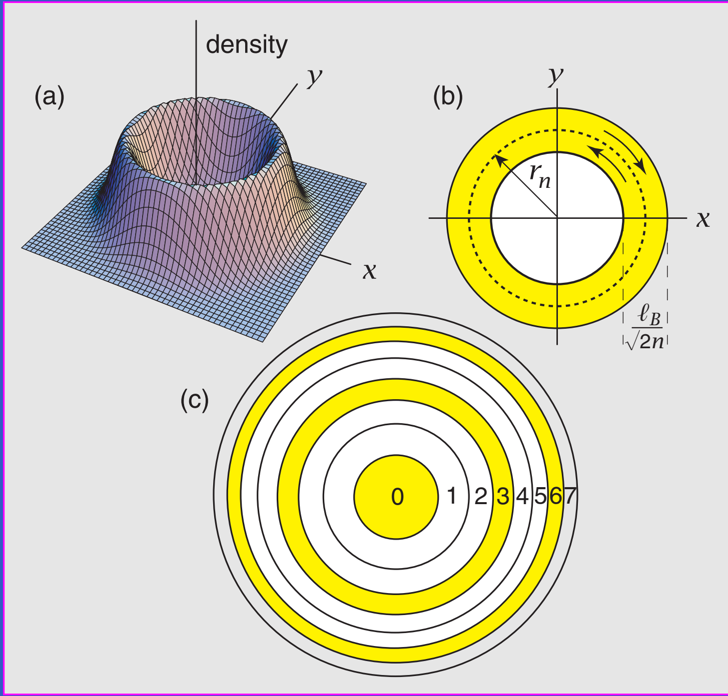

Landau site

- The Fock space $\mathbb{H}_W$ is made of the states

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle, \quad n = 0, 1, 2, \cdots$$

QH system is governed by noncommutative geometry

Weyl Ordering

- Weyl-ordering of a classical quantity $f(\boldsymbol{r})$

$$W[f] = \frac{1}{(2\pi)^2}\int d^2q d^2r\, e^{i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{X})} f(\boldsymbol{r})$$

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- We apply this to the electron density $f(\boldsymbol{r}) = \delta^2(\boldsymbol{x} - \boldsymbol{r}) \equiv \rho(\boldsymbol{x})$
  Here, $\boldsymbol{r}$ is the electron trajectory, while $\boldsymbol{x}$ is the coordinate (parameter)

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- Field-theoretical density reads

$$\hat{\rho}(\boldsymbol{q}) \equiv \langle \Psi | W[\rho(\boldsymbol{q})] | \Psi \rangle = \frac{1}{2\pi} \sum_{mn} \langle m | e^{-i\boldsymbol{q}\boldsymbol{X}} | n \rangle c^\dagger(m) c(n)$$

with electron field $\Psi(\boldsymbol{x})$ confined within the LLL

$$\Psi(\boldsymbol{x}) = \langle \boldsymbol{x} | \Psi \rangle = \sum_n \langle \boldsymbol{x} | n \rangle c(n)$$

Density Operators with Spin (Monolayer)

- The observables in the monolayer system are the densities

$$\hat{\rho}(\boldsymbol{q}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{q}\boldsymbol{X}}|n\rangle c_\alpha^\dagger(m)c_\alpha(n)$$

$$\hat{S}_a(\boldsymbol{q}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{q}\boldsymbol{X}}|n\rangle c_\alpha^\dagger(m)\frac{1}{2}(\boldsymbol{\tau}_a)_{\alpha\beta}\,c_\beta(n)$$

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- The density algebra $W_\infty(2)$ is the SU(2) extension of W_∞

$$[\hat{\rho}(\boldsymbol{p}),\hat{\rho}(\boldsymbol{q})] = \frac{i}{\pi}\hat{\rho}(\boldsymbol{p}+\boldsymbol{q})\sin\left(\ell_B^2\frac{\boldsymbol{p}\wedge\boldsymbol{q}}{2}\right)$$
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Spin rotation $\Rrightarrow$ U(1) density modulation $\Rrightarrow$ Increase of Coulomb energy

Landau-site Hamiltonian

Coulomb Hamiltonian $\Rrightarrow$ LLL projection $\Rrightarrow$ Landau-site Hamiltonian

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Coulomb Hamiltonian ⇛ LLL projection ⇛ Landau-site Hamiltonian

- LLL-projection of Coulomb Hamiltonian

$$H_{\mathrm{C}} = \pi \int d^2q\, V(\boldsymbol{q})\rho(-\boldsymbol{q})\rho(\boldsymbol{q}) \Rrightarrow \hat{H}_{\mathrm{C}} = \sum_{mnij} V_{mnij} \sum_{\sigma,\tau} c_\sigma^\dagger(m) c_\sigma(n) c_\tau^\dagger(i) c_\tau(j)$$

$$V_{mnij} = \frac{1}{4\pi} \int d^2k\, V_{\mathrm{D}}(\boldsymbol{k}) \langle m|e^{i\boldsymbol{X}\boldsymbol{k}}|n\rangle \langle i|e^{-i\boldsymbol{X}\boldsymbol{k}}|j\rangle, \qquad V_{\mathrm{D}}(\boldsymbol{q}) = \frac{e^2}{4\pi\varepsilon|\boldsymbol{q}|} e^{-\ell_B^2 \boldsymbol{q}^2/2}$$

  - V_{nnjj} yields the direct interaction, while V_{njjn} the exchange interaction

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- Homogeneous eigen states $\Rrightarrow$ degenerate ground states

$$\hat{H}_{\mathrm{C}}|\mathrm{g}\rangle = -\epsilon_{\mathrm{X}} N_\Phi |\mathrm{g}\rangle, \qquad \epsilon_{\mathrm{X}} = \frac{1}{4\pi}\int d^2k\, V_{\mathrm{D}}(\boldsymbol{k}) = \frac{1}{2}\sqrt{\frac{\pi}{2}} \frac{e^2}{4\pi\varepsilon\ell_B}$$

$$|\mathrm{g}\rangle = \prod_n \left[\sin\theta\, c_\uparrow^\dagger(n)|0\rangle + \cos\theta\, c_\downarrow^\dagger(n)|0\rangle\right]$$

Exchange Interaction

- Exchange interaction is extracted by using the algebraic identiy for SU(N)

$$\delta_{\sigma\beta}\delta_{\tau\alpha} = \frac{1}{2}\sum_{A}^{N^2-1} \lambda^A_{\sigma\tau}\lambda^A_{\alpha\beta} + \frac{1}{N}\delta_{\sigma\tau}\delta_{\alpha\beta}$$

- Projected Coulomb Hamiltonian

$$\hat{H}_{\mathrm{C}} = \sum_{mnij} V_{mnij} \sum_{\sigma,\tau} c^\dagger_\sigma(m)c_\sigma(n)c^\dagger_\tau(i)c_\tau(j) = -\pi\int d^2p\, V_{\mathrm{D}}(\boldsymbol{p})\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})$$

$$\Rightarrow \hat{H}_{\mathrm{X}} = -2\sum_{mnij} V_{mnij}[S_a(m,j)S_a(i,n) + \frac{1}{2N}\rho(m,j)\rho(i,n)]$$

$$\hat{H}_{\mathrm{X}} = -\pi\int d^2p\, V_{\mathrm{X}}(\boldsymbol{p})\left[\hat{\boldsymbol{S}}(-\boldsymbol{p})\hat{\boldsymbol{S}}(\boldsymbol{p}) + \tfrac{1}{4}\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})\right]$$

with $V_{\mathrm{X}}(\boldsymbol{p}) = \frac{\ell_B^2}{\pi}\int d^2k\, e^{-i\ell_B^2\boldsymbol{p}\wedge\boldsymbol{k}} e^{-\ell_B^2\boldsymbol{p}^2/2} V(\boldsymbol{k}) = \frac{\sqrt{2\pi}e^2\ell_B}{4\pi\varepsilon} I_0(\ell_B^2\boldsymbol{p}^2/4)e^{-\ell_B^2\boldsymbol{p}^2/4}$

Effective Field Theory

$$\widehat{H}_{\mathsf{X}} = -\pi \int d^2p \, V_{\mathsf{X}}(\boldsymbol{p}) \left[\widehat{\boldsymbol{S}}(-\boldsymbol{p}) \widehat{\boldsymbol{S}}(\boldsymbol{p}) + \frac{1}{4} \widehat{\rho}(-\boldsymbol{p}) \widehat{\rho}(\boldsymbol{p}) \right]$$

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- For sufficient smooth configuration we make the derivative expansion

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with $\rho_\Phi = \frac{\rho_0}{\nu} = \frac{1}{2\pi\ell_B^2}, \qquad V_{\mathrm{X}}(0) = 4\ell_B^2\epsilon_{\mathrm{X}}, \qquad J_s = \frac{1}{8\pi}\epsilon_{\mathrm{X}}$

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$$\hat{H}_X = -\pi \int d^2p\, V_X(\boldsymbol{p}) \left[\hat{\boldsymbol{S}}(-\boldsymbol{p})\hat{\boldsymbol{S}}(\boldsymbol{p}) + \frac{1}{4}\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})\right]$$

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- Exchange interaction leads to a nonlinear sigma model as an effective theory

$$\mathcal{H}_X^{\text{spin}} = 2J_s\partial_k\mathcal{S}(\boldsymbol{x})\partial_k\mathcal{S}(\boldsymbol{x})$$

with $\boldsymbol{S}(\boldsymbol{x}) = \frac{\rho_0}{\nu}\mathcal{S}(\boldsymbol{x})$

Nonlinear Sigma Model

$$H_{\text{eff}} = 2J_s \int d^2x \, \partial_k \boldsymbol{S}^{\text{sky}}(\boldsymbol{x}) \partial_k \boldsymbol{S}^{\text{sky}}(\boldsymbol{x})$$

Nonlinear Sigma Model

$$H_{\text{eff}} = 2J_s \int d^2x\, \partial_k \mathcal{S}^{\text{sky}}(\boldsymbol{x})\partial_k \mathcal{S}^{\text{sky}}(\boldsymbol{x})$$

- Topological solitons are **Skyrmions** associated with $\pi_2(CP^1) = \mathbb{Z}$

$$Q^{\text{sky}}(\boldsymbol{x}) = \frac{1}{\pi}\varepsilon_{abc}\varepsilon^{ij}\mathcal{S}_a\partial_i\mathcal{S}_b\partial_j\mathcal{S}_c = \frac{1}{\pi}\frac{4(\kappa\ell_B)^2}{[r^2+4(\kappa\ell_B)^2]^2}$$

- Skyrmion configuration (its size is fixed to minimize the energy)

$$\mathcal{S}_x^{\text{sky}} = \sqrt{1-\sigma_{\text{sky}}^2}\cos\theta, \quad \mathcal{S}_y^{\text{sky}} = -\sqrt{1-\sigma_{\text{sky}}^2}\sin\theta, \quad \mathcal{S}_z^{\text{sky}} = \sigma_{\text{sky}}$$

with $$\sigma_{\text{sky}}(\boldsymbol{x}) = \frac{r^2-4(\kappa\ell_B)^2}{r^2+4(\kappa\ell_B)^2}$$

  - The skyrmion scale κ is arbitral in 2-dimensional nonlinear sigma model

Spin Wave (Goldstone Mode)

- Effective Hamiltonian in the presence of the Zeeman effect

$$\mathcal{H}_{\text{eff}} = 2J_s \partial_k \mathcal{S}(\boldsymbol{x}) \partial_k \mathcal{S}(\boldsymbol{x}) - \rho_0 \Delta_Z \mathcal{S}_z(\boldsymbol{x})$$

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- For small fluctuation, i.e., pseudo-Goldstone mode

$$\mathcal{H}_{\text{eff}} = \frac{1}{2} J_s (\partial_k \sigma)^2 + \frac{1}{2} J_s (\partial_k \vartheta)^2 + \frac{\rho_0 \Delta_{\text{Z}}}{4} (\sigma^2 + \vartheta^2)$$

with

$$\mathcal{S}_x(\boldsymbol{x}) = \frac{1}{2}\sigma(\boldsymbol{x}), \quad \mathcal{S}_y(\boldsymbol{x}) = \frac{1}{2}\sqrt{1 - \sigma^2(\boldsymbol{x})} \sin \vartheta(\boldsymbol{x}), \quad \mathcal{S}_z(\boldsymbol{x}) = \frac{1}{2}\sqrt{1 - \sigma^2(\boldsymbol{x})} \cos \vartheta(\boldsymbol{x})$$

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- Quantum coherence develops if coherence length $\gg$ magnetic length

$$\xi_{\text{spin}} = \sqrt{\frac{2\rho_s}{\rho_0 \Delta_{\text{Z}}}} = \frac{7.33}{\sqrt{B_\perp}} \ell_B$$

CP^1 Skyrmion

- Due to the $W_\infty(2)$ algebra the spin rotation induces a density modulation

$$[\hat{S}_a(\boldsymbol{p}), \hat{\rho}(\boldsymbol{q})] = \frac{i}{\pi}\hat{S}_a(\boldsymbol{p}+\boldsymbol{q}) \sin\left(\frac{\ell_B^2}{2}\boldsymbol{p}\wedge\boldsymbol{q}\right)$$

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- A Skyrmion excitation induces a density modulation

$$\rho^{\text{sky}}(\boldsymbol{x}) \equiv \langle \mathfrak{S}_{\text{sky}} | \hat{\rho}(\boldsymbol{x}) | \mathfrak{S}_{\text{sky}} \rangle \simeq Q^{\text{sky}}(\boldsymbol{x}) - \rho_0$$

with the topological charge

$$Q^{\text{sky}}(\boldsymbol{x}) = \frac{1}{\pi}\varepsilon_{abc}\varepsilon^{ij}\mathcal{S}_a\partial_i\mathcal{S}_b\partial_j\mathcal{S}_c$$

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- Effective Hamiltonian for nonperturbative excitations

$$H_{\mathrm{eff}} = 2J_s\int d^2x\,\partial_k\boldsymbol{\mathcal{S}}^{\mathrm{sky}}(\boldsymbol{x})\partial_k\boldsymbol{\mathcal{S}}^{\mathrm{sky}}(\boldsymbol{x}) - \rho_0\Delta_{\mathrm{Z}}\int d^2x\,\mathcal{S}_z^{\mathrm{sky}}(\boldsymbol{x}) + \tfrac{1}{2}\int d^2x d^2y\,\rho^{\mathrm{sky}}(\boldsymbol{x})V(\boldsymbol{x}-\boldsymbol{y})\rho^{\mathrm{sky}}(\boldsymbol{y})$$

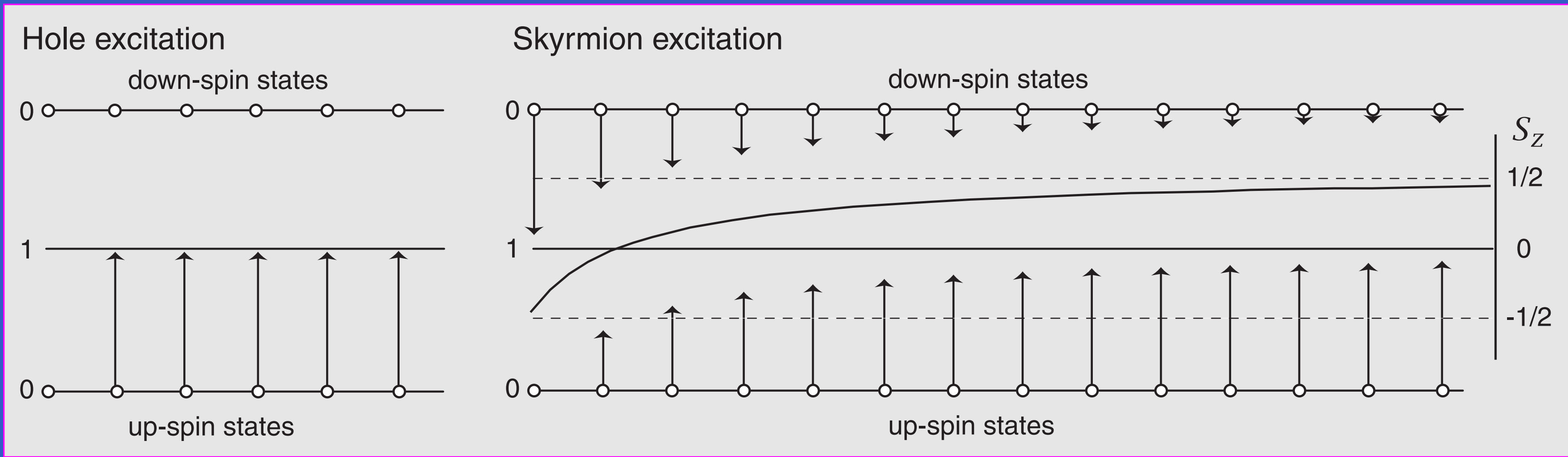
CP^1 Skyrmion in Microscopic Theory

- A microscopic Skyrmion state

$$|\mathfrak{S}_{\text{sky}}\rangle = \prod_{n=0}^{\infty}\left[u(n)c_{\downarrow}^{\dagger}(n) + v(n)c_{\uparrow}^{\dagger}(n+1)\right]|0\rangle, \qquad u^2(n) + v^2(n) = 1$$

- A hole excitation is a special one: $u(n) = 0$, $v(n) = 1$ for all n

$$|\mathfrak{S}_{\text{sky}}\rangle = \prod_{n=1}^{\infty} c_{\uparrow}^{\dagger}(n)|0\rangle$$



CP^1 Skyrmion in Microscopic Theory (continued)

- Density and spin modulations

$$\rho^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int d^2q \sum_{mn}\langle m|e^{-i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{X})}|n\rangle\langle\mathfrak{S}_{\mathsf{sky}}|c^\dagger(m)c(n)|\mathfrak{S}_{\mathsf{sky}}\rangle$$

$$S_a^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int d^2q \sum_{mn}\langle m|e^{-i\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{X})}|n\rangle\langle\mathfrak{S}_{\mathsf{sky}}|c^\dagger(m)\frac{\tau_a}{2}c(n)|\mathfrak{S}_{\mathsf{sky}}\rangle$$

$\Rrightarrow$

$$\rho^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{2\pi} - \frac{1}{2\pi}\frac{1}{\omega^2+1}\, e^{-\frac{1}{2}r^2}M(\omega^2;\omega^2+2;r^2/2)$$

$$S_z^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{4\pi} - \frac{1}{4\pi}e^{-\frac{1}{2}r^2}M(\omega^2;\omega^2+1;r^2/2) - \frac{1}{4\pi}\frac{\omega^2}{\omega^2+1}e^{-\frac{1}{2}r^2}M(\omega^2+1;\omega^2+2;r^2/2)$$

$$S_x^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{4\pi}\frac{\sqrt{2}\omega x}{\omega^2+1}e^{-\frac{1}{2}r^2}M(\omega^2+1;\omega^2+2;r^2/2)$$

$$S_y^{\mathsf{sky}}(\boldsymbol{x}) = \frac{1}{4\pi}\frac{-\sqrt{2}\omega y}{\omega^2+1}e^{-\frac{1}{2}r^2}M(\omega^2+1;\omega^2+2;r^2/2)$$

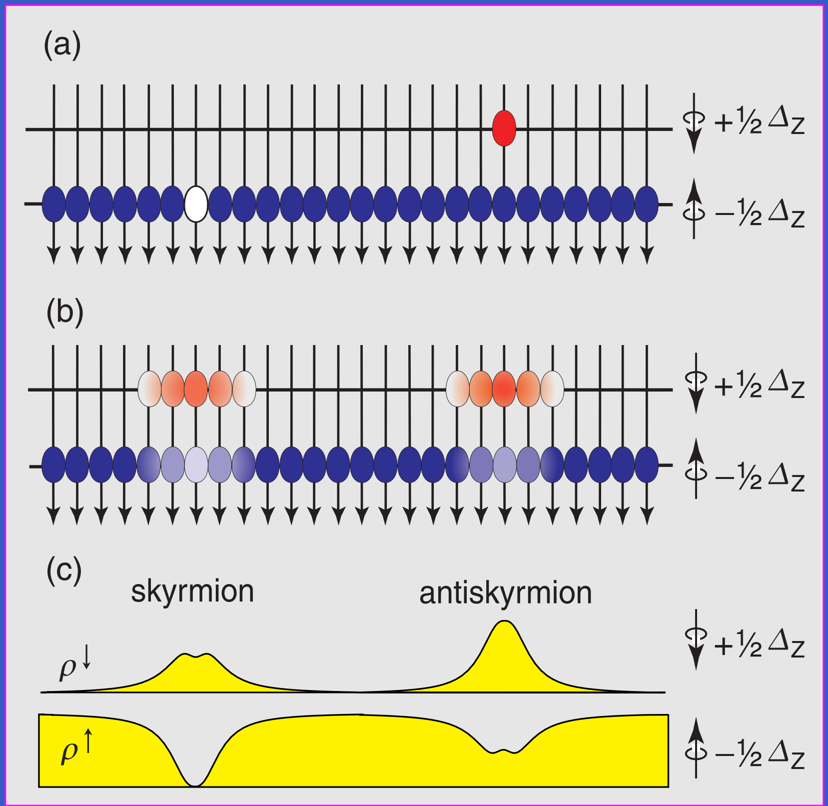
M is the Kummer function

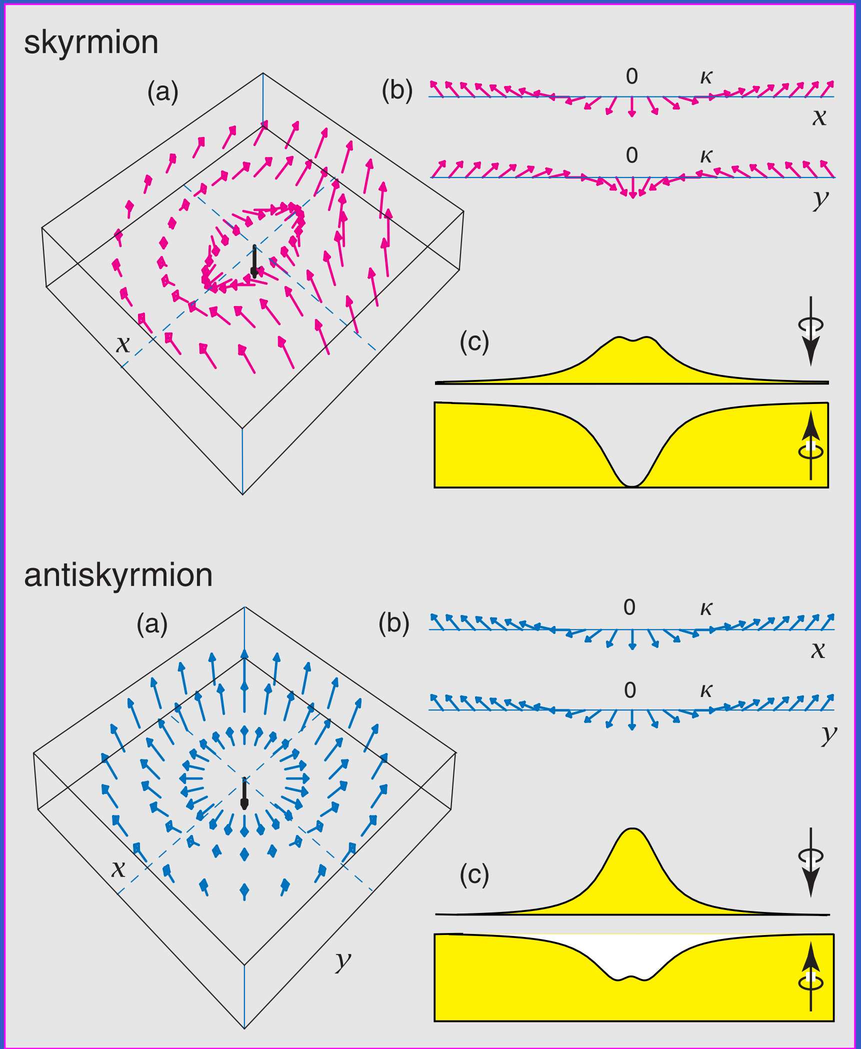
anzats: $u^2(n) = \dfrac{\omega^2}{n+1+\omega^2}$, $\quad v^2(n) = \dfrac{n+1}{n+1+\omega^2}$, $\quad \omega = \sqrt{2}\kappa\ell_B$ (κ: Skyrmion scale)

CP^1 Skyrmion and Spin Flip

- Skyrmion has a fixed scale to optimize the Coulomb and Zeeman energies
  - Skyrmion flips many spins coherently

$$N^{\text{spin}} = -\frac{1}{2}\int d^2x \left[2S_z^{\text{sky}}(\boldsymbol{x}) - \rho_0\right]$$





Activation Energy and Skyrmions (Experiment)

- Skyrmion excitation energy ($E_{\text{C}}^0 = e^2/4\pi\varepsilon\ell_B$)

$$E_{\text{sky}} \simeq \left\{ \frac{1}{4}\sqrt{\frac{\pi}{2}} + \frac{0.26}{2\kappa} + 2\tilde{g}\kappa^2 \ln\left(\frac{\sqrt{2\pi}}{32\tilde{g}} + 1\right) \right\} E_{\text{C}}^0$$

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- Skyrmion size ($\tilde{g} = \Delta_Z / E_C^0$)

$$\kappa \simeq \frac{1}{2}\left\{ \tilde{g}\ln\left(\frac{\sqrt{2\pi}}{32\tilde{g}} + 1\right)\right\}^{-1/3} \simeq 1$$

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$N^{\rm spin} \simeq 3.5$ at $B = 7$ Tesla

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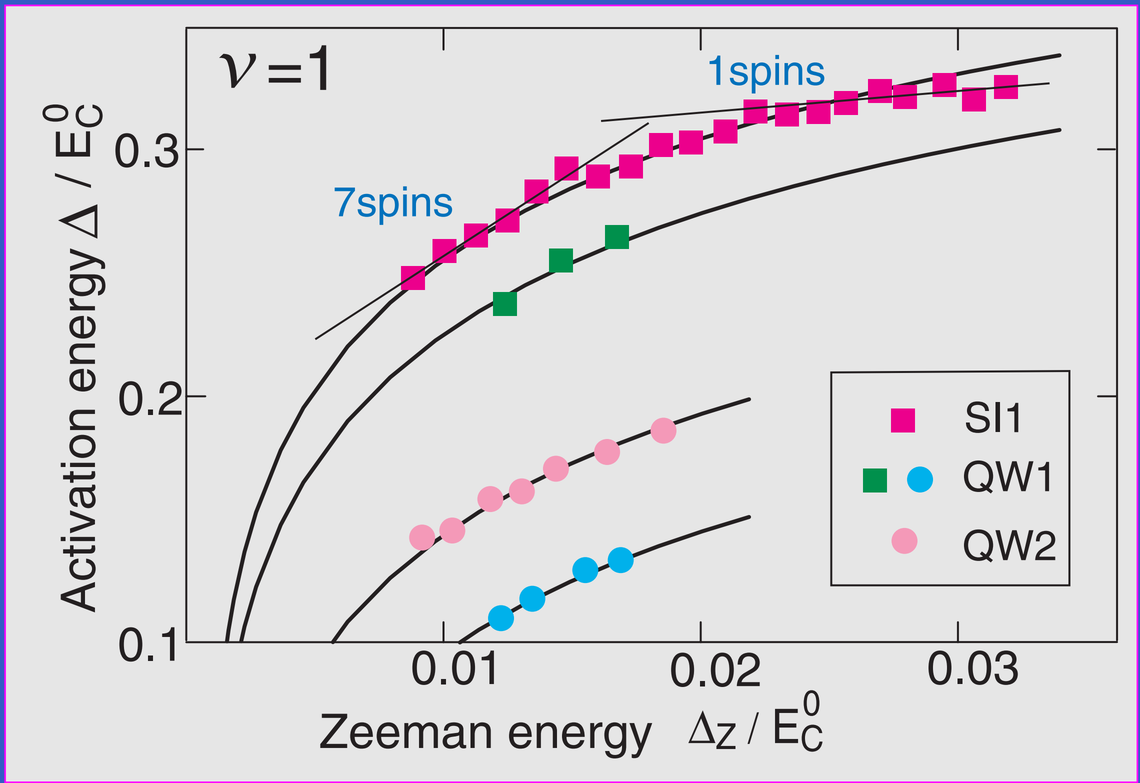

Schmeller et al, PRL75(1995)4290

Multicomponent QH System

Monolayer QH System with SU(N) Symmetry

- Kinetic Hamiltonian for N-component electrons with SU(N) symmetry

$$H_{\mathrm{K}} = \frac{1}{2M}\Psi^{\dagger}(P_x - iP_y)(P_x + iP_y)\Psi + \frac{1}{2}\hbar\omega_{\mathrm{c}} \qquad \text{with} \quad P_k = -i\hbar\partial_k + eA_x^{\mathrm{ext}}$$

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- There are N^2 densities: $\rho(\boldsymbol{x}) = \Psi^\dagger(\boldsymbol{x})\Psi(\boldsymbol{x})$, $\quad I_A(\boldsymbol{x}) = \frac{1}{2}\Psi^\dagger(\boldsymbol{x})\lambda_A\Psi(\boldsymbol{x})$

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- Projected Coulomb Hamiltonian

$$\hat{H}_{\mathrm{C}} = -\pi\int d^2p\, V_{\mathrm{D}}(\boldsymbol{p})\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})$$

$$\Rightarrow \boxed{\hat{H}_{\mathrm{X}} = -\pi\int d^2p\, V_{\mathrm{X}}(\boldsymbol{p})\left[\hat{\boldsymbol{I}}(-\boldsymbol{p})\hat{\boldsymbol{I}}(\boldsymbol{p}) + \frac{1}{4}\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})\right]}$$

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- Low-energy effective Hamiltonian is SU(N) nonlinear sigma model

$$\boxed{H_{\mathrm{X}}^{\mathrm{eff}} = 2J_s\int d^2x\, \partial_k\mathfrak{I}(\boldsymbol{x})\partial_k\mathfrak{I}(\boldsymbol{x}) \qquad \text{with} \quad \hat{\boldsymbol{I}}(\boldsymbol{x}) = \rho_\Phi\mathfrak{I}(\boldsymbol{x})}$$

  - One Landau site can accomodante k electrons ($k \le N$)

Grassmannian Field

- SU(N) nonlinear sigma model ⇛ with spontaneous symmetry breakdown

$$H_{\mathrm{X}}^{\mathrm{eff}} = 2J_s \int d^2x \; \partial_k \boldsymbol{\mathfrak{I}}(\boldsymbol{x}) \partial_k \boldsymbol{\mathfrak{I}}(\boldsymbol{x})$$

  - There are $N^2 - 1$ isospin densities $\mathfrak{I}(\boldsymbol{x})$, but how may independent ones?

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- Dynamical fields are **Grassmannian fields** at $\nu = k$ taking values on $\text{G}^{N,k}$

$$G^{N,k} = \frac{U(N)}{U(k) \otimes U(N-k)} = \frac{SU(N)}{U(1) \otimes SU(k) \otimes SU(N-k)}, \qquad G^{N,1} \equiv CP^{N-1}$$

  - Dimension is $2k(N-k)$ ⇒ $2k(N-k)$ Goldstone modes

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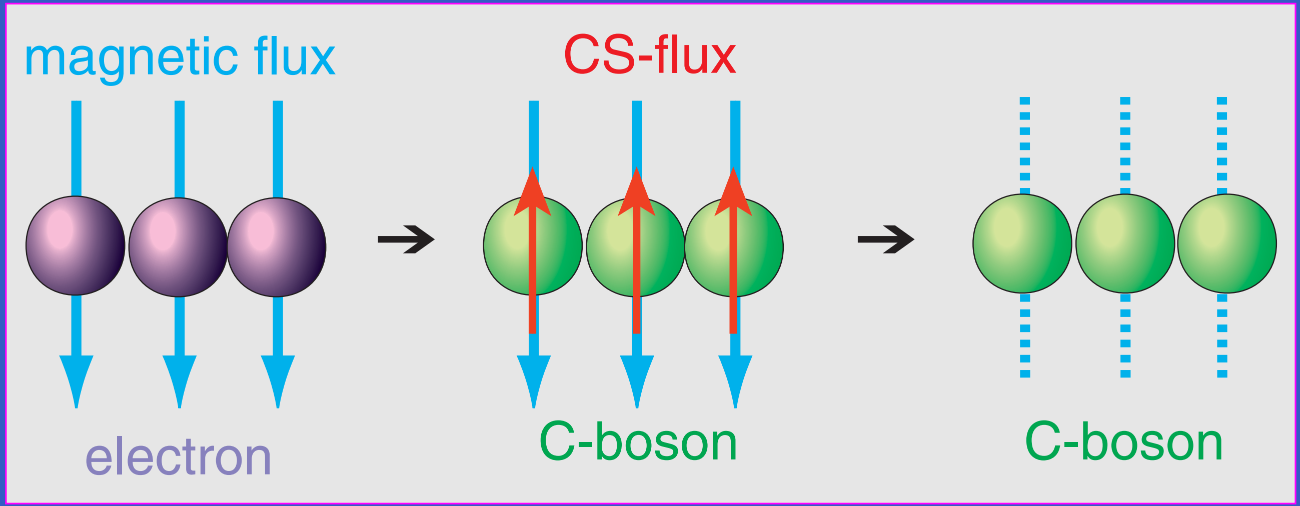
- Topological solitons are $G^{N,k}$ Skyrmions at $\nu = k$

  according to $\pi_2\left(G^{N,k}\right) = \mathbb{Z}$

Charge-Isospin Separation

- Spinless electrons are bosonized by attaching a flux $\Rrightarrow$ **composite bosons**

$$\psi(\boldsymbol{x}) = e^{-i\Theta(\boldsymbol{x})}\phi(\boldsymbol{x}) \qquad \text{with} \quad \varepsilon_{jk}\partial_j\partial_k\Theta(\boldsymbol{x}) = 2\pi\rho(\boldsymbol{x})$$



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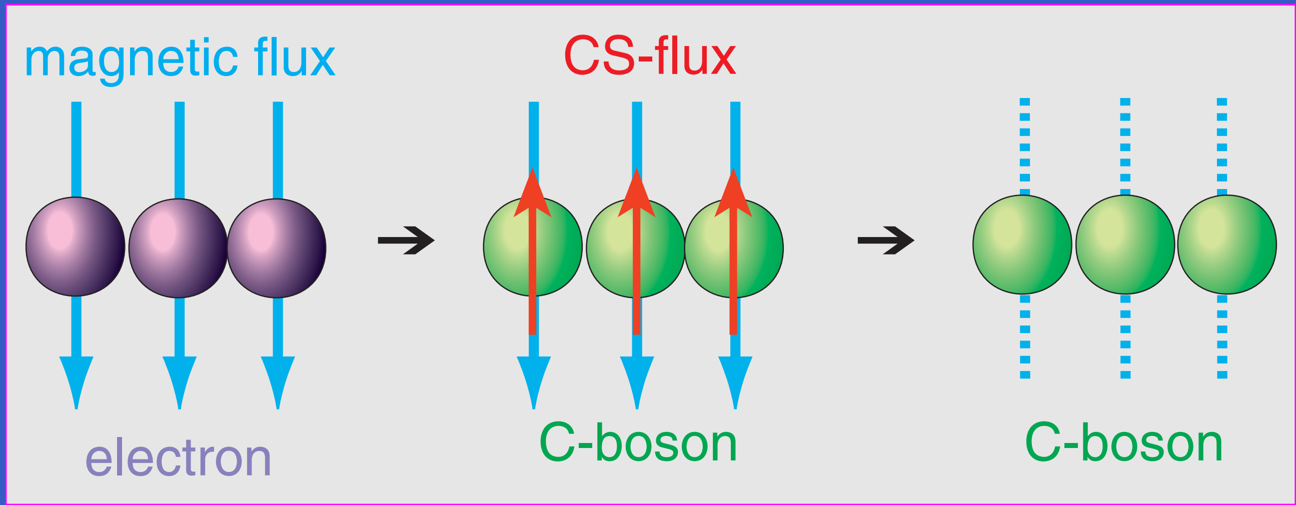

- U(N) electrons are also bosonized

$$\psi_\alpha(\boldsymbol{x}) = e^{-i\Theta(\boldsymbol{x})}\phi(\boldsymbol{x})n_\alpha(\boldsymbol{x}) \qquad \text{with} \quad \sum n_\alpha^\dagger(\boldsymbol{x})n_\alpha(\boldsymbol{x}) = 1$$

  - There are $N-1$ complex degree of freedom in n_α $\Rightarrow$ **CP^{N-1} field**

Effective Field Theory with CP^{N-1} Field at $\nu = 1$

- Relation between isospin field and CP^{N-1} field: $\mathcal{J}_A(\boldsymbol{x}) = \boldsymbol{n}^\dagger(\boldsymbol{x}) \frac{\lambda_A}{2} \boldsymbol{n}(\boldsymbol{x})$

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- Relation between isospin field and CP^{N-1} field: $\mathfrak{I}_A(\boldsymbol{x}) = \boldsymbol{n}^\dagger(\boldsymbol{x})\frac{\lambda_A}{2}\boldsymbol{n}(\boldsymbol{x})$
- Equivalence between **the SU(N) sigma model** and **the CP^{N-1} model**

$$\mathcal{H}_\mathrm{X} = 2J_s \sum_{A=1}^{N^2-1} [\partial_k \mathfrak{I}_A]^2 = 2J_s \sum_{\alpha=1}^{N} \left(\partial_j n^{\alpha\dagger} + iK_j n^{\alpha\dagger}\right)\left(\partial^j n^\alpha + iK_j n^\alpha\right)$$

with $K_\mu = -i\sum_\alpha n^{\alpha\dagger}\partial_\mu n^\alpha$

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with $K_\mu = -i\sum_\alpha n^{\alpha\dagger}\partial_\mu n^\alpha$

- There are N-fold degeneracy in the ground state
  - One ground state is chosen spontaneously with a CP^{N-1} Skyrmion on it

$$\boldsymbol{n}_{\mathrm{g}}(\boldsymbol{x}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{n}_{\mathrm{sky}}(\boldsymbol{x}) = \begin{pmatrix} \kappa_{N-1} \\ \vdots \\ \kappa_1 \\ z \end{pmatrix}$$

Effective Field Theory with G^{N,k} Field at $\nu = k$

- At $\nu = k$ there are N electrons in one Landau site
  - We need k CP^{N-1} fields to describe such a system: Grassmannian field

Grassmannian field: $Z(\boldsymbol{x}) = (\boldsymbol{n}_1, \boldsymbol{n}_2, \cdots, \boldsymbol{n}_k)$

Isospin field: $\mathcal{I}_A(\boldsymbol{x}) = \mathrm{Tr}\left[Z^\dagger(\boldsymbol{x})\frac{\lambda_A}{2}Z(\boldsymbol{x})\right] = \frac{1}{2}\sum_{i=1}^{k}\boldsymbol{n}_i^\dagger(\boldsymbol{x})\lambda_A\boldsymbol{n}_i(\boldsymbol{x})$

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$$\mathcal{H}_\mathrm{X} = 2J_s\sum_{A=1}^{N^2-1}[\partial_k\mathcal{I}_A]^2 = 2J_s\mathrm{Tr}\left[\left(\partial_j Z - iK_j Z\right)^\dagger\left(\partial_j Z - iK_j Z\right)\right]$$

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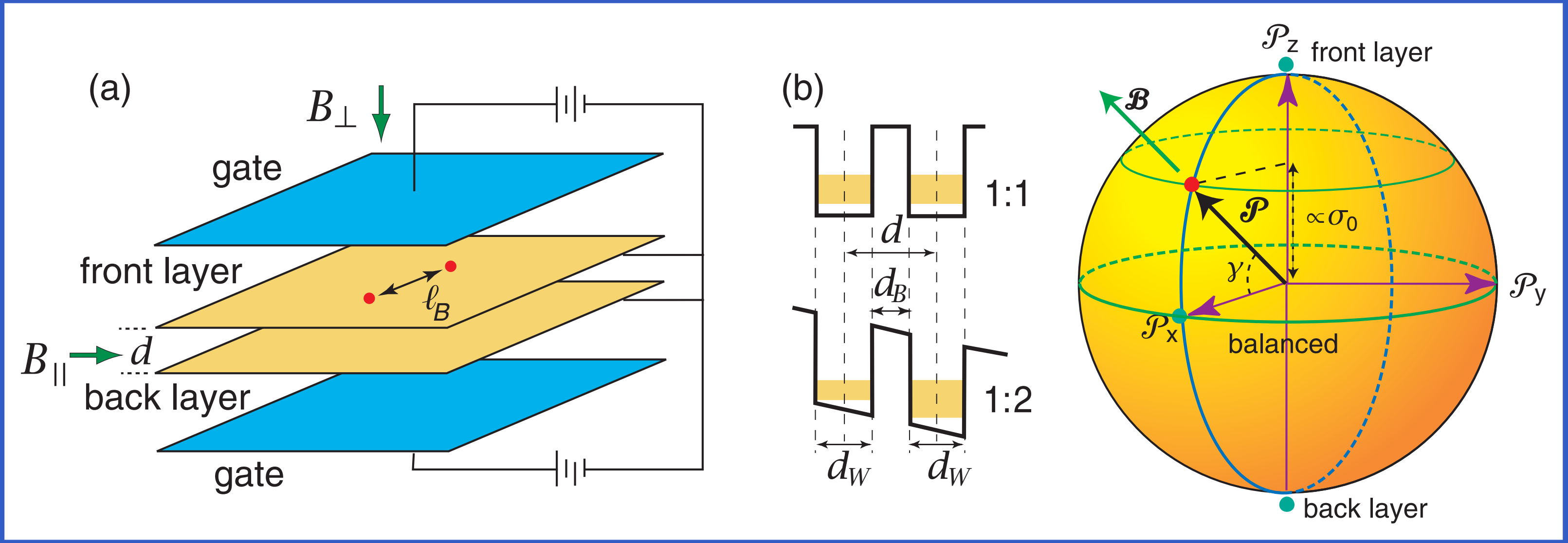
Bilayer QH System

Pseudospins in Bilayer System



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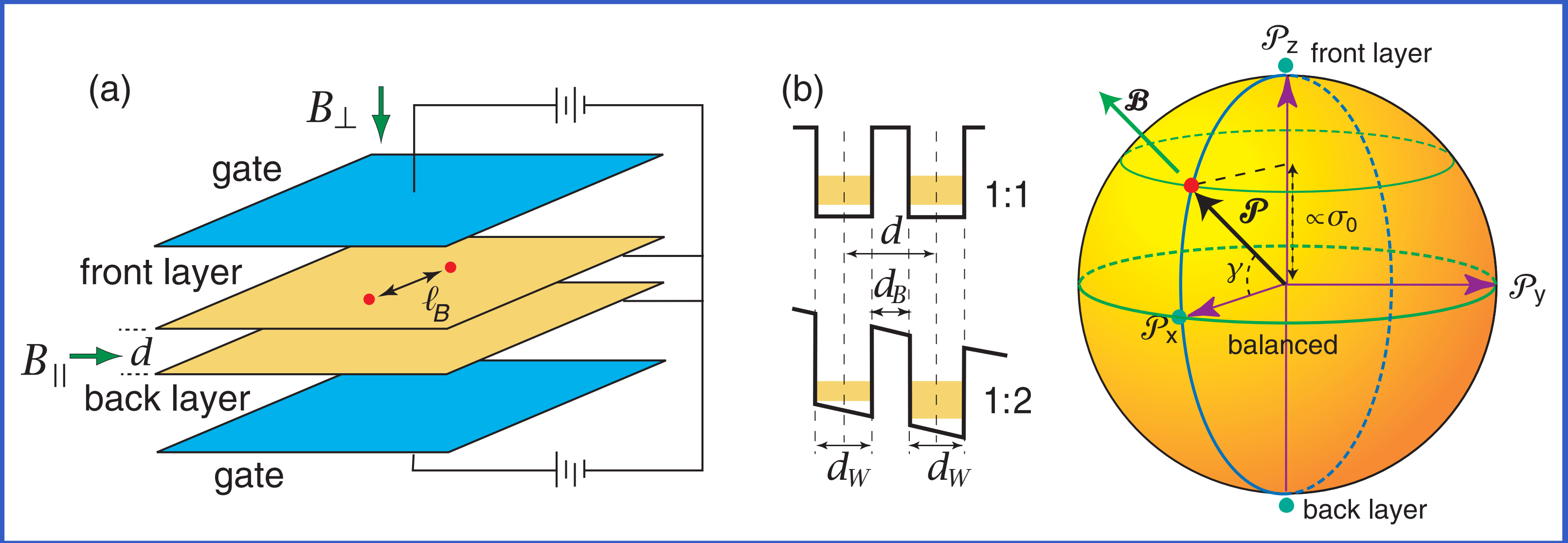


- Pseudospin (ppin) $\boldsymbol{P}$ is a powerful tool to elucidate bilayer system

$$H_{pz} = \sum_i \left[-\Delta_{SAS} P_x(i,i) - eV_{bias} P_z(i,i) \right]$$

$$\langle P_z \rangle = \frac{1}{2}\sigma_0, \quad \langle P_z \rangle = \frac{1}{2}\sqrt{1-\sigma_0^2}$$

- Imbalace parameter P_z is controlled experimentally

$$\sigma_0 \equiv \frac{\rho^{\text{front}} - \rho^{\text{back}}}{\rho^{\text{front}} + \rho^{\text{back}}} = 2P_z, \qquad eV_{\text{bias}} = \frac{\sigma_0}{\sqrt{1-\sigma_0^2}} \Delta_{SAS}$$

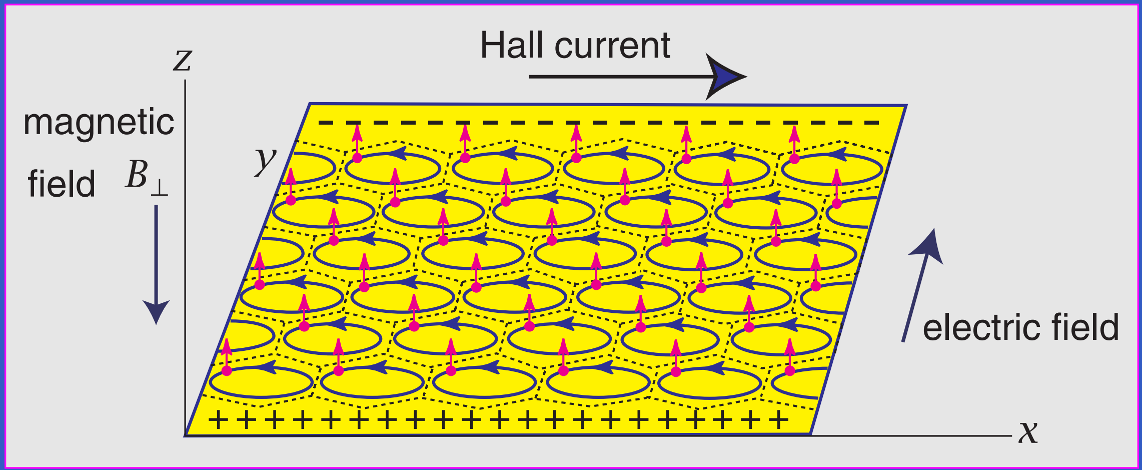
Quantum Hall Ferromagnets: Monolayer with SU(2) (review)

- **Coulomb Hamiltonian ⇛ LLL-projection ⇛ Landau-site Hamiltonian**
  - Coulomb interactions induce spin coherence

$$H_{\mathrm{C}} = \pi \int d^2q\, \rho(-\boldsymbol{p}) V(\boldsymbol{p}) \rho(\boldsymbol{p}) \quad \Rrightarrow$$

$$\boxed{H_{\mathrm{X}}^{\mathrm{spin}} = -\pi \int d^2p\, V_{\mathrm{X}}(\boldsymbol{p}) \left[\hat{\boldsymbol{S}}(-\boldsymbol{p}) \hat{\boldsymbol{S}}(\boldsymbol{p}) + \tfrac{1}{4} \hat{\rho}(-\boldsymbol{p}) \hat{\rho}(\boldsymbol{p}) \right]}$$

$$V_{\mathrm{X}}(\boldsymbol{p}) = \frac{\ell_B^2}{\pi} \int d^2k\, e^{-i\ell_B^2 \boldsymbol{p}\wedge\boldsymbol{k}} e^{-\ell_B^2 \boldsymbol{p}^2/2} V(\boldsymbol{k}), \qquad V(\boldsymbol{k}) = \frac{e^2}{4\pi\varepsilon|\boldsymbol{k}|}$$



Quantum Hall Ferromagnets: Spinless Bilayer with SU(2)

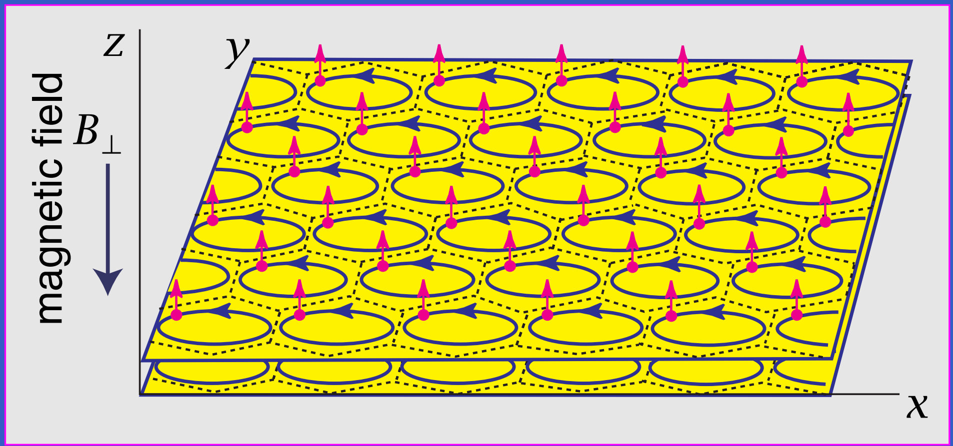
- Coulomb Hamiltonian ⇛ LLL-projection ⇛ Landau-site Hamiltonian
  - Coulomb interactions induce **interlayer coherence** in bilayer system

$$H_{\mathrm{C}} = \pi \int d^2q \, [\rho(-\boldsymbol{p})V^+(\boldsymbol{p})\rho(\boldsymbol{p}) + 2P_z(-\boldsymbol{p})V^-(\boldsymbol{p})P_z(\boldsymbol{p})] \quad \Rrightarrow$$

$$H_{\mathrm{X}}^{\mathrm{ppin}} = -\pi \int d^2p \, [V_{\mathrm{X}}^d(\boldsymbol{p}) \textstyle\sum_{a=xy} \hat{P}_a(-\boldsymbol{p})\hat{P}_a(\boldsymbol{p}) + 2V_{\mathrm{X}}^-(\boldsymbol{p})\hat{P}_z(-\boldsymbol{p})\hat{P}_z(\boldsymbol{p}) + \frac{1}{4}V_{\mathrm{X}}(\boldsymbol{p})\hat{\rho}(-\boldsymbol{p})\hat{\rho}(\boldsymbol{p})]$$

$$V_{\mathrm{X}}^{\pm}(\boldsymbol{p}) = \frac{\ell_B^2}{\pi}\int d^2k \, e^{-i\ell_B^2 \boldsymbol{p}\wedge\boldsymbol{k}} e^{-\ell_B^2\boldsymbol{p}^2/2} V^{\pm}(\boldsymbol{k}), \quad V^{\pm}(\boldsymbol{k}) = \frac{e^2}{8\pi\varepsilon|\boldsymbol{k}|}\left(1 \pm e^{-d/\ell_B}\right)$$

$$V_{\mathrm{X}}^{\pm}(\boldsymbol{p}) = \frac{1}{2}\left[V_{\mathrm{X}}(\boldsymbol{p}) \pm V_{\mathrm{X}}^d(\boldsymbol{p})\right]$$



Effective Hamiltonian in Spinless Bilayer with SU(2)

$$H_{\mathrm{X}}^{\mathrm{ppin}} = -\pi \int d^2p \, [V_{\mathrm{X}}^{d}(\boldsymbol{p}) \sum_{a=xy} \widehat{P}_a(-\boldsymbol{p})\widehat{P}_a(\boldsymbol{p}) + 2V_{\mathrm{X}}^{-}(\boldsymbol{p})\widehat{P}_z(-\boldsymbol{p})\widehat{P}_z(\boldsymbol{p}) + \frac{1}{4}V_{\mathrm{X}}(\boldsymbol{p})\widehat{\rho}(-\boldsymbol{p})\widehat{\rho}(\boldsymbol{p})]$$

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$$H_{\mathsf{X}}^{\mathsf{ppin}} = -\pi \int d^2p \, [V_{\mathsf{X}}^{d}(\boldsymbol{p}) \sum_{a=xy} \widehat{P}_a(-\boldsymbol{p})\widehat{P}_a(\boldsymbol{p}) + 2V_{\mathsf{X}}^{-}(\boldsymbol{p})\widehat{P}_z(-\boldsymbol{p})\widehat{P}_z(\boldsymbol{p}) + \frac{1}{4}V_{\mathsf{X}}(\boldsymbol{p})\widehat{\rho}(-\boldsymbol{p})\widehat{\rho}(\boldsymbol{p})]$$

Landau-site Hamiltonian ⇛ Derivative Expansion ⇛ Effective Field Theory

Effective Hamiltonian in Spinless Bilayer with SU(2)

$$H_{\mathrm{X}}^{\mathrm{ppin}} = -\pi \int d^2p \, [V_{\mathrm{X}}^d(\boldsymbol{p}) \textstyle\sum_{a=xy} \widehat{P}_a(-\boldsymbol{p})\widehat{P}_a(\boldsymbol{p}) + 2V_{\mathrm{X}}^-(\boldsymbol{p})\widehat{P}_z(-\boldsymbol{p})\widehat{P}_z(\boldsymbol{p}) + \tfrac{1}{4}V_{\mathrm{X}}(\boldsymbol{p})\widehat{\rho}(-\boldsymbol{p})\widehat{\rho}(\boldsymbol{p})]$$

Landau-site Hamiltonian ⇛ Derivative Expansion ⇛ Effective Field Theory

- Coulomb interactions induce **interlayer coherence** in bilayer system

$$H_{\mathrm{C}}^{\mathrm{ppin}} \simeq 2\int d^2x \left(\sum_{a=x,y} J_s^d [\partial_k \mathcal{P}_a(\boldsymbol{x})]^2 + J_s [\partial_k \mathcal{P}_z(\boldsymbol{x})]^2 \right)$$

$$\frac{J_s^d}{J_s} = -\sqrt{\frac{2}{\pi}}\frac{d}{\ell_B} + \left(1 + \frac{d^2}{\ell_B^2}\right) e^{d^2/2\ell_B^2} \mathrm{erfc}\left(d/\sqrt{2}\ell_B\right), \qquad J_s = \frac{1}{16\pi}\sqrt{\frac{\pi}{2}}\frac{e^2}{4\pi\varepsilon\ell_B}$$

Effective Hamiltonian in Spinless Bilayer with SU(2)

$$H_{\mathsf{X}}^{\mathsf{ppin}} = -\pi \int d^2p\, [V_{\mathsf{X}}^{d}(\boldsymbol{p}) \sum_{a=xy} \widehat{P}_a(-\boldsymbol{p})\widehat{P}_a(\boldsymbol{p}) + 2V_{\mathsf{X}}^{-}(\boldsymbol{p})\widehat{P}_z(-\boldsymbol{p})\widehat{P}_z(\boldsymbol{p}) + \tfrac{1}{4}V_{\mathsf{X}}(\boldsymbol{p})\widehat{\rho}(-\boldsymbol{p})\widehat{\rho}(\boldsymbol{p})]$$

Landau-site Hamiltonian ⇛ Derivative Expansion ⇛ Effective Field Theory

- Coulomb interactions induce **interlayer coherence** in bilayer system

$$H_{\mathsf{C}}^{\mathsf{ppin}} \simeq 2\int d^2x \left(\sum_{a=x,y} J_s^d [\partial_k \mathcal{P}_a(\boldsymbol{x})]^2 + J_s[\partial_k \mathcal{P}_z(\boldsymbol{x})]^2 \right)$$

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- Topological solitons **ppin-Skyrmions**: $\pi_2(\mathrm{CP}^1) = \mathbb{Z}$

  ⇛ flipping many peudospins coherently

Bilayer Quantum Hall Ferromagnets with SU(4)

- Physical Variables

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \qquad S_a(\mathbf{x}) = \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{spin}}\psi(\mathbf{x})$$
$$P_a(\mathbf{x}) = \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{ppin}}\psi(\mathbf{x}), \quad R_{ab}(\mathbf{x}) = \frac{1}{2}\psi^\dagger(\mathbf{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\psi(\mathbf{x})$$

with

$$\tau_x^{\text{spin}} = \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_x \end{pmatrix} \quad \tau_y^{\text{spin}} = \begin{pmatrix} \tau_y & 0 \\ 0 & \tau_y \end{pmatrix} \quad \tau_z^{\text{spin}} = \begin{pmatrix} \tau_z & 0 \\ 0 & \tau_z \end{pmatrix}$$
$$\tau_x^{\text{ppin}} = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad \tau_y^{\text{ppin}} = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix} \quad \tau_z^{\text{ppin}} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

Bilayer Quantum Hall Ferromagnets with SU(4)

- Physical Variables

$$\rho(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}), \qquad S_a(\boldsymbol{x}) = \tfrac{1}{2}\psi^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\psi(\boldsymbol{x})$$
$$P_a(\boldsymbol{x}) = \tfrac{1}{2}\psi^\dagger(\boldsymbol{x})\tau_a^{\text{ppin}}\psi(\boldsymbol{x}), \quad R_{ab}(\boldsymbol{x}) = \tfrac{1}{2}\psi^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\psi(\boldsymbol{x})$$

with

$$\tau_x^{\text{spin}} = \begin{pmatrix} \tau_x & 0 \\ 0 & \tau_x \end{pmatrix} \quad \tau_y^{\text{spin}} = \begin{pmatrix} \tau_y & 0 \\ 0 & \tau_y \end{pmatrix} \quad \tau_z^{\text{spin}} = \begin{pmatrix} \tau_z & 0 \\ 0 & \tau_z \end{pmatrix}$$
$$\tau_x^{\text{ppin}} = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad \tau_y^{\text{ppin}} = \begin{pmatrix} 0 & -i1_2 \\ i1_2 & 0 \end{pmatrix} \quad \tau_z^{\text{ppin}} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

- Effective Hamiltonian describes SU(4) coherence with **SU(4) Skyrmions**

$$\mathcal{H}_{\text{X}} = J_s^d\left(\sum[\partial_k\mathcal{S}_a]^2 + [\partial_k\mathcal{P}_a]^2 + [\partial_k\mathcal{R}_{ab}]^2\right)$$
$$+2J_s^-\left(\sum[\partial_k\mathcal{S}_a]^2 + [\partial_k\mathcal{P}_z]^2 + [\partial_k\mathcal{R}_{az}]^2\right)$$

CP^3 Skyrmions in Bilayer Quantum Hall Ferromagnets

- Charge-isospin separation ⇒ CP^3 field $\boldsymbol{n}$

$$
\begin{aligned}
\rho(\boldsymbol{x}) &= \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}), & \mathcal{S}_a(\boldsymbol{x}) &= \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\boldsymbol{n}(\boldsymbol{x}) \\
\mathcal{P}_a(\boldsymbol{x}) &= \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{ppin}}\boldsymbol{n}(\boldsymbol{x}), & \mathcal{R}_{ab}(\boldsymbol{x}) &= \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\boldsymbol{n}(\boldsymbol{x})
\end{aligned}
$$

CP^3 Skyrmions in Bilayer Quantum Hall Ferromagnets

- Charge-isospin separation ⇛ CP^3 field $\boldsymbol{n}$

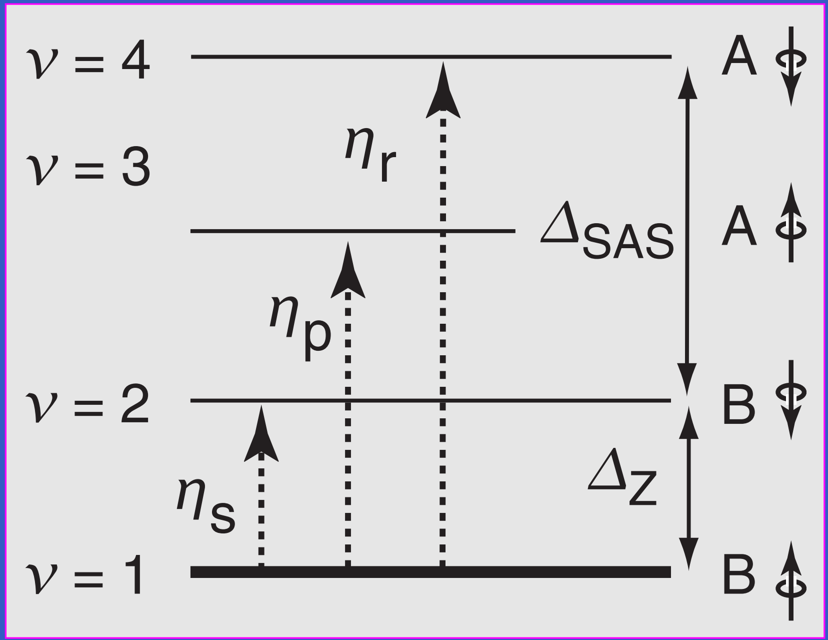
$$\rho(\boldsymbol{x}) = \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}), \qquad \mathcal{S}_a(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\boldsymbol{n}(\boldsymbol{x})$$

$$\mathcal{P}_a(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{ppin}}\boldsymbol{n}(\boldsymbol{x}), \qquad \mathcal{R}_{ab}(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{n}^\dagger(\boldsymbol{x})\tau_a^{\text{spin}}\tau_a^{\text{ppin}}\boldsymbol{n}(\boldsymbol{x})$$

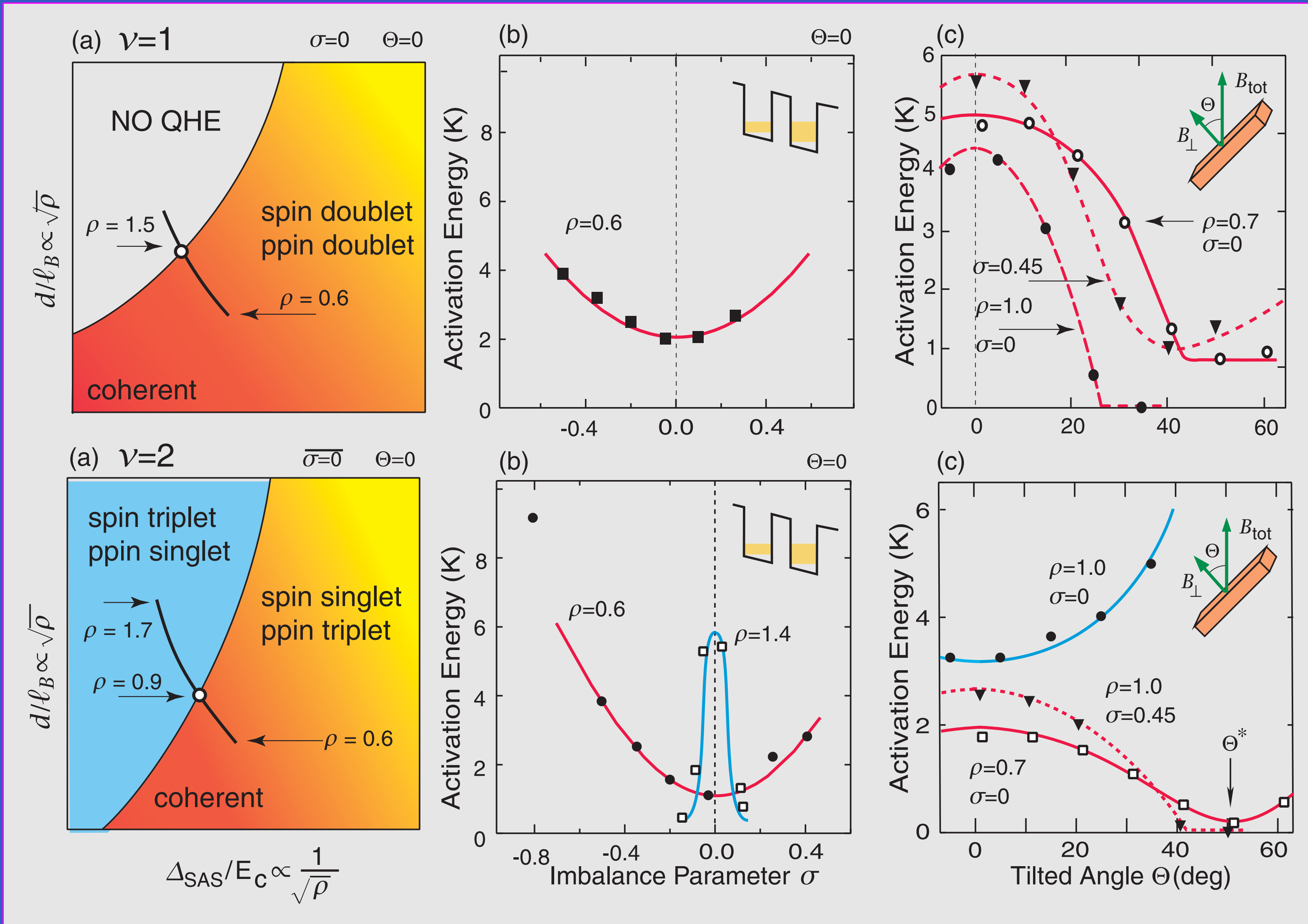
- CP^3 Skyrmions

$$\begin{pmatrix} n^{\mathrm{A}\downarrow} \\ n^{\mathrm{A}\uparrow} \\ n^{\mathrm{B}\downarrow} \\ n^{\mathrm{B}\uparrow} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{\mathrm{g}} \Rightarrow \begin{pmatrix} \kappa_{\mathrm{r}} \\ \kappa_{\mathrm{p}} \\ \kappa_{\mathrm{s}} \\ z \end{pmatrix}_{\mathrm{sky}}$$

$$\begin{pmatrix} n^{\mathrm{f}\uparrow}(\boldsymbol{x}) \\ n^{\mathrm{f}\downarrow}(\boldsymbol{x}) \\ n^{\mathrm{b}\uparrow}(\boldsymbol{x}) \\ n^{\mathrm{b}\downarrow}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} z\sqrt{1+\sigma_0} + \kappa_p\sqrt{1-\sigma_0} \\ \kappa_s\sqrt{1+\sigma_0} + \kappa_r\sqrt{1-\sigma_0} \\ z\sqrt{1-\sigma_0} - \kappa_p\sqrt{1+\sigma_0} \\ \kappa_s\sqrt{1-\sigma_0} - \kappa_r\sqrt{1+\sigma_0} \end{pmatrix}$$



Experiments in Bilayer QH States ($\nu = 1\&2$)



✔Sawada et al, PRL80(1998)4534

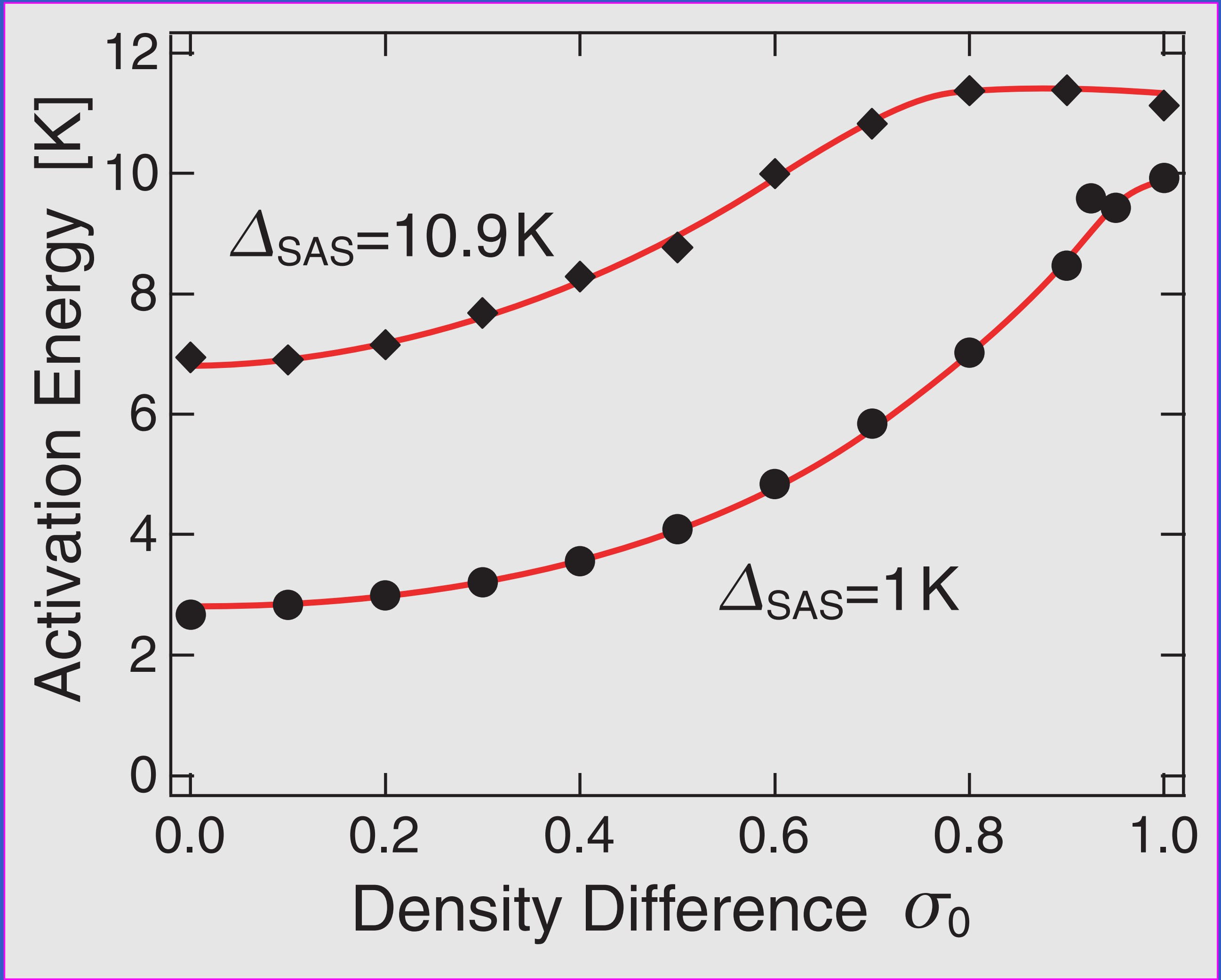
Experimental Results I ($\nu = 1$)

- Bilayer QH system
  $\Rrightarrow$ **Isospin SU(4)**
  ✔Ezawa, PRL82(1999)3512

- Zeeman effect
  $\Rrightarrow$ **Spin $\mathcal{S}_z$**

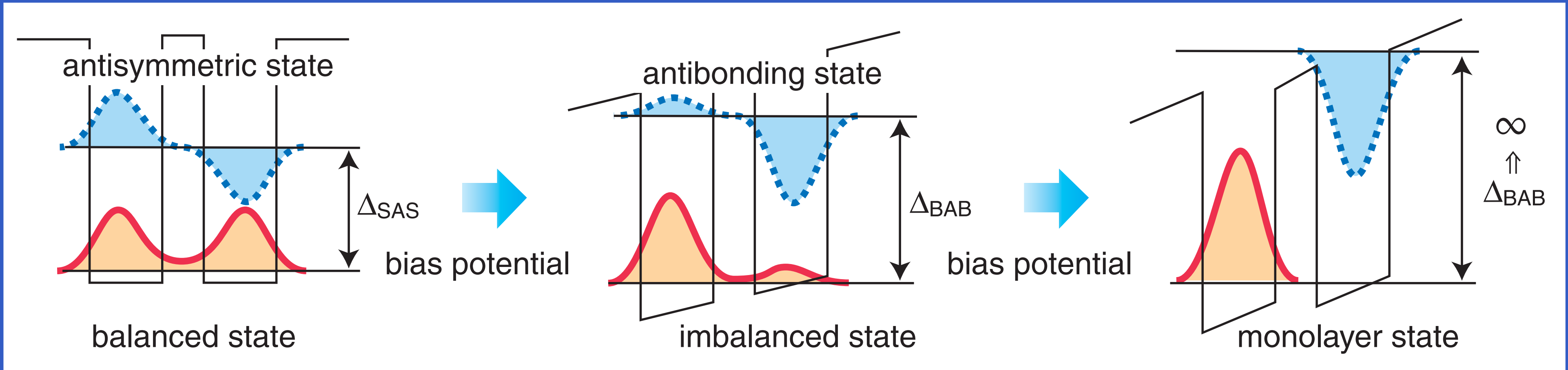
- Bias voltage
  $\Rrightarrow$ **Pseudospin $\mathcal{P}_z$**

Data by changing the density balance



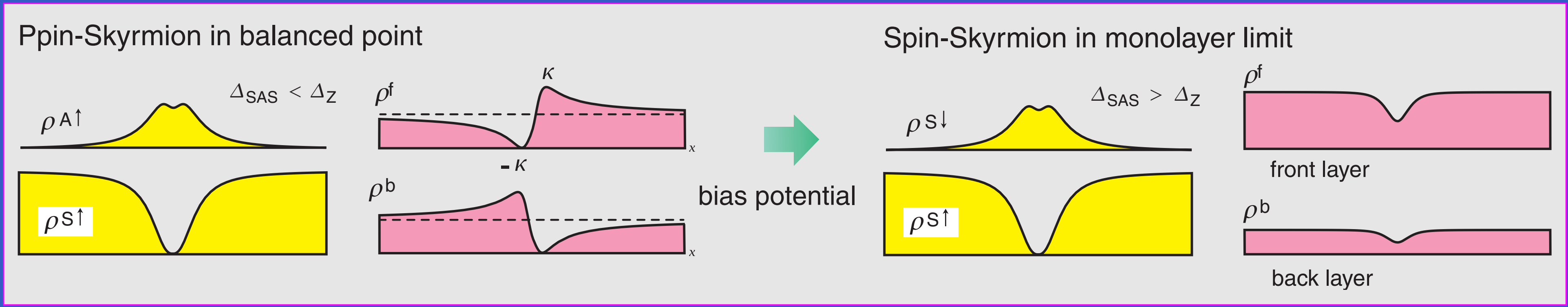
✔Terasawa et al. Physica E (2004)52

Bilayer to Monolayer QH Systems ($\nu = 1$)



- System is controled continuously from balanced point to monolayer limit
  - Pseudopin textures in bilayer $\Rrightarrow$ Spin textures in monolayer
  - Pseudospin SU(2) $\Rrightarrow$ SU(4) $\Rrightarrow$ Spin SU(2)
  - Ppin CP^1 Skyrmion $\Rrightarrow$ CP^3 Skyrmion $\Rrightarrow$ Spin CP^1 Skyrmion
  - Topological charge is the same: : $\pi_2(CP^1) = \pi_2(CP^3) = \mathbb{Z}$
- imbalance parameter: $\sigma_0 \equiv \dfrac{\rho^{\text{front}} - \rho^{\text{back}}}{\rho^{\text{front}} + \rho^{\text{back}}} = 2\mathcal{P}_z$

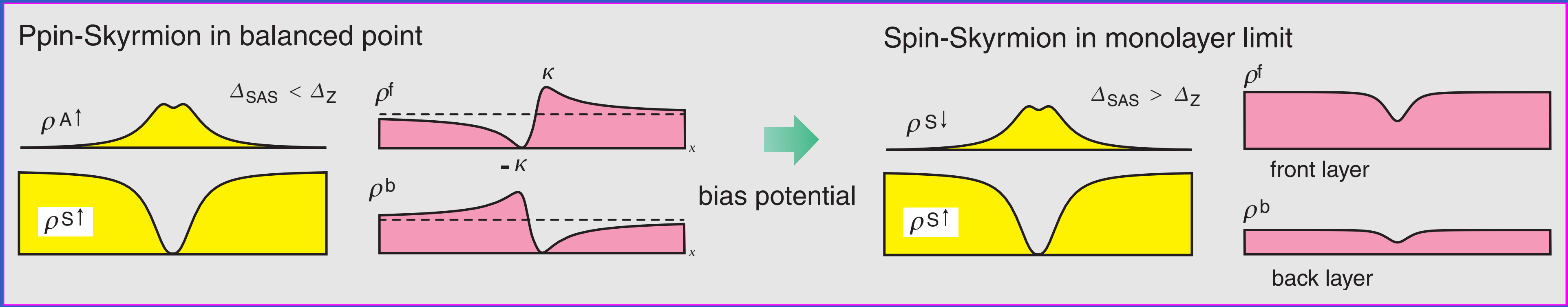
CP^3 Skyrmion vs Imbalance Parameter ($\nu = 1$)



- **CP^3 Skyrmion** interpolates ppin Skyrmion to spin Skyrmion continuously

$$n_{\text{sky}}(\boldsymbol{x}) = \begin{pmatrix} n^{A\downarrow} \\ n^{A\uparrow} \\ n^{B\downarrow} \\ n^{B\uparrow} \end{pmatrix} = \begin{pmatrix} \kappa_r \\ \kappa_p \\ \kappa_s \\ z \end{pmatrix}, \qquad \begin{pmatrix} n^{f\uparrow}(\boldsymbol{x}) \\ n^{f\downarrow}(\boldsymbol{x}) \\ n^{b\uparrow}(\boldsymbol{x}) \\ n^{b\downarrow}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} z\sqrt{1+\sigma_0} + \kappa_p\sqrt{1-\sigma_0} \\ \kappa_s\sqrt{1+\sigma_0} + \kappa_r\sqrt{1-\sigma_0} \\ z\sqrt{1-\sigma_0} - \kappa_p\sqrt{1+\sigma_0} \\ \kappa_s\sqrt{1-\sigma_0} - \kappa_r\sqrt{1+\sigma_0} \end{pmatrix}$$

CP^3 Skyrmion vs Imbalance Parameter ($\nu = 1$)



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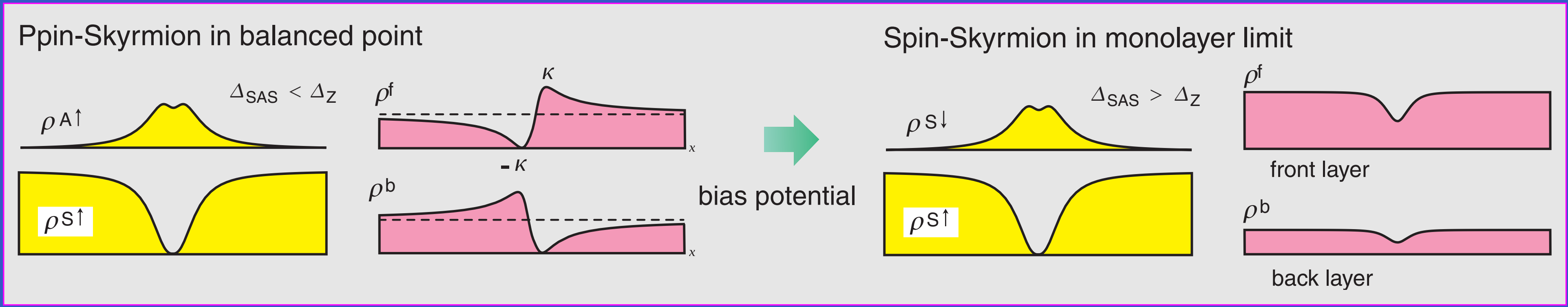
- ppin CP^1 Skyrmion ($\kappa_{\text{spin}} = \kappa_{\text{res}} = 0; \kappa_{\text{ppin}} \neq 0$) is excited at balanced point

CP^3 Skyrmion vs Imbalance Parameter ($\nu = 1$)



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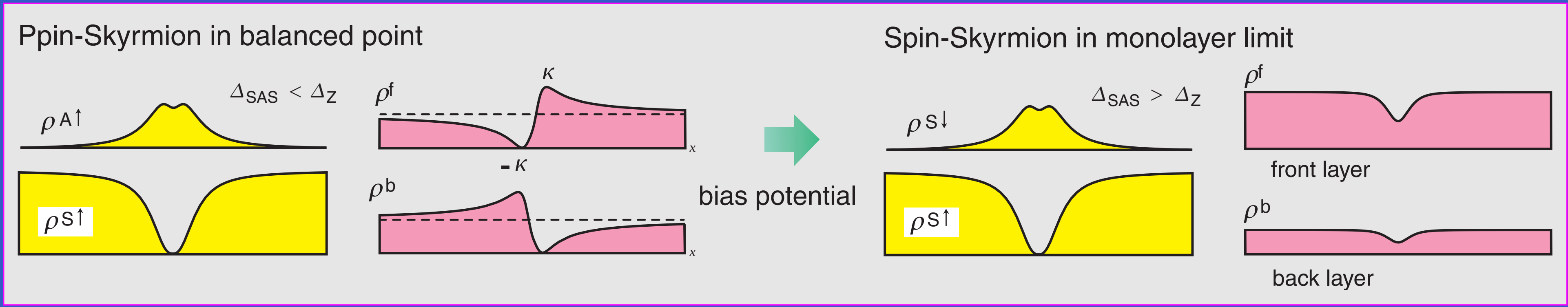
- ppin CP^1 Skyrmion ($\kappa_{\text{spin}} = \kappa_{\text{res}} = 0; \kappa_{\text{ppin}} \neq 0$) is excited at balanced point

- isospin CP^3 Skyrmion ($\kappa_{\text{spin}} \neq 0; \kappa_{\text{ppin}} \neq 0; \kappa_{\text{res}} \simeq 0$) is excited in general

CP^3 Skyrmion vs Imbalance Parameter ($\nu = 1$)

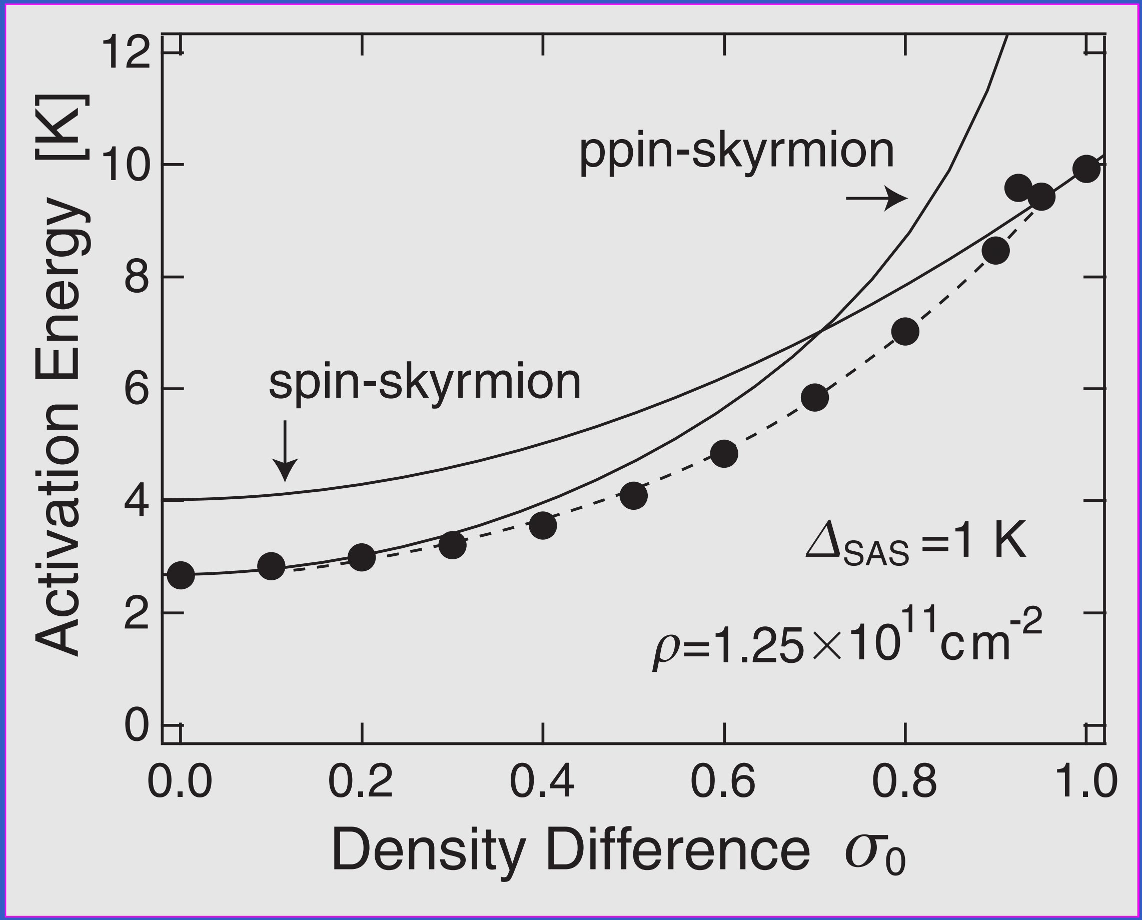


- CP^3 Skyrmion interpolates ppin Skyrmion to spin Skyrmion continuously

$$n_{\text{sky}}(\boldsymbol{x}) = \begin{pmatrix} n^{A\downarrow} \\ n^{A\uparrow} \\ n^{B\downarrow} \\ n^{B\uparrow} \end{pmatrix} = \begin{pmatrix} \kappa_r \\ \kappa_p \\ \kappa_s \\ z \end{pmatrix}, \quad \begin{pmatrix} n^{f\uparrow}(\boldsymbol{x}) \\ n^{f\downarrow}(\boldsymbol{x}) \\ n^{b\uparrow}(\boldsymbol{x}) \\ n^{b\downarrow}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} z\sqrt{1+\sigma_0} + \kappa_p\sqrt{1-\sigma_0} \\ \kappa_s\sqrt{1+\sigma_0} + \kappa_r\sqrt{1-\sigma_0} \\ z\sqrt{1-\sigma_0} - \kappa_p\sqrt{1+\sigma_0} \\ \kappa_s\sqrt{1-\sigma_0} - \kappa_r\sqrt{1+\sigma_0} \end{pmatrix}$$

- ppin CP^1 Skyrmion ($\kappa_{\text{spin}} = \kappa_{\text{res}} = 0; \kappa_{\text{ppin}} \neq 0$) is excited at balanced point
- isospin CP^3 Skyrmion ($\kappa_{\text{spin}} \neq 0; \kappa_{\text{ppin}} \neq 0; \kappa_{\text{res}} \simeq 0$) is excited in general
- spin CP^1 Skyrmion ($\kappa_{\text{ppin}} = \kappa_{\text{res}} = 0; \kappa_{\text{spin}} \neq 0$) is excited in monolayer limit

Activation Energy vis Density Difference ($\nu = 1$)

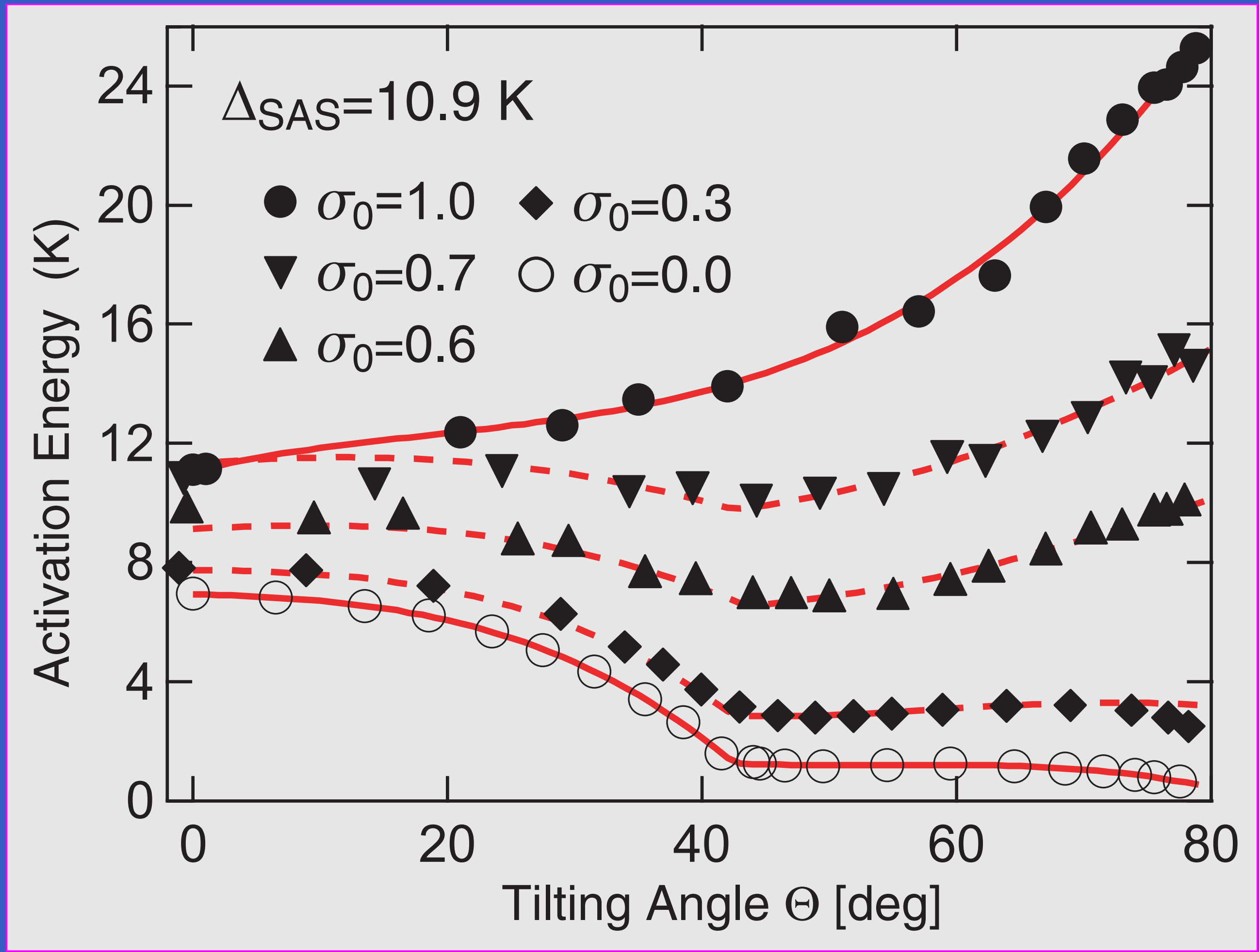


- ppin CP^1 Skyrmion ($\kappa_{spin} = \kappa_{res} = 0; \kappa_{ppin} \neq 0$) is excited at balanced point
- isospin CP^3 Skyrmion ($\kappa_{spin} \neq 0; \kappa_{ppin} \neq 0; \kappa_{res} \simeq 0$) is excited in general
- spin CP^1 Skyrmion ($\kappa_{ppin} = \kappa_{res} = 0; \kappa_{spin} \neq 0$) is excited in monolayer limit

Experimental Results II ($\nu = 1$)

- Bilayer QH system
  $\Rrightarrow$ **Isospin SU(4)**
  ✔Ezawa, PRL82(1999)3512

Data by applying the parallel field

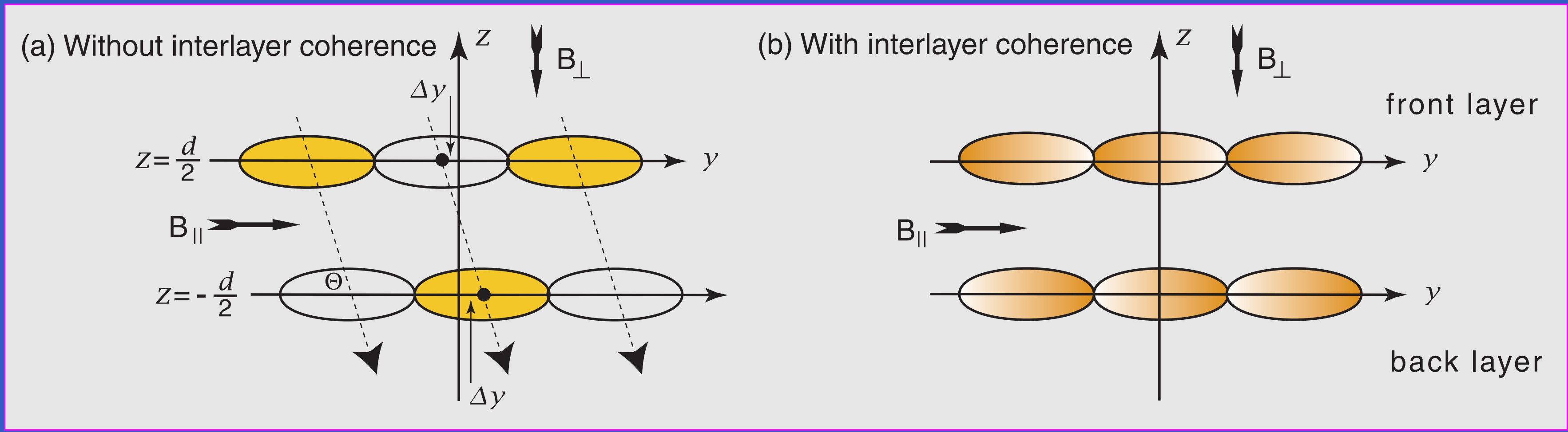


✔Terasawa et al. Physica E (2004)52

- Zeeman effect
  $\Rrightarrow$ **Spin $\mathcal{S}_z$**

- Bias voltage
  $\Rrightarrow$ **Pseudospin $\mathcal{P}_z$**

Parallel Magnetic Field in Bilayer System



- The effect of parallel magnetic field

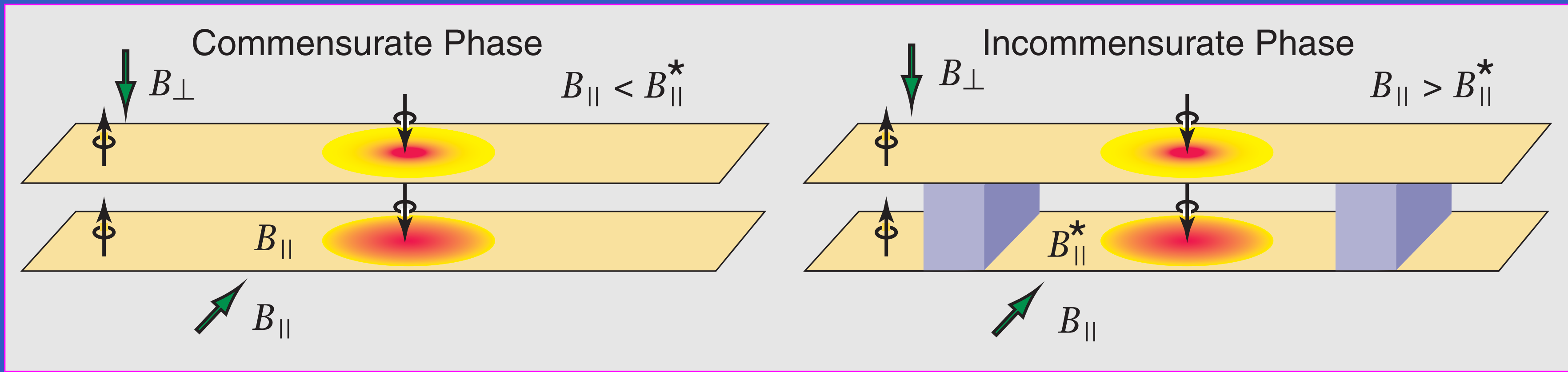
$$\psi^{\alpha}(\boldsymbol{x}; B_{\|}) = \exp\left[\mp(y - \bar{y}_k^0)\delta_{\mathrm{m}} - \frac{1}{8}\delta_{\mathrm{m}}^2 \ell_B^2\right]\psi^{\alpha}(\boldsymbol{x})$$ without coherence

$$\psi^{\alpha}(\boldsymbol{x}; B_{\|}) = \exp\left(\mp\frac{i}{2}\delta_{\mathrm{m}} x\right)\psi^{\alpha}(\boldsymbol{x})$$ with coherence

- CP^3 Skyrmion is modified only by phase in coherent phase

$$n_{\mathrm{sky}}(x; B_{\|}) = \exp\left(\mp\tfrac{i}{2}\delta_{\mathrm{m}} x\right) n_{\mathrm{sky}}(x), \qquad \delta_{\mathrm{m}} = \frac{edB_{\|}}{\hbar}$$

Activation Energy of CP^3 Skyrmions ($\nu = 1$)



- Excitations require the exchange, Coulomb, Zeeman and tunneling energies

$$E_{\text{sky}} = E_{\text{X}}^{\Theta=0} + E_{\text{self}}^{+}(\kappa) + \frac{1}{2}\varepsilon_{\text{cap}}(1-\sigma_0^2)N_{\text{ppin}}(\kappa_p) + N_{\text{spin}}(\kappa_s)\Delta_{\text{Z}}^{\Theta} + N_{\text{ppin}}(\kappa_p)\Delta_{\text{SAS}}^{\Theta}$$

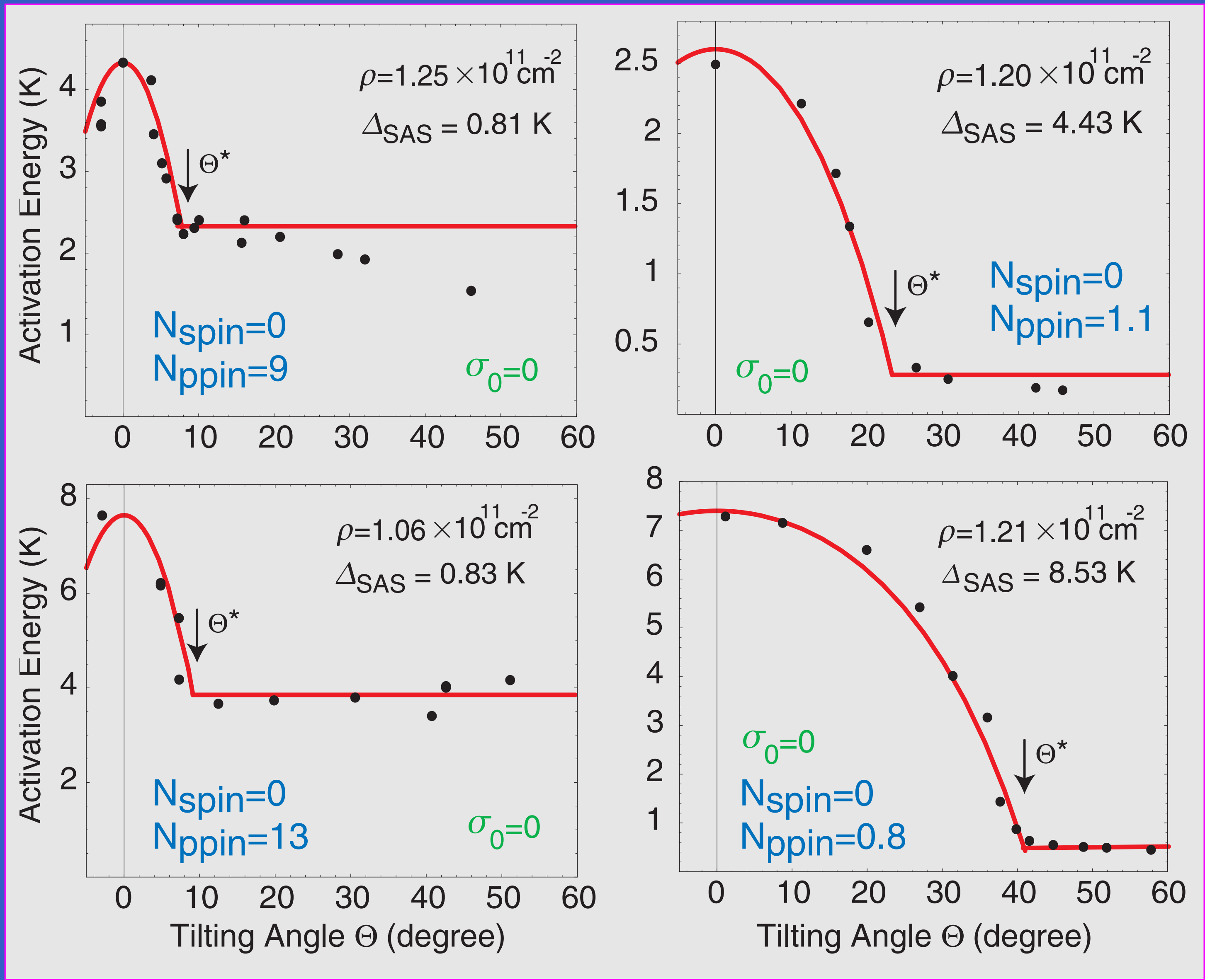
$$\Delta_{\text{Z}}^{\Theta} = g^*\mu_B B_\perp\sqrt{1+\tan^2\Theta}, \quad N_{\text{spin}}(\kappa_s) \propto \kappa_s^2, \quad N_{\text{ppin}}(\kappa_s) \propto \kappa_p^2, \quad \kappa^2 \equiv \kappa_s^2 + \kappa_p^2$$

$$\Delta_{\text{SAS}}^{\Theta} = \begin{cases} \frac{1}{\sqrt{1-\sigma_0^2}}\Delta_{\text{SAS}} - \frac{2\pi d^2 J_s^d}{\ell_B^2}(1-\sigma_0^2)\tan^2\Theta & \text{for} \quad \Theta < \Theta^* \\ \frac{1}{\sqrt{1-\sigma_0^2}}\Delta_{\text{SAS}} - \frac{2\pi d^2 J_s^d}{\ell_B^2}(1-\sigma_0^2)\tan^2\Theta^* & \text{for} \quad \Theta > \Theta^* \end{cases}$$

where Θ^* is commensurate-incommensurate transition point, $\tan\Theta^* = B_{||}^*/B_\perp$

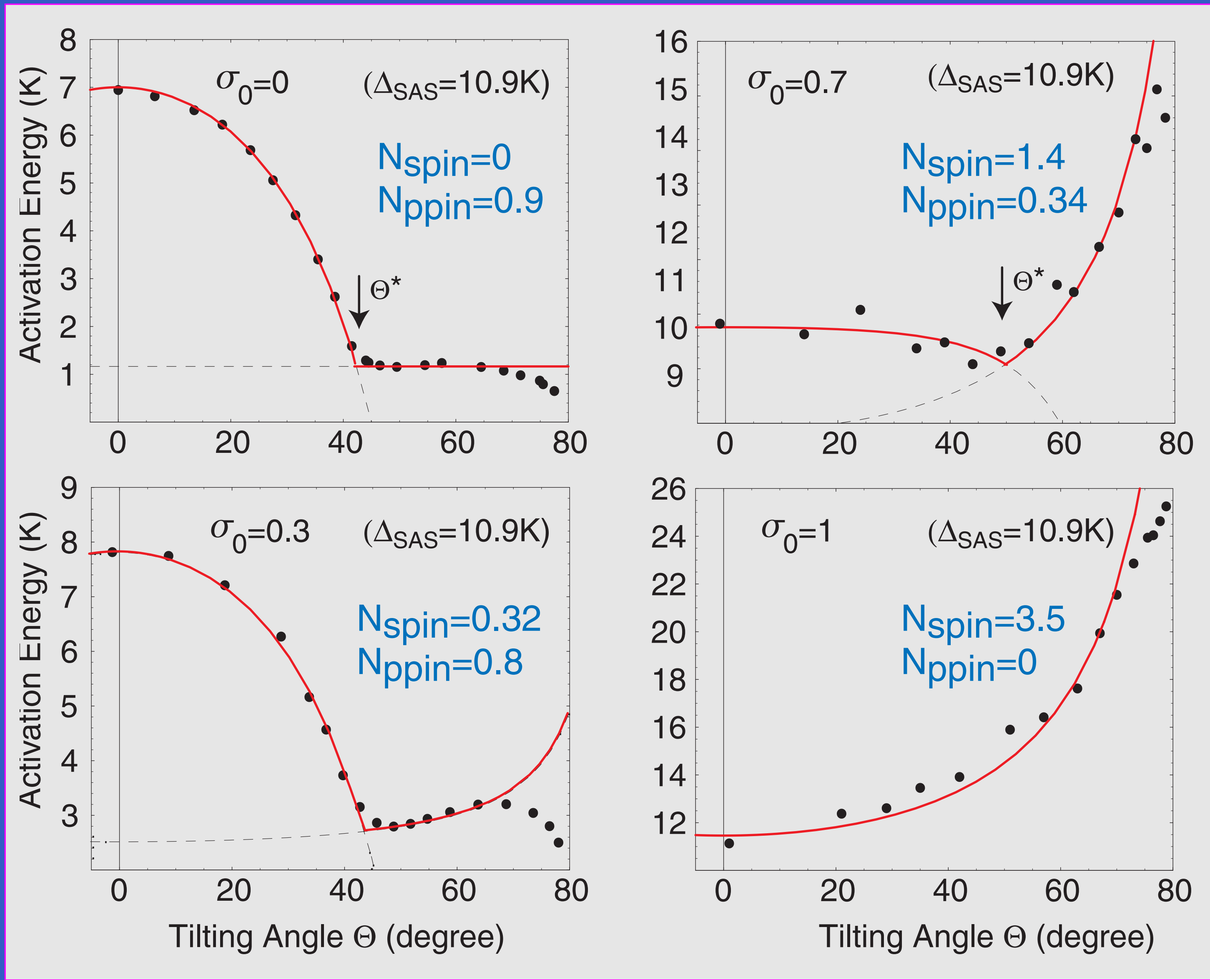
✔Ezawa et al. PRB70(2004)125304

Excitation of Ppin CP^1 Skyrmions ($\nu = 1$)



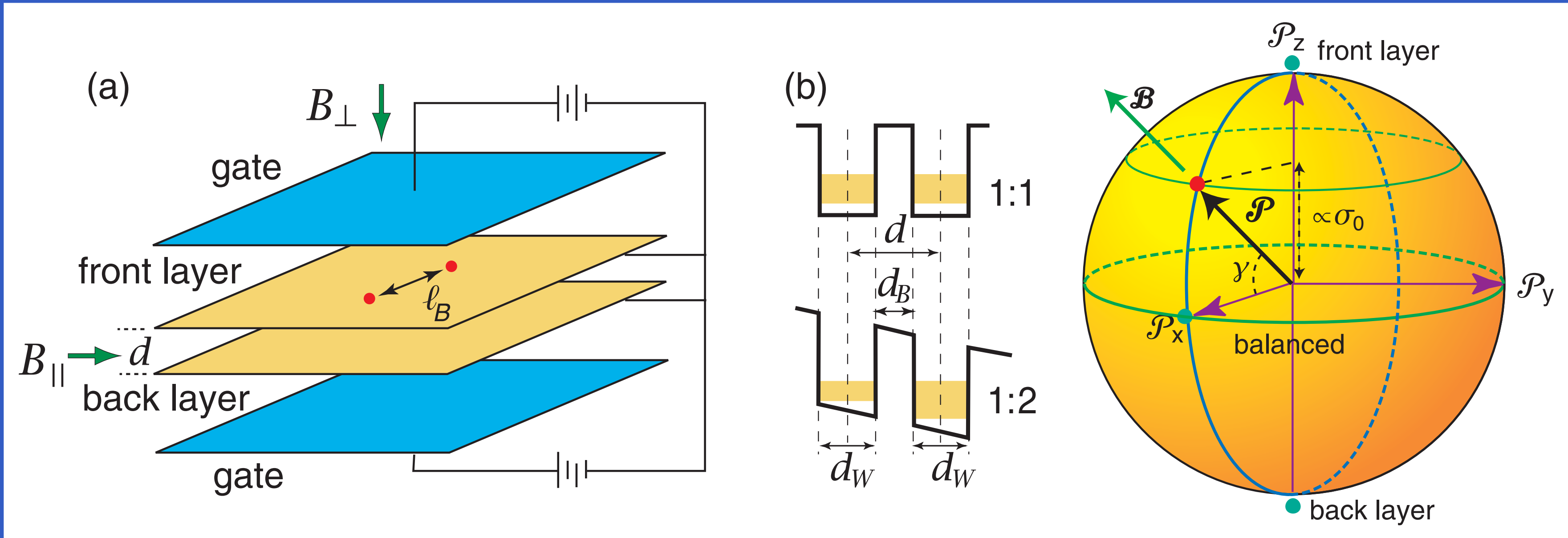
Murphy et al., PRL72(1994)728

Excitation of CP^3 Skyrmions ($\nu = 1$)

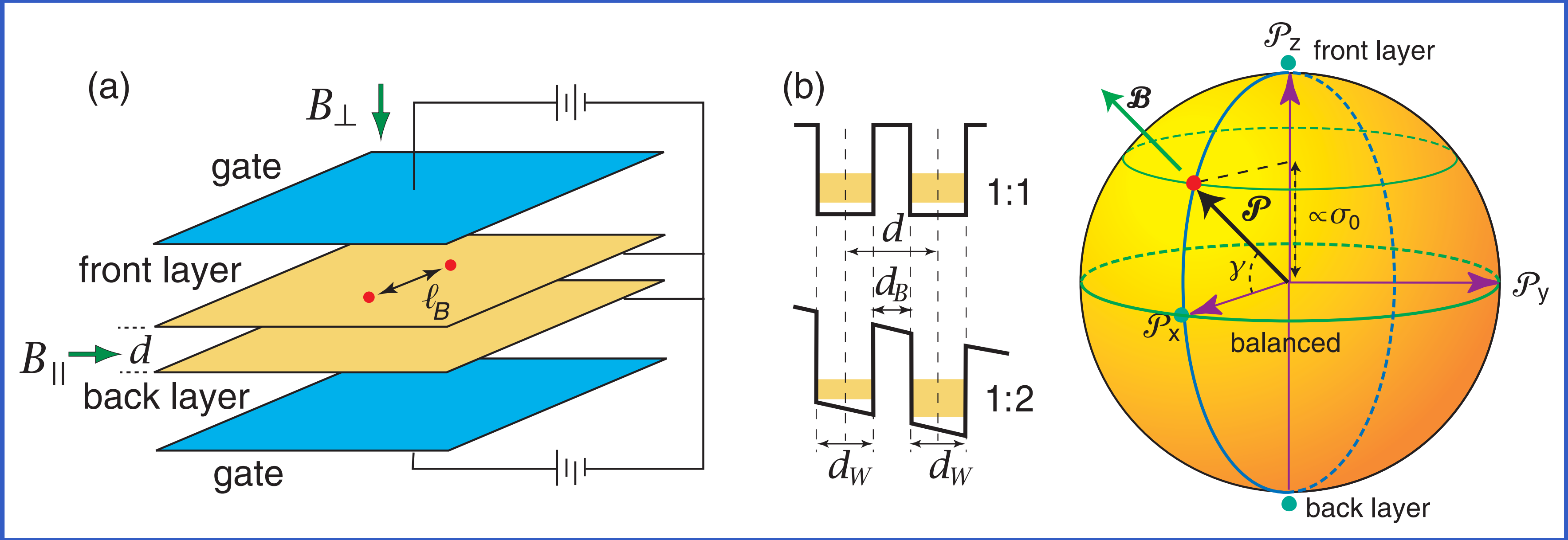


✔Terasawa et al. Physica E (2004)52

Bilayer QH States ($\nu = 2$)

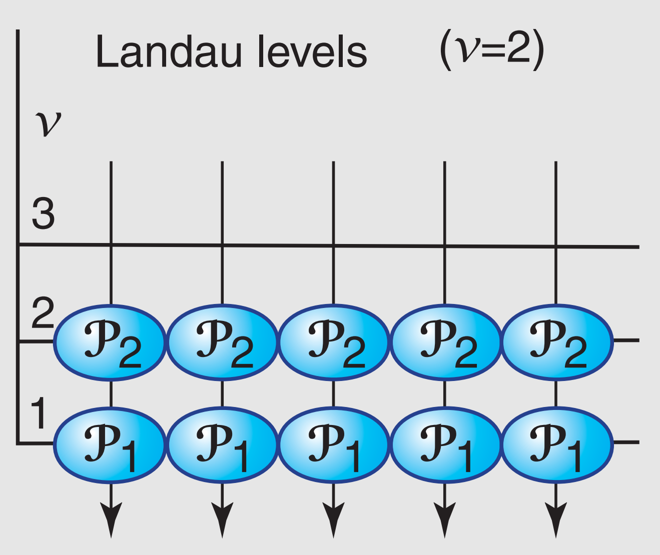


Bilayer QH States ($\nu = 2$)



- Composition of two spins (two pseudospins)

$$\nu = 2 : \quad 2 \otimes 2 = 1 \oplus 3$$



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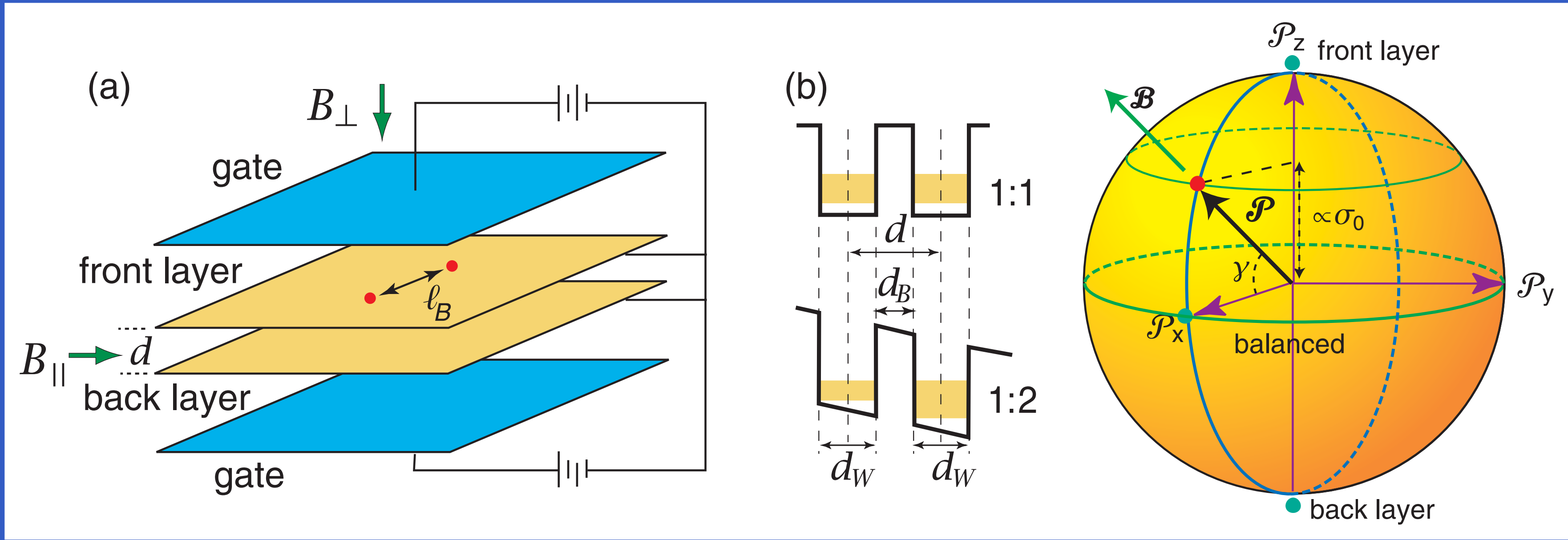

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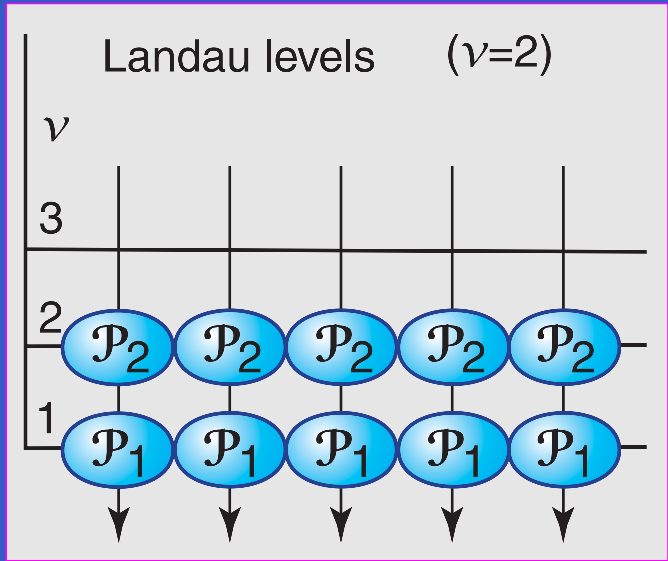
- Two types of QH states
  - spin-singlet and ppin-triplet **ppin-phase**
  - spin-triplet and ppin-singlet **spin-phase**



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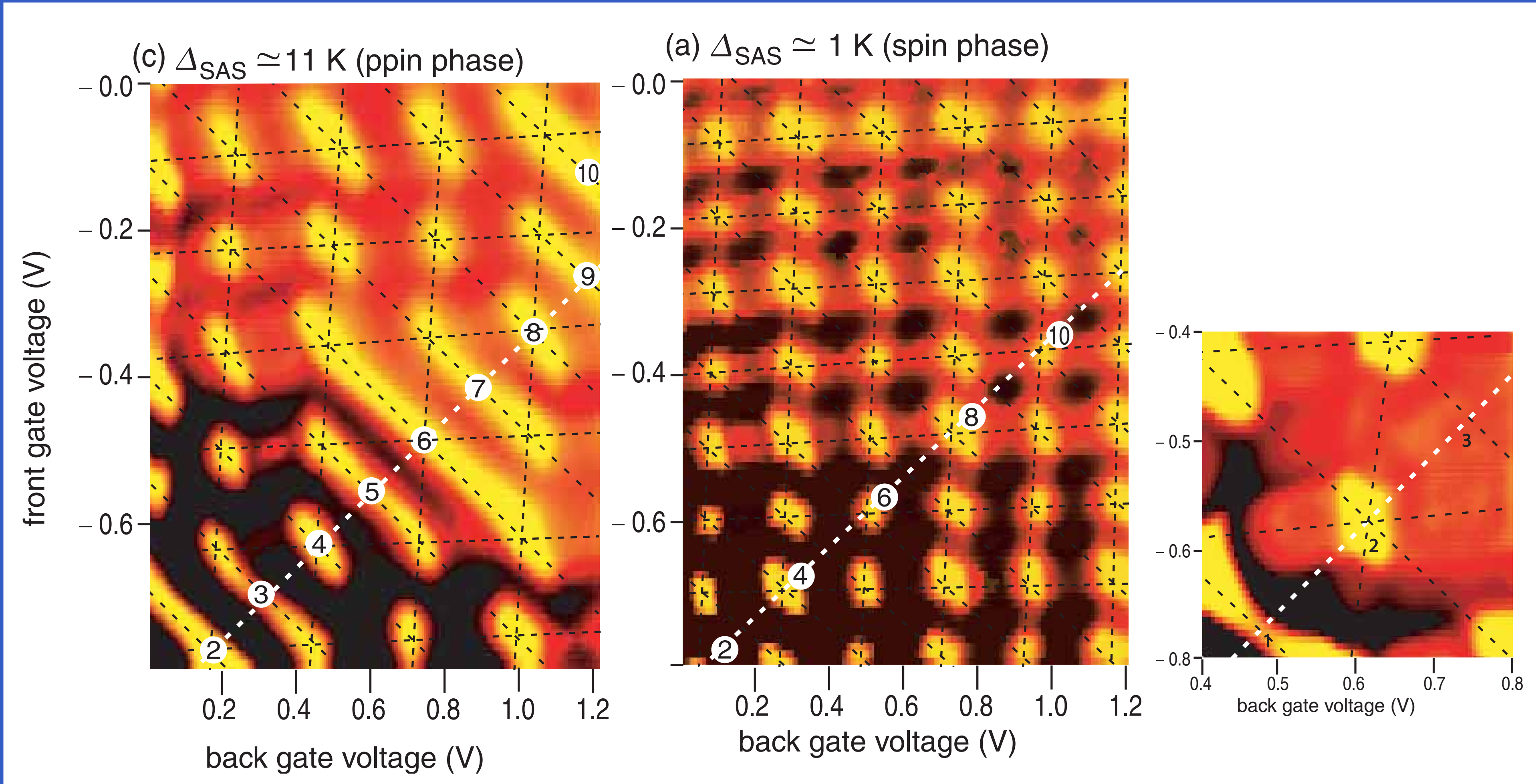

✔ Muraki et al, SSC 112(1999)625

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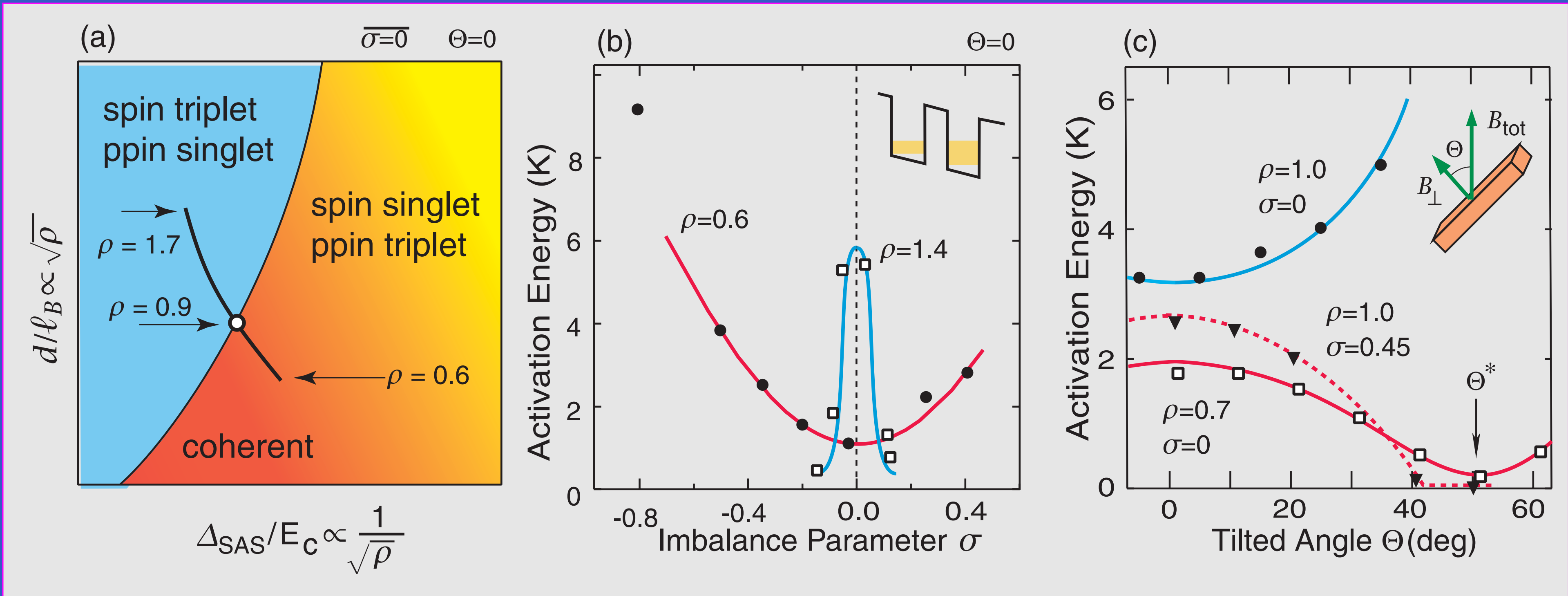

- Two types of QH states
  - spin-singlet and ppin-triplet **ppin-phase**
  - spin-triplet and ppin-singlet **spin-phase**
- Is the spin-phase unique?
  - Is it an uncorrelated two-monolayer system or a genuine bilayer system

Grassmannian Soliton

- In zero tunnelling gap (two electrons distinguishable)
  - Two layers behave independently $\Rrightarrow$ two CP^1 fields
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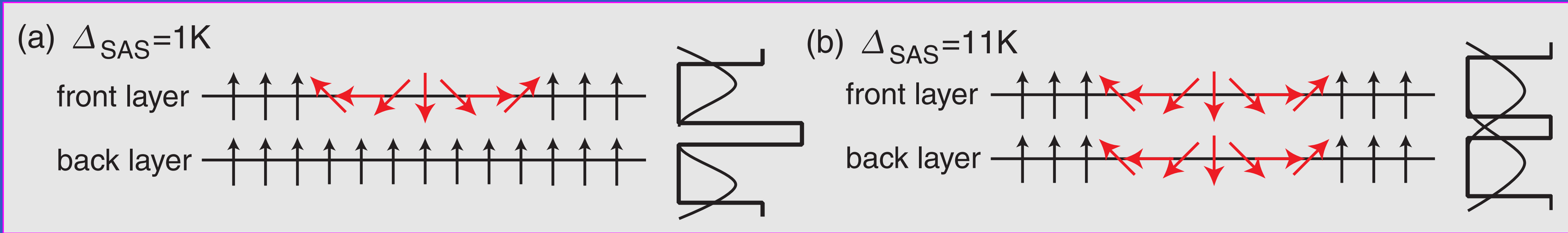

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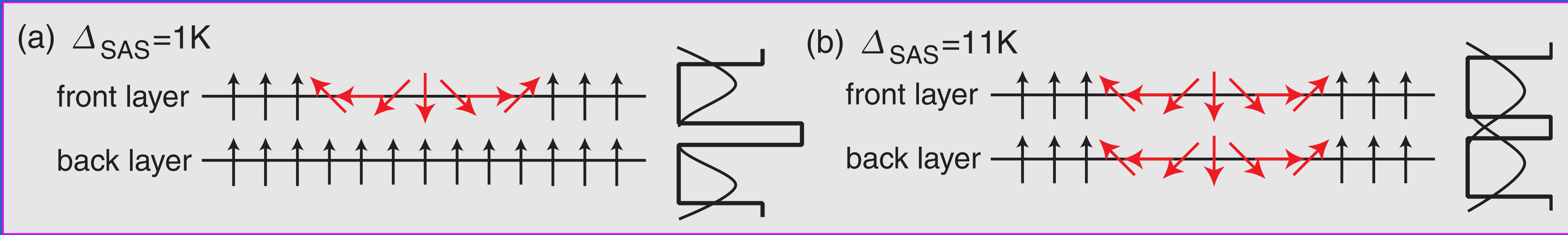
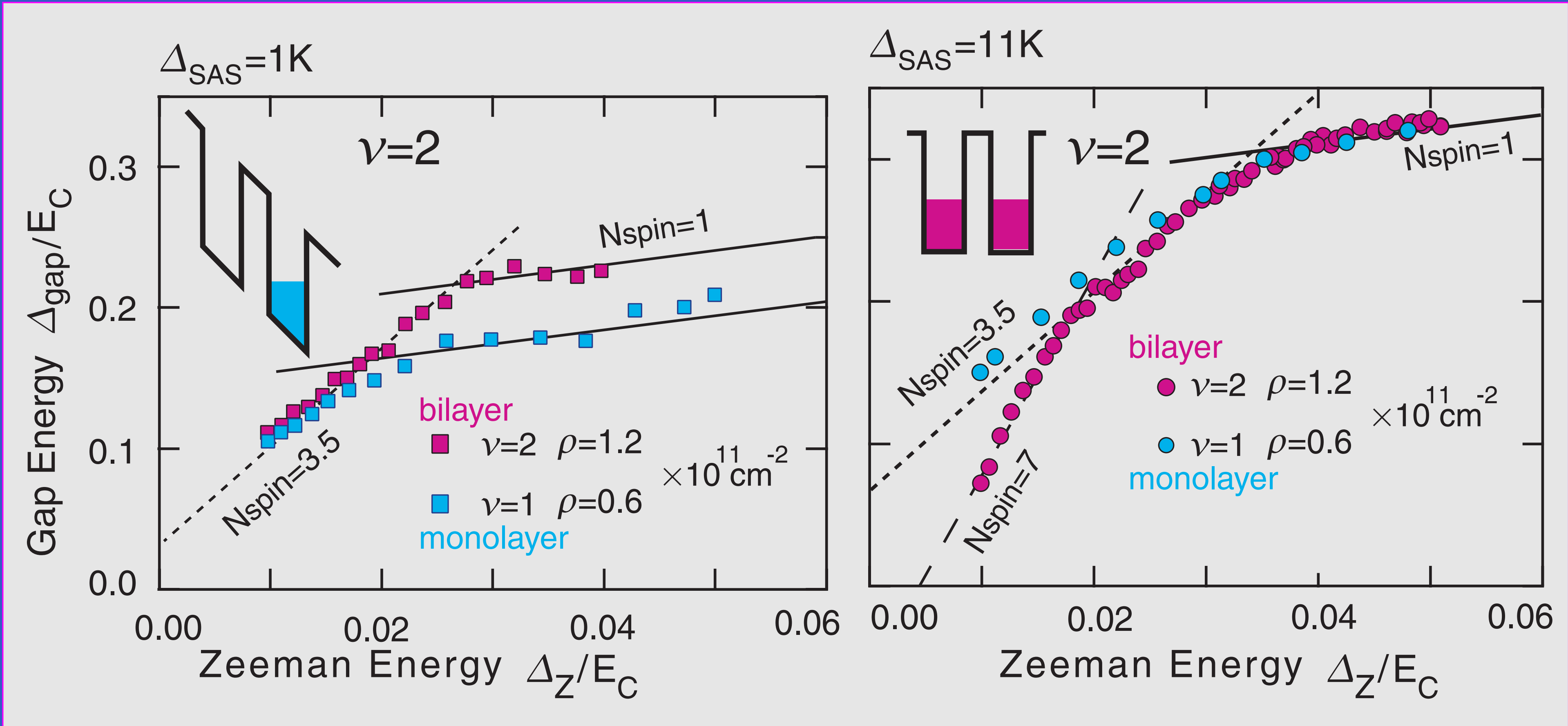


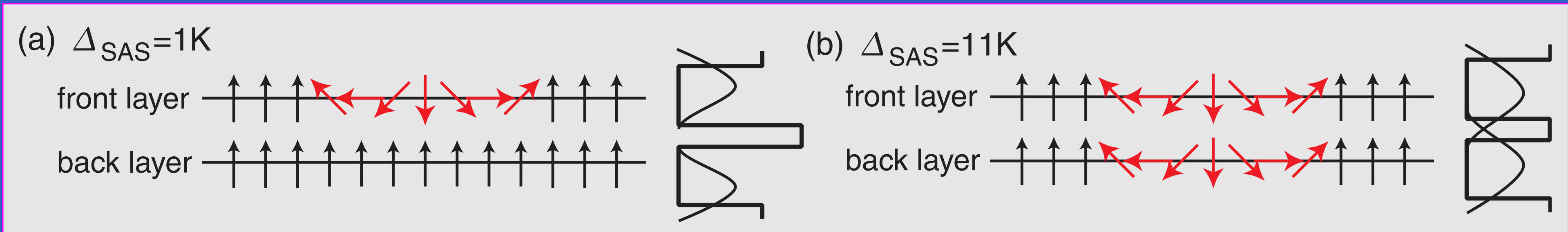
- In large tunnelling gap (two electrons indistinguishable)
  - Two layers behave coherently $\Rrightarrow$ one Grassmannian $G^{4,2}$ field
  - one $G^{4,2}$ Skyrmion with charge $2e$ is excited
  - flipped spin number twice as much as that of CP^1 Skyrmion
  - Grassmannian solitons arise based on

$$\pi_2(G^{N,k}) = \mathbb{Z}, \quad G^{N,k} = SU(N)/[U(1) \otimes SU(N) \otimes SU(N-k)$$

CP^1 versus $G^{4,2}$ Skyrmions (experiment)

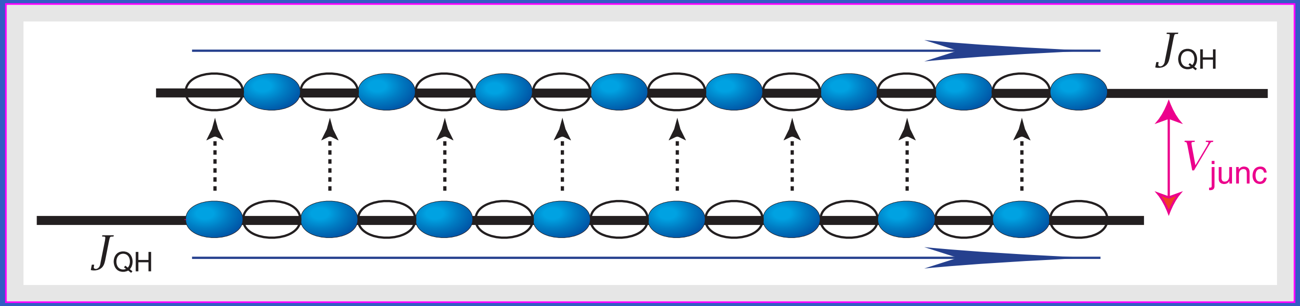


✔Kumada et al, JPSJ69(2000)3178



Josephson-like Effects

- **Josephson tunneling current** (predicted by Ezawa&Iwazaki, 1992)



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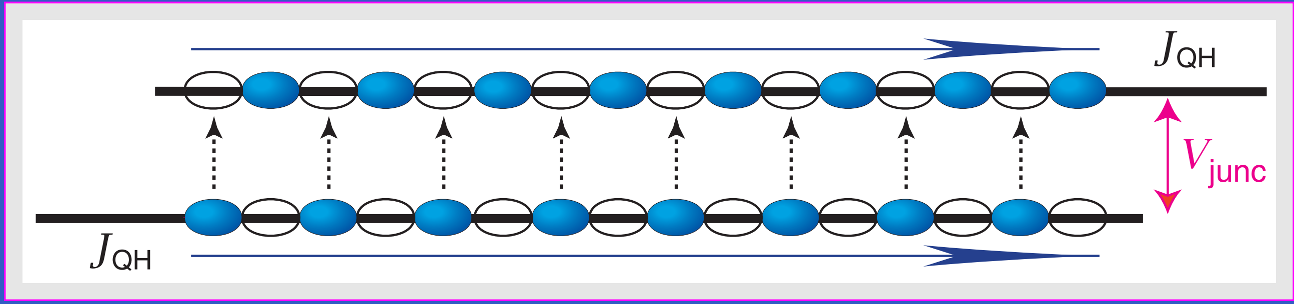
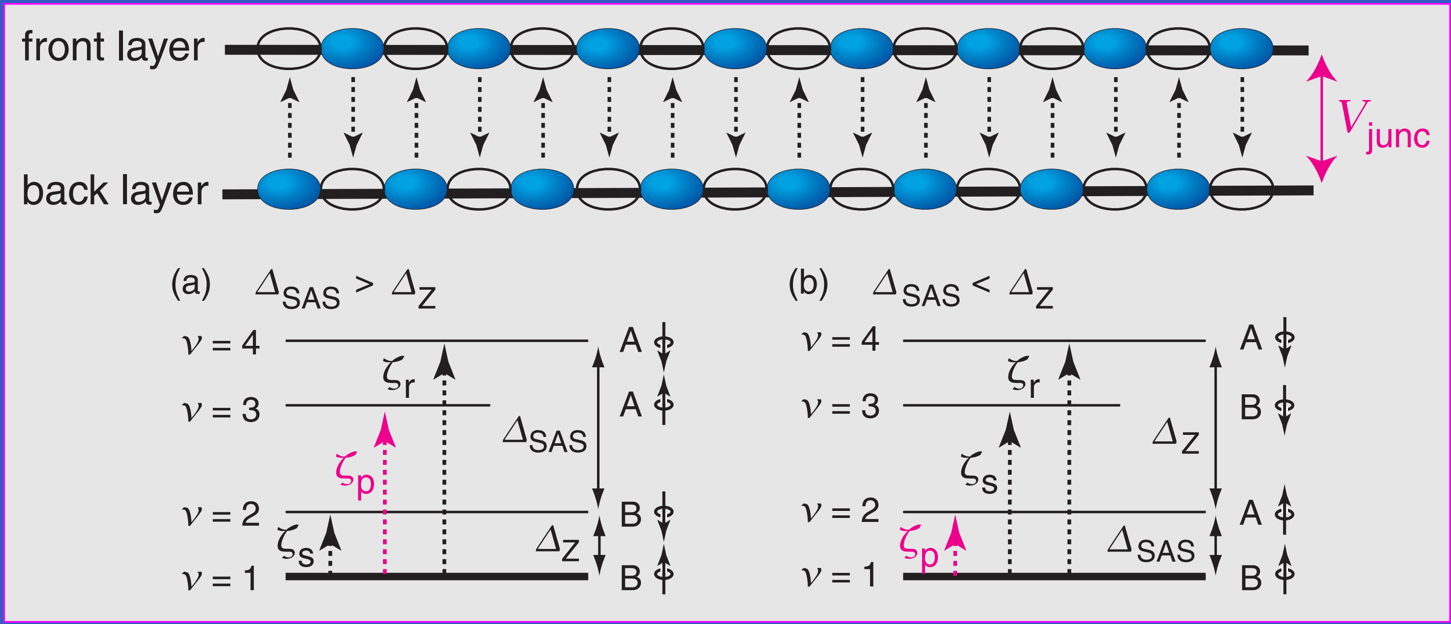
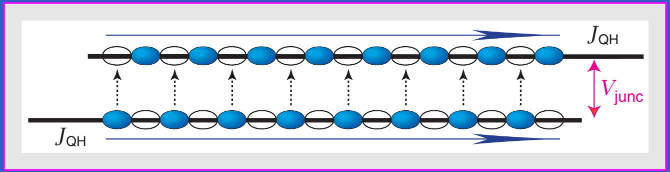


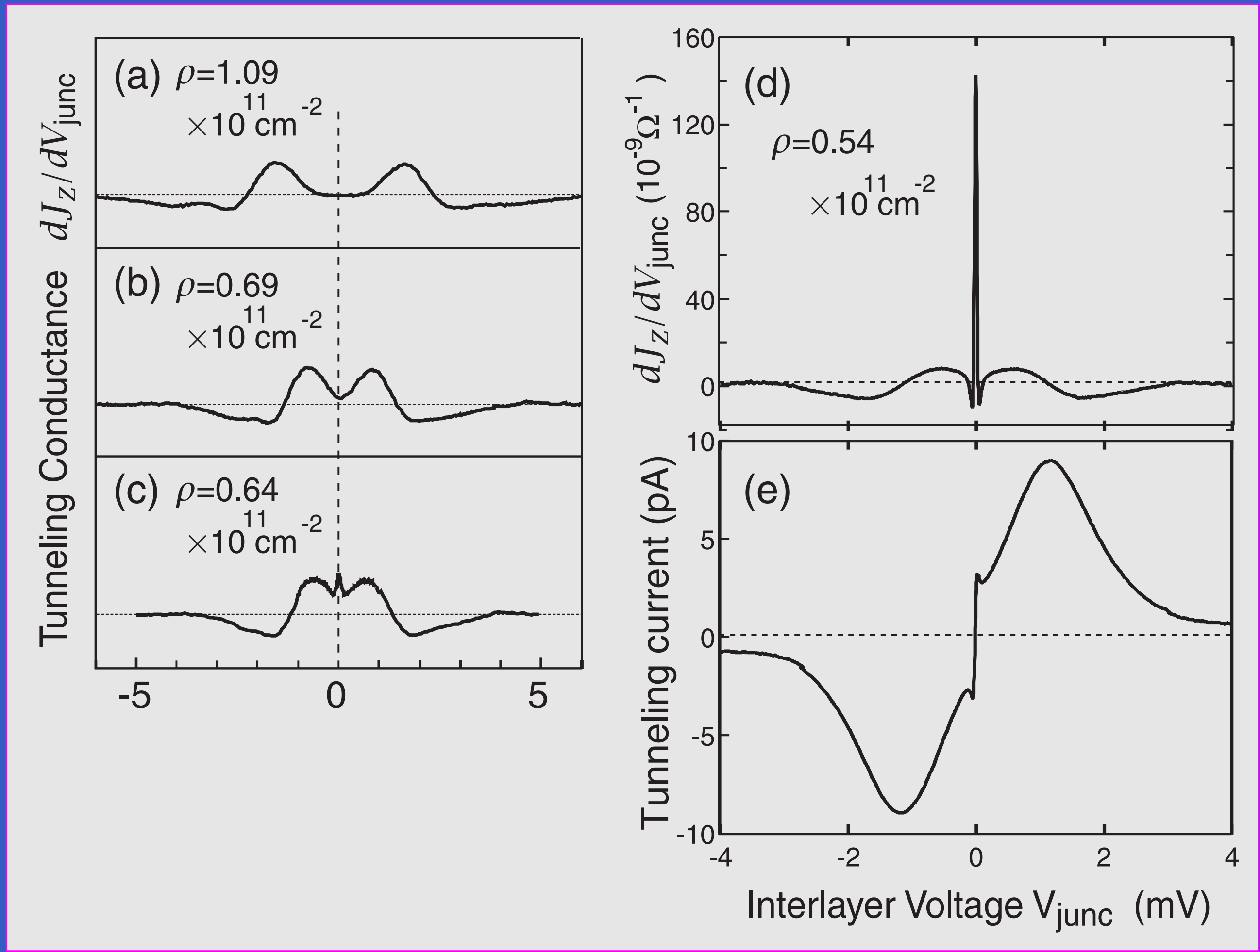
- **Plasmon excitations expected** (detectable by microwave)



Josephson-like Tunneling (Experiments)

- Layer coherence
  $\Rrightarrow$
  Coherent tunneling
  (Ezawa-Iwazaki, 1993)





Spielman, Eisenstein, PRL84(2000)5808;PRL87(2001)36803

Conclusions

- An ideal system realizing
  - ⇛ **noncommutatvie geometry**
- Noncommutative geometry
  - ⇛ **quantum coherence**
- SU(2) spin coherence
  - ⇛ **CP^1 Skyrmions**
- SU(4) isospin coherence
  - ⇛ **CP^3 and $G^{4,2}$ Skyrmions**
- Interlayer coherence
  - ⇛ Josephson-like phenomena
- **Condensation of single charge e**
  - ⇛ Statistical transformation

2nd Version Comming Soon !!