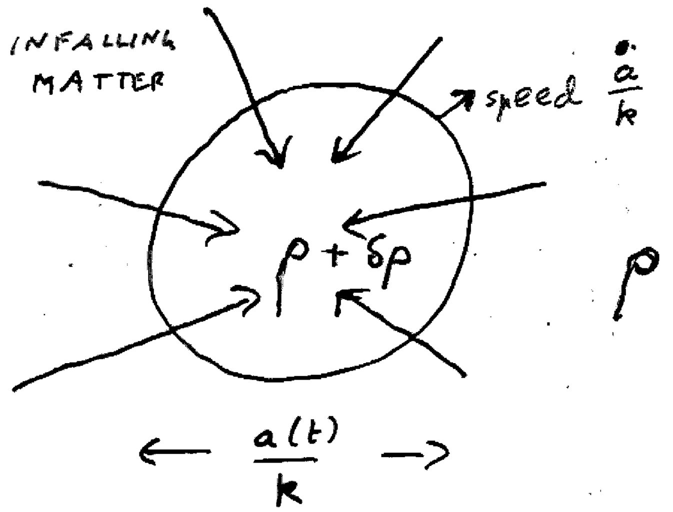
# THE ORIGIN OF STRUCTURE

Q How did the first galaxies form?

A By gravitational collapse of slightly overdense regions.




Q When does this process start?

A At the epoch of HORIZON ENTRY

$$a(t)\, H(t) \,/\, k = 1$$

$$\dot{a}(t)/k = 1$$

$[c=1]$

# Understanding the Primordial Density Perturbation (PDP)

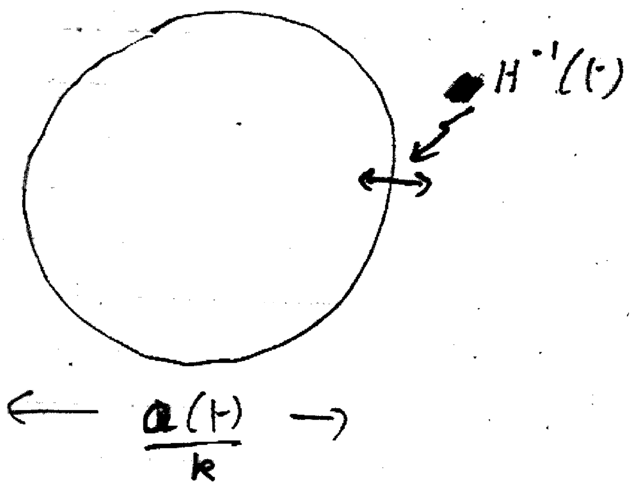
Lecture notes on my website (at Lancaster University Physics)

M. ROOS 3rd EDITION ?

1. Simple description of the PDP
2. Fourier analysis, conserved ~~pe~~ curvature perturbation $\zeta$.
3. Inflation and the quantum-to-classical transition.
4. Slow roll inflation, generating $\zeta$.
5. Inflation models and particle physics.

Q Why doesn't the process begin earlier?

A Because there isn't time for particles to move into the region




In one Hubble time, no particle can travel farther than Hubble distance (particle speed $< c=1$). (Exact maximum distance travelled in total is the particle horizon as discussed earlier.)

Q At what epoch does a galaxy-forming region enter the horizon?

A ~~At~~ At temperature given by

$$\left(\frac{k_B T}{1\,\mathrm{MeV}}\right)^3 \sim 10^{-2}\,\frac{M_\odot}{M}$$

where $M$ is the galaxy mass and $M_\odot$ is the mass of the sun. Galaxies weigh $10^6$ to $10^{12}\,M_\odot$, so horizon entry is <u>after nucleosynthesis</u>.

Q What's the composition of the Universe at horizon entry?

A Like at present; CDM, baryonic matter, photons, neutrinos. [The cosmological constant is negligible, because it's constant whereas the other energy densities go like $a^{-3}$ (CDM and baryons) or $a^{-4}$ (photons and neutrinos).]

Q How does the galaxy-forming region evolve before horizon entry?

A Like some separate unperturbed universe, with locally-defined scale factor $a(\underline{x}, t)$

$\uparrow$ galaxy position

$$\rho \propto a^{-4}$$

$$n_i \propto a^{-3} \quad [i = CDM, B, \gamma, \nu]$$

Q How does the separate Universe differ from the unperturbed one?

A Apparantly, only by a time difference, so

$$\frac{1}{4}\frac{\delta\rho}{\rho} = \frac{1}{3}\frac{\delta n_i}{n_i} \quad \left[= -\frac{\delta a}{a}\right]$$

the ADIABATIC CONDITION

$n_i(\rho)$ unique functions throughout spacetime

Q. When a galaxy-forming region enters the horizon, do all 4 types of particle flow into it?

A. No. The attraction of gravity competes with the random particle motion. Need to consider each case separately.

a). CDM (Cold Dark Matter)

Negligible random motion, negligible collisions. So only gravity operates

$$\Rightarrow \left(\frac{\delta n_{CDM}}{n_{CDM}}\right) \text{ increases with time}$$

b). Neutrinos

Random speed $\simeq c$, negligible collisions (after $k_B T \sim 1\,\text{MeV}$), speed too big for gravity to have a significant effect

$$\Rightarrow \frac{\delta n_\nu}{n_\nu} \text{ decreases exponentially}$$

($\nu$ motion washes out the inhomogeneity in $n_\nu$).

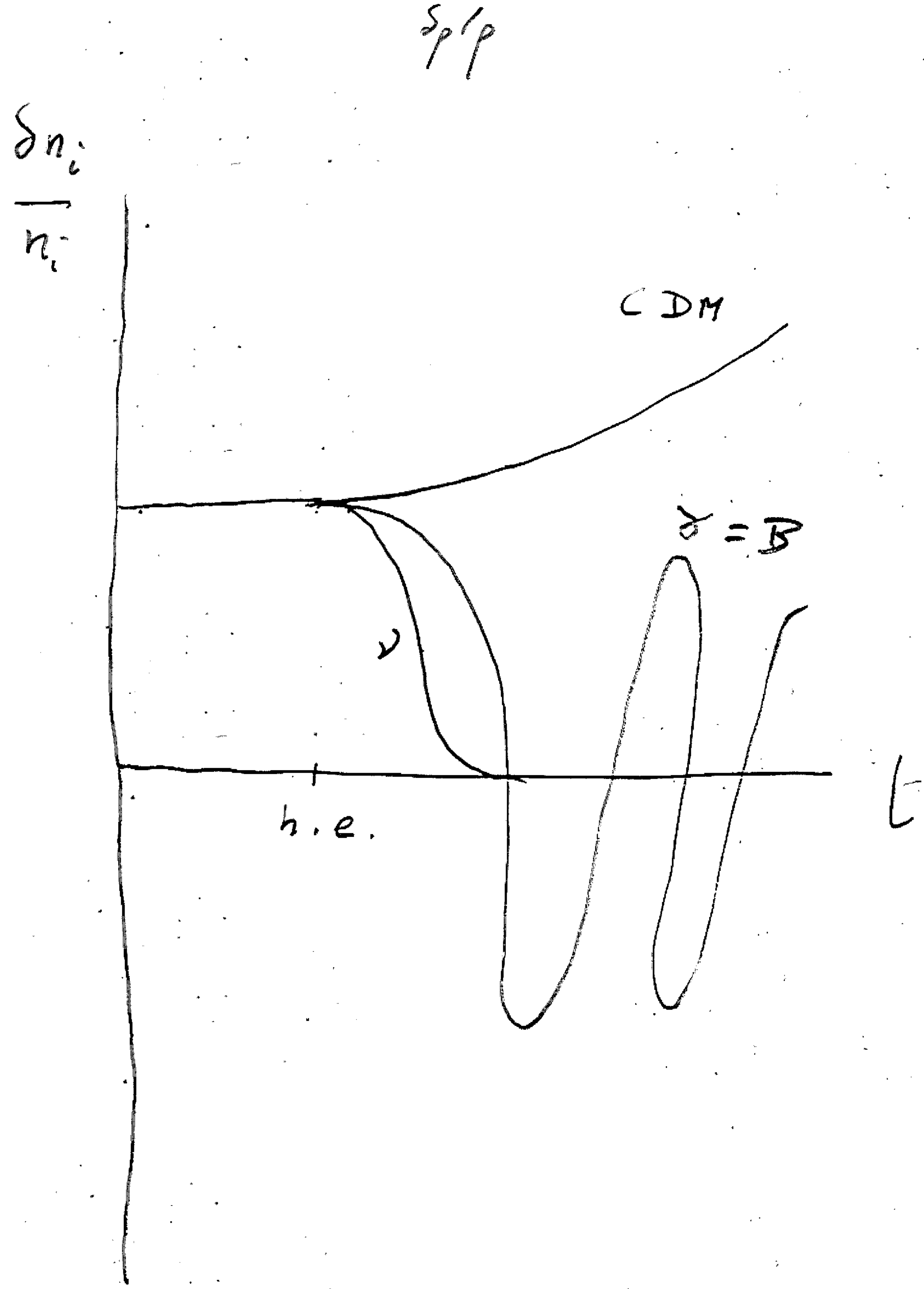
c) Photons. Random speed $c$, but unlike neutrinos they frequently collide with electrons. Hence the photon gas can support 'sound waves' ('acoustic waves' i.e., waves caused by pressure variations). So $\delta n_\gamma / n_\gamma$ oscillates after horizon entry

d) Baryons ( = nuclei )

Tied to electrons by electrostatic force (must have $n_p(\underline{x}) = n_e(\underline{x})$ in each local region). But electrons tied to photons by collisions. Hence

$$\frac{\delta n_B}{n_B} = \frac{\delta n_\gamma}{n_\gamma}$$

even after horizon entry. This 'baryon-photon gas' is oscillating as a whole.



$\delta\rho/\rho$
$\frac{\delta n_i}{n_i}$
CDM
$\gamma = B$
$\nu$
h.e.
t

Q What's the 'speed of sound' in the baryon-photon gas?

A Recall for non-relativistic gas speed is $c_s = \sqrt{\frac{P}{\rho_m}}$ ← mass density

Can easily show that for relativistic gas (~~speed~~ random motion ~ c) this becomes $c_s = \sqrt{\frac{P}{\rho/c^2}}$ ← energy density.

The photons have random speed c and dominate both P and $\rho$, so

$$c_s = \frac{1}{\sqrt{3}} c$$

We'll take $c_s \sim c$ for estimates.

So

$$\boxed{\text{wavelength} \sim c \times \text{period}}$$

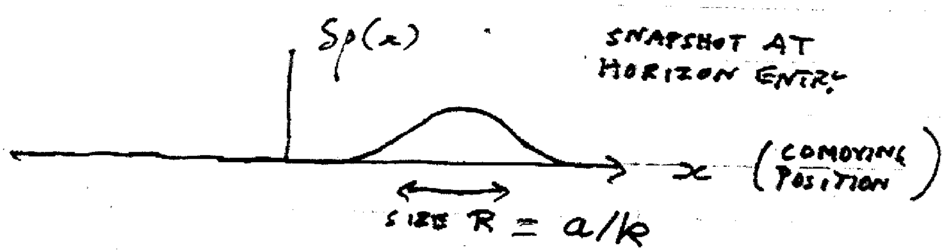
for these 'sound waves'.

Q Are the 'sound' waves in the baryon-photon gas standing waves or travelling waves?

A Standing waves, because the time-delay mechanism that originates them doesn't have any preferred direction.

Q Is the oscillation in a given galaxy-forming region exactly sinusoidal?

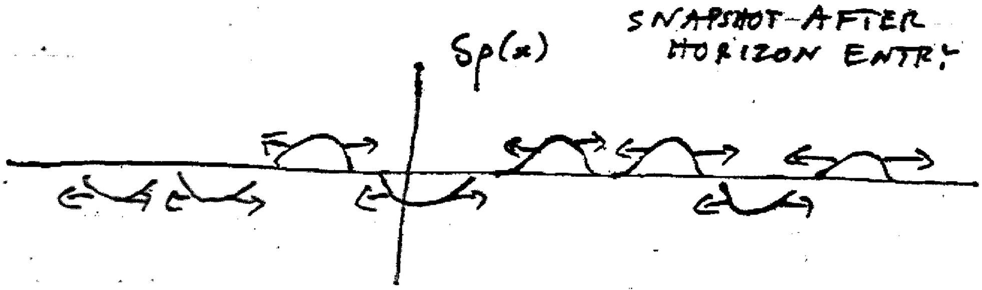
A No, because the density is a superposition of pure sine (and cosine) waves




But dominant ~~wavelength is ~ R, giving period ~ R/c~~. wavenumber is $k/a$

Q After the oscillation starts (at horizon entry) doesn't the amplitude decrease rapidly, as the overdense region radiates energy in the sound waves?

A No, because space is filled with overdense (and underdense) regions so each region has no net energy loss. (standing waves!)




(Each over- or under-dense region on it's own would radiate energy, but it cancels to give standing waves).

NOTE: WE JUST PLOTTED REGIONS OF GIVEN SIZE. THE TRUE $\rho(x)$ IS A SUPERPOSITION OF THESE, WITH DIFFERENT SIZES. MATHEMATICALLY CORRECT DESCRIPTION IS FOURIER ANALYSIS.

## STOCHASTIC PROPERTIES

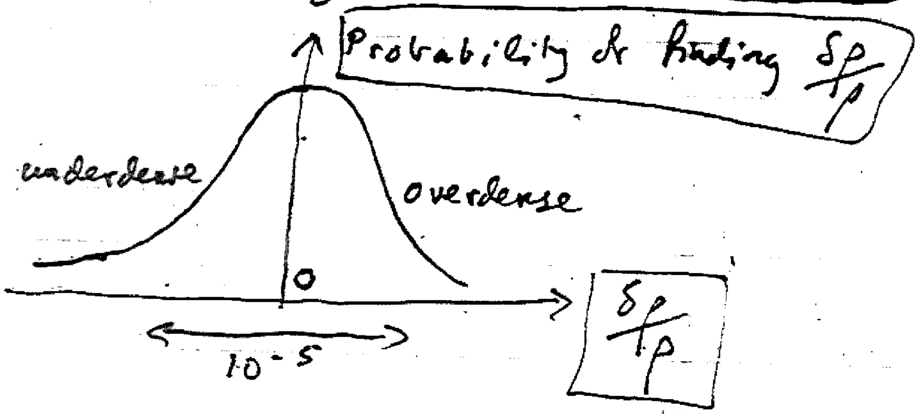
Q What is the typical overdensity of a galaxy-forming region at horizon entry?

A $\frac{\delta\rho}{\rho} \sim 10^{-5}$, independently of the galaxy size

Q What about a randomly chosen region?

A The density ~~cont~~ contrast $\delta\rho/\rho$ has a Gaussian distribution, with width $\sim 10^{-5}$, independently of the size of the region.

$$\boxed{\underset{\text{GALAXY}}{10^{-2}\,\text{Mpc}} < k^{-1} < \underset{H_0^{-1}}{10^{4}\,\text{Mpc}}}$$




JARGON: "The primordial density perturbation is Gaussian and scale-invariant"

## 2. FOURIER ANALYSIS AND THE CURVATURE PERTURBATION

Use comoving coordinate $\underline{x}$,

(physical distance $\underline{r} = a(t)\underline{x}$)

Fixed time $t$.

C

A <u>perturbation</u> $g(\underline{x})$ is any quantity which would vanish in the Robertson-Walker (unperturbed) Universe.

$$g(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \int g(\underline{k}) e^{i\underline{k}\underline{x}} d^3\underline{k}$$

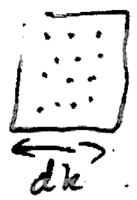
book Liddle & Lyth

# Gaussian perturbations

We imagine an ensemble of universes, ours is typical (each perturbation a 'random field'). Gaussian means the $g(\underline{k})$ are uncorrelated except for $\langle g(\underline{k})\rangle = 0$

$$\langle g^*(\underline{k})\, g(\underline{k}')\rangle = \delta^3(\underline{k}-\underline{k}') \times \frac{2\pi}{k^3}\, \mathcal{P}_g(k)$$

$\langle g^* g g^* g\rangle$




SPECTRUM of g

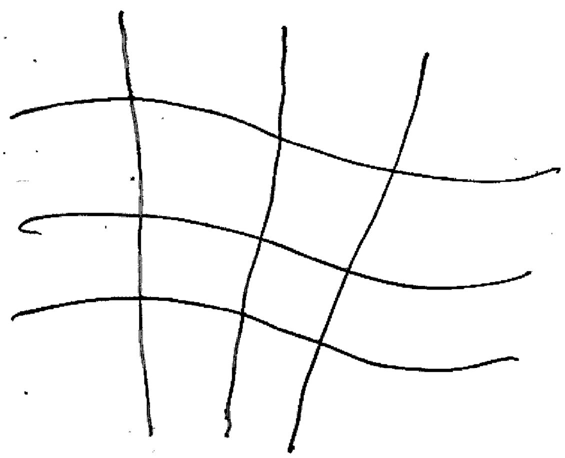
NOTE

$$\langle g^2(\underline{x})\rangle = \int_0^\infty \mathcal{P}_g(k)\,\frac{dk}{k}$$

$\langle g^4\rangle$

## Gauge choice

To define the perturbations, generally we need to specify a <u>slicing</u> and <u>threading</u> of spacetime, corresponding to some coordinate system



slices: fixed $t$

threads: fixed $\underline{x}$

We're not interested in threading.
Consider choices of slicing.

(i) Uniform energy density $\delta\rho = 0$

(ii) Flat slicing (zero curvature scalar, Euclidean geometry except for gravitational waves).

## The curvature perturbation $\zeta$

Using only spacetime geometry (not gravity) we can define $\zeta$ in two equivalent ways

(i) specifies the metric perturbation on slices of uniform density ($\delta\rho = 0$) $\zeta(\underline{x}, t)$

$$dl^2 = a^2(t)\, \delta_{ij}\, dx^i dx^j (1 + \zeta(\underline{x}))$$

(ii) specifies $\delta\rho$ on flat slices

$$\zeta = \frac{\delta\rho}{3(\rho + P)} \qquad \text{[DHL + Wands]}$$

## Conservation of $\zeta$

We are considering only cosmological scales (small galaxies and bigger), and only outside the horizon

$$\frac{k}{aH} \ll 1$$

($a(t)/k$ is the scale).

Then, we can show that $\zeta$ is conserved if there is a unique function $P(\rho)$ throughout spacetime. [DHL + Wands, et al]

[ADIABATIC CONDITION FOR PRESSURE]

Proof involves only energy conservation $dE = -P\,dV$

not gravity.

Adiabatic condition for pressure is obviously satisfied during radiation domination ($p = \frac{1}{3}\rho$) and matter domination ($p = 0$).

# Observing the primordial density perturbation: one number and null results

(i) $\mathcal{P}_\zeta^{\frac{1}{2}}(k) = 2 \times 10^{-5}$

independent of $k$ (flat spectrum).

(ii) Define spectral index $n$:

$$n - 1 = \cancel{d \mathcal{P}_\zeta} \; \frac{d\,(\ln \mathcal{P}_\zeta)}{d\,(\ln k)}$$

$n = 0.97 \pm 0.03$

(iii) Non-gaussianity?

For simplicity, assume

$$\zeta(\underline{x}) = \zeta_{GAUSS}(\underline{x}) + f_{NL}\, \zeta^2_{GAUSS}(\underline{x})$$

$|f_{NL}| < 100$

[note $\zeta \sim 10^{-5}$, so non-gaussian fraction is $\lesssim 10^{-3}$

(vii) Tensor perturbation
(gravitational waves).

Inflation *might* generate this perturbation, contributing to the low multipoles of the CMB some fraction $\sqrt{r}$.

Observation : $r \lesssim 10^{-1}$

(iv, v, vi) Isocurvature perturbation,

Recall the adiabatic condition

$$\frac{\delta n_{CDM}}{n_{CDM}} = \frac{\delta n_B}{n_B} = \frac{\delta n_\gamma}{n_\gamma} = \frac{\delta n_\nu}{n_\nu}$$

(i.e. $n_i(\rho)$, unique).

If not satisfied, ~~could~~ some "isocurvature" perturbations are nonzero:

$$S_{CDM} = \frac{\delta n_{CDM}}{n_{CDM}} - \frac{\delta n_\gamma}{n_\gamma}$$

$$S_B = \frac{\delta n_B}{n_B} - \frac{\delta n_\gamma}{n_\gamma}$$

$$S_\nu = \frac{\delta n_\nu}{n_\nu} - \frac{\delta n_\gamma}{n_\gamma}$$

Observation: $\left|\frac{S_i}{3}\right| \lesssim 10^{-1}$

## The origin of the primordial density perturbation

From the vacuum fluctuation of some scalar field.

Which field?

1982 The inflaton field.

2001 Some other field, called the curvaton.

## Observation may decide between the inflaton and curvaton paradigms

CMB anisotropy (WMAP, CBI etc.)
Galaxy surveys (SDSS, 2dF etc.)
Weak gravitational lensing, Lyman-alpha forest, ...

### Parameters to be fitted

| Parameter | Observed Value | Inflaton Paradigm | Curvaton Paradigm |
|---|---|---|---|
| $\mathcal{P}_\zeta^{1/2}(\ln k)$ | $2 \times 10^{-5}$ | yes | yes |
| $n - 1 \equiv [\ln(\mathcal{P}_\zeta)]'$ | $\simeq 0$ | yes | — |
| $n'$ | $\simeq 0$ | yes | — |
| Tensor fraction $r$ | $\simeq 0$ | yes | — |
| Non-gaussianity $f_{\rm NL}$ | $\simeq 0$ | — | yes |
| $s_{\rm cdm} \equiv S_{\rm cdm\mathbf{k}}/\zeta_{\mathbf{k}}$ | $\simeq 0$ | — | yes |
| $s_B \equiv S_{B\mathbf{k}}/\zeta_{\mathbf{k}}$ | $\simeq 0$ | — | yes |
| $s_\nu \equiv S_{\nu\mathbf{k}}/\zeta_{\mathbf{k}}$ | $\simeq 0$ | — | yes |

## 3. Inflation and the quantum to classical transition

Inflation: an era of repulsive gravity, $\ddot{a} > 0$.

We want to explain the origin of the primordial density perturbation, and why the Universe is so homogeneous. This is impossible (or difficult?) without inflation because each comoving region of the Universe is outside the horizon, $\frac{aH}{k} > 1$, at early times. (The horizon problem.)

Inflation also solves two other problems.

<u>Flatness problem</u>

Why is geometry Euclidean?

Friedmann eq.

$$H^2 = \frac{\rho}{3M_p^2} - \frac{K}{a^2}$$

divide by $H^2$, define

$$\Omega = \frac{\rho}{3M_p^2 H^2}$$

$$\Omega(t) - 1 = \frac{K}{(aH)^2} = \frac{K}{\dot{a}^2}$$

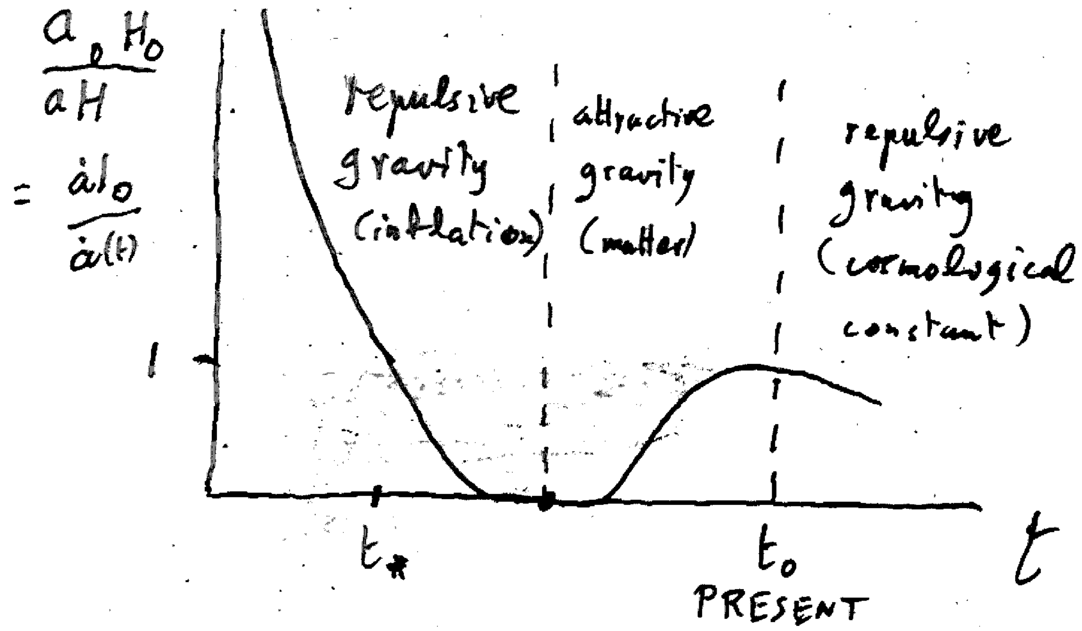
$\frac{K}{(aH)^2}$ measures non-flatness
(over one Hubble distance).

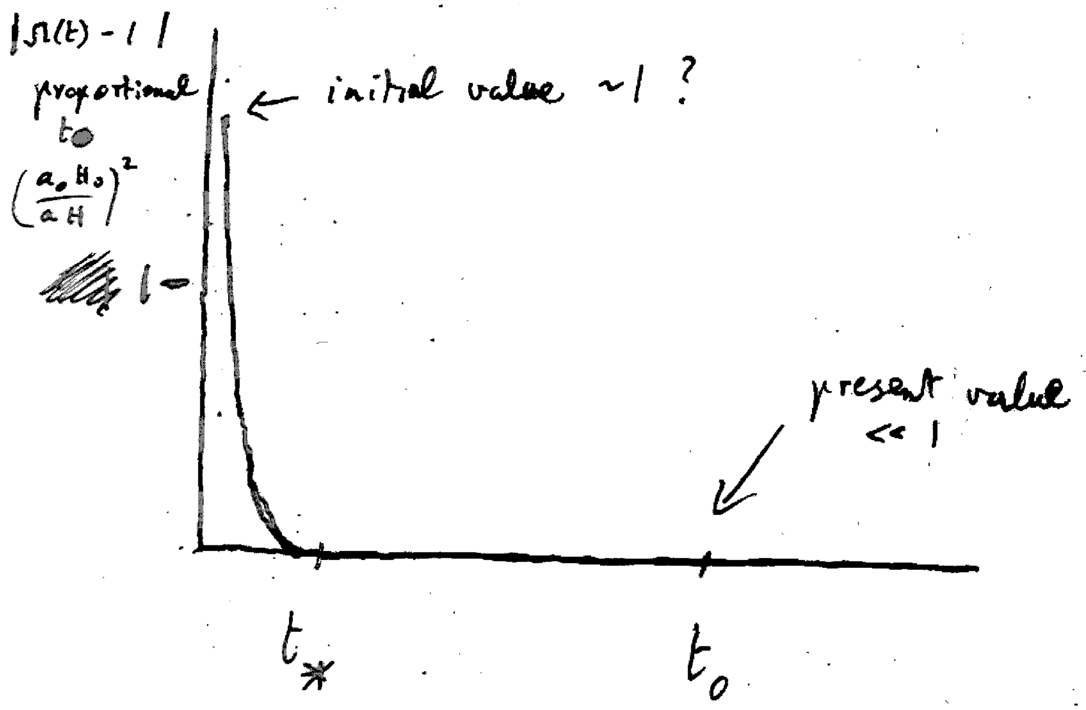
Without inflation, $|\Omega - 1|$ must be exponentially small in the beginning.

We want inflation to last long enough so that the whole observable Universe is inside the horizon initially: i.e., there is an epoch during inflation of <u>horizon exit</u> for the obs. Univ., at some time $t_*$:

$$\frac{a_* H_*}{a_0 H_0} = 1 \quad \left( = \frac{\dot{a}(t_*)}{\dot{a}(t_0)} \right)$$

[the observable Universe has present size $H_0^{-1}$, so size earlier is $\frac{a(t)}{k}$ with $k = H_0$, exits horizon when $\frac{a H_*}{k} = 1$, and we inserted $a_0 = 1$ for elegance]




IN THESE GRAPHS, $t_*$ IS THE TIME WHEN THE OBSERVABLE UNIVERSE LEAVES THE HORIZON, $a_* H_* = a_0 H_0$

## Unwanted relic problem (eg. monopole problem).

Without inflation, the temperature at early times may be very big, which may create unwanted relics. With inflation, maximum temperature afterwards ("reheat temp.") is

$$T_{RH} \lesssim \rho_*^{1/4} \qquad \boxed{T^4 < \rho < \rho_*}$$

[because $\rho(t)$ always decreases, and $\rho \gtrsim T^4$ in thermal equilibrium]. Also inflation exponentially dilutes density of relics created <u>before</u> inflation.

We'll see that almost exponential inflation is needed to generate the curvature perturbation.

$$a(t) \propto e^{\text{const.}\, t}$$

Since $H \equiv \dot{a}/a$,

$$a(t) \propto e^{H_* t}$$

$H_*$ is constant value of $H$, ~~the t~~ De-Sitter Universe.

We need this at $t = t_*$, and for a few Hubble after (while cosmological scales leave the horizon). For simplicity, assume ~~at~~ for

$$t_* < t < t_{end} \leftarrow \text{END OF INFLATION}$$

## <u>Observational constraints on the energy scale of inflation $\rho_*$</u>

(i) We don't observe in CMB gravitational waves generated by inflation:

$$\rho_*^{1/4} \lesssim 10^{16}\ \text{GeV}$$

well <u>below</u> the Planck scale $M_P = 10^{18}$ GeV

(ii) Inflation must end before nucleosynthesis

$$\rho_*^{1/4} \gtrsim 10\ \text{MeV}$$

But SUSY suggests much stronger constraint

$$\rho_*^{1/4} \gtrsim 10^{10}\ \text{GeV}$$

SUSY BREAKING SCALE

# How much inflation?

What is

$$N \equiv \ln \frac{a_{end}}{a_*}$$

(number of e-folds)?

## Simple estimate

Assume radiation domination from end of inflation to the present.

$$a(t) \propto t^{1/2}, \quad H \propto \frac{1}{t},$$

$$aH \propto 1/a \propto T$$

$$\frac{a_* H_*}{a_0 H_0} = \frac{\rho_*^{1/4}}{T_0} \quad \text{———} (1)$$

$\boxed{a_{end}}$

$T_0 \Leftarrow 3K = 3 \times 10^{-4}\,\mathrm{eV}$

Using definition of $N$,

~~(1) becomes~~

$$\frac{a_{end}\, H_*}{a_*\ \ H_*} = e^N \quad (2)$$

(use $a_* H_* = a_0 H_0$ (definition of $t_*$)

Combine (1) and (2)

$$\boxed{N \sim \ln \frac{\rho_*^{1/4}}{T_0}}$$

Using bounds on $\rho_*$,

$$24 \lesssim N \lesssim 66$$

Precise calculation gives

$$N = 62 \underbrace{\boxed{+ \ln \frac{\rho_*^{1/4}}{10^{16}\,\mathrm{GeV}} - N_0}}_{<0}$$

$N_0 > 0$ allows for departures from radiation domination.

# The quantum-to-classical transition

At the <u>classical</u> level, inflation eventually makes the Universe completely homogeneous (cf. no-hair theorem for black hole formation). [but transplanckian problem: our Universe starts off inside the horizon?]

<u>But</u> the <u>quantum</u> fluctuation of light scalar fields is converted to a classical perturbation (cf. Hawking radiation from black holes).

Assume that during inflation some scalar field has Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$$

$$ds^2 = dt^2 - a^2(t)\,\delta_{ij}\,dx^i dx^j$$

$$\phi(\underline{x}, t) = \phi(t) + \delta\phi(\underline{x}, t)$$

$[\phi(t)]$ $\quad \ddot{\phi} + 3H\dot{\phi} + V' = 0$

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[\left(\frac{k}{a}\right)^2 + V''\right]\delta\phi = 0$$

$[\delta\phi(\underline{k})]$

~~We assume $V''$~~

We assume inflation, $a \propto e^{H_* t}$

and assume $V'' \ll H_*^2$ (light field).

Go to conformal time

$$d\tau = \frac{dt}{a}$$

Define $\quad u = a\,\delta\phi(\underline{k})$

Can choose

IGNORING $V''$

$$\tau = -\frac{1}{aH_*} < 0$$

$$\frac{d^2u}{d\tau^2} + \left(k^2 - 2(aH_*)^2\right)u = 0$$

Well before horizon exit, can ignore expansion, $k^2 \gg (aH_*)^2$.

and $k^2 \gg a^2 V''$

Find

$$\mathcal{L} \simeq \frac{1}{2}\eta^{\mu\nu}\partial_\mu u\, \partial_\nu u$$

Massless free field in flat spacetime.

Quantise in Heisenberg picture

$$u(\tau) = \frac{1}{\sqrt{2k}}\left[e^{-ik\tau} a(\underline{k}) + e^{ik\tau} a^{\dagger}(\underline{k})\right]$$

$$[a(\underline{k}), a^{\dagger}(\underline{k}')] = \delta^3(\underline{k}-\underline{k}')$$

Assume vacuum state

$$a(\underline{k})| \rangle = 0$$

Now follow evolution of $u(\tau)$ through horizon exit. ~~Eqn of~~ Field equation requires

$$u(\tau) = w(k,\tau)\, a(\underline{k}) + w^* a^{\dagger}(\underline{k})$$

$w$ satisfies field equation

$$\ddot{w} + [k^2 - 2(aH_*)^2]w = 0$$

Solve with initial condition

$$w = \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

Get

$$w = -\frac{1}{\sqrt{2k}} \left( \frac{k}{aH_*} - i \right) \frac{e^{-ik\tau}}{\tau}$$

$$[\tau = -1/aH_*]$$

~~Now~~ A few Hubble times after horizon exit, $* \boxed{k\tau \ll 1}$

$$w \simeq i \frac{1}{\sqrt{2k}} \frac{1}{\tau}$$

and

$$u(\tau) \simeq \frac{i}{\sqrt{2k}} \frac{1}{\tau} \left[ a(\underline{k}) - a^\dagger(\underline{k}) \right]$$

It someone measures $u$ precisely, it will remain precise, $\propto \frac{1}{\tau}$

[Starobinsky]

~~$u = a\,\delta\phi$~~

$$\delta\phi(\underline{x}) = \frac{u(\underline{x})}{a}$$

is a Gaussian perturbation

(independent Fourier components)

with spectrum

$$\mathcal{P}^{\frac{1}{2}}_{\delta\phi}(k) = \frac{H_*}{2\pi}$$

[Recall the definition of $\mathcal{P}_g$ in terms of $\langle g(\underline{k})\, g(\underline{k}')\rangle$]

independent of $k$ for exact De-Sitter spacetime.

<u>But</u> we should not go <u>too many</u> Hubble times after inflation, or $V''$ may become important (recall field equation involves $\left[\left(\frac{k}{a}\right)^2 + V''\right]$).

The time-independent state of the Universe $| \rangle$ does <u>not</u> correspond to a precise value of $\delta\phi(\underline{k})$, but to a probability distribution.

But the observed Universe <u>does</u> correspond to precise values of $\delta\phi(\underline{k})$, hence precise $\delta\phi(\underline{x})$

$$| \rangle = \sum_{\delta\phi(\underline{k})} | \text{Universes like ours with definite } \delta\phi(\underline{k}) \rangle$$

The ensemble considered by astronomers <u>before</u> inflation was proposed

Who measured our Universe? (Schrödinger cat problem).

## Slow roll inflation

Almost the only known way to get exponential inflation uses (i) Einstein gravity

(ii) An almost flat potential as a function of the inflation field $\phi$;

$$V(\underset{\substack{\uparrow \\ \text{flat direction}}}{\phi}, \text{ other fields})$$

In other directions, V either ~~min~~ fixes the fields or is much flatter than in the $\phi$ direction. [In general $\phi$ could have several components, eg. Kodata & Stewart 2003. Assume one component here.]

Ignore perturbation in $\phi$ for the moment. $\boxed{V' \equiv \frac{\partial V}{\partial \phi}}$

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

and can find from Lagrangian

$$\rho = \frac{1}{2}\dot{\phi}^2 + V$$

$$P = \frac{1}{2}\dot{\phi}^2 - V$$

assume these expressions give <u>total</u> $\rho$ and $P$ (other $\dot{\phi}_n$ are negligible).

Assume <u>slow-roll</u> approximation

$$\dot{\phi} \simeq - \frac{V'}{3H} \quad (\text{neglect } \ddot{\phi})$$

~~Expected to hold if $V$ satisfies the <u>flatness conditions</u> $\epsilon \ll 1$, $|\eta| \ll 1$~~

$$\cancel{\epsilon = \frac{1}{2} M_p^2 (V'/V)^2}$$

$$\cancel{\eta = M_p^2 V''/V}$$

and $\rho \simeq V \quad [\dot{\phi}^2 \ll V]$

Expected to hold if $V$ satisfies the flatness conditions

$$\epsilon \equiv \frac{1}{2} M_p^2 (V'/V)^2 \ll 1$$
$$|\eta| \equiv | M_p^2 V''/V| \ll 1 .$$

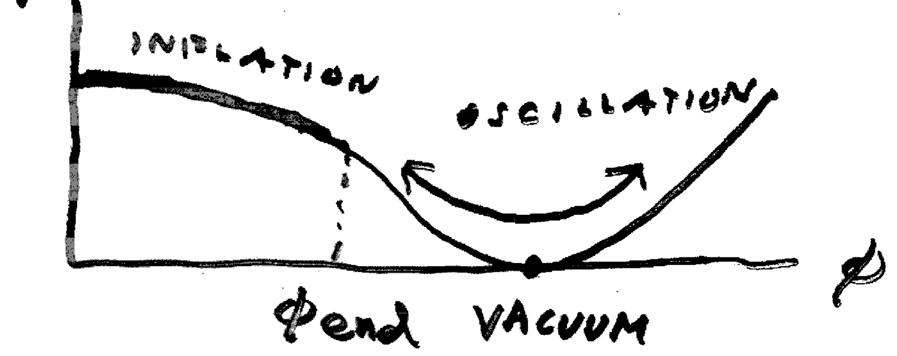
## The end of inflation

It occurs when $\phi$ reaches some value $\phi_{end}$. Two kinds of model

(i) Single-field (= non-hybrid) (1982)

Potential $V$ generated only by the displacement of $\phi$ from its VEV

INFLATION ENDS WHE $\epsilon$ or $\eta \sim 1$



The oscillation of $\phi$ ends inflation because

$$\langle P \rangle = \langle \tfrac{1}{2}\dot{\phi}^2 - V \rangle = 0$$

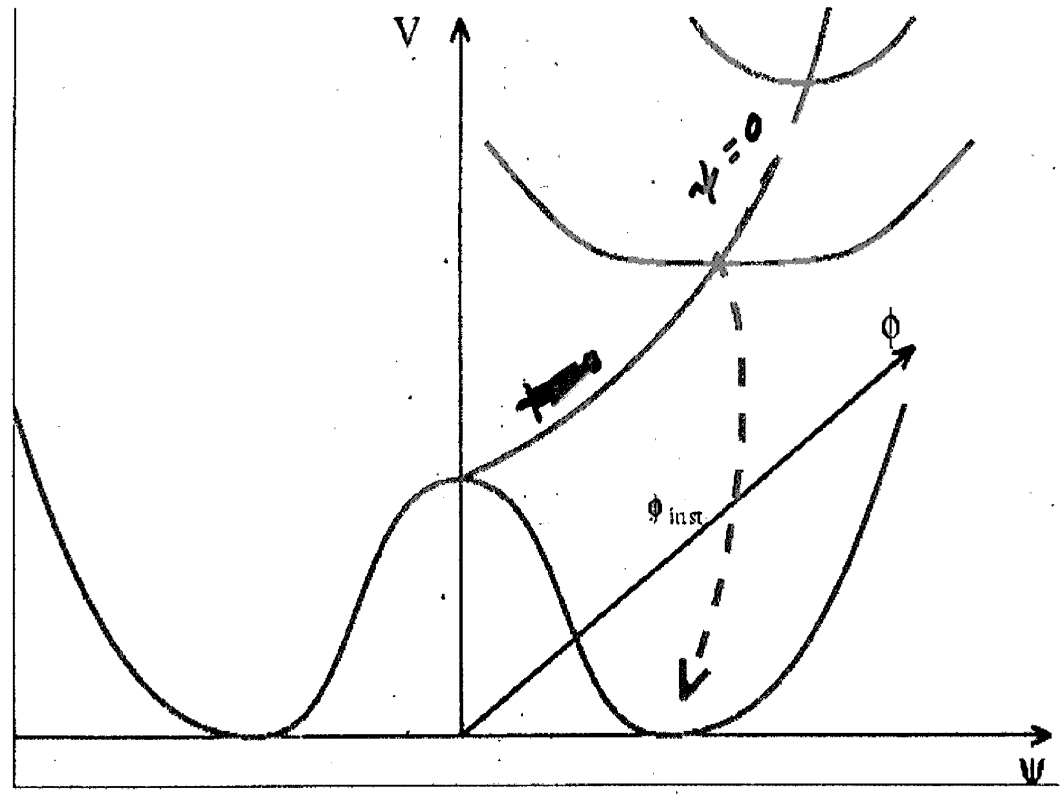
$$\langle K.E. \rangle = \langle P.E. \rangle$$

equivalent to matter domination, $a(t) \propto t^{2/3}$

$$\ddot{a} < 0 \, .$$

<u>(ii) Hybrid inflation (1991)</u>

$V$ comes mostly from the displacement of some 'waterfall field' $\chi$. Inflation ends when $\chi$ is destabilised


V
ψ
φ
φ inst
ψ=0

Original version of the potential is

$$V = V_0 - \frac{1}{2} m_\psi^2 \psi^2 + \lambda \phi^2 \psi^2 + \frac{1}{2} m_\phi^2 \phi^2$$

When $\lambda \phi^2 > \frac{1}{2} m_\psi^2$,

$\psi$ trapped at $\psi = 0$,

$$V = V_0 + \frac{1}{2} m^2 \phi^2,$$

inflates if $m$ is small.

Inflation ends at

$$\lambda \phi_{end}^2 = \frac{1}{2} m_\psi^2,$$

then $\psi$ and $\phi$ have coupled oscillations.

## Calculating $\phi_* \equiv \phi(t_*)$

Slow-roll and flatness conditions give [Recall N = # e-folds] [N ~ 50]

$$N = \frac{1}{M_p^2} \int_{\phi_{end}}^{\phi_*} \frac{d\phi}{\sqrt{2\epsilon}}$$

## The curvature perturbation $\zeta$ generated by the inflaton perturbation $\delta\phi$

Recall $\zeta = \dfrac{\delta\rho}{3(\rho + P)}$ ← ON FLAT SLICES

and $\delta\rho = V' \delta\phi$

Using slow-roll

$$\zeta = \frac{3H_*^2}{V'} \delta\phi$$

Can show that the previous calculation applies to $\delta\phi$ on flat slices, so

$$\boxed{\mathcal{P}_\zeta^{\frac{1}{2}} = \frac{3H_*^2}{V'} \frac{H_*}{2\pi}}$$

A FEW HUBBLE TIMES AFTER HORIZON EXIT

According to the inflaton paradigm, this is the observed perturbation $\mathcal{P}_\zeta^{\frac{1}{2}} = 2\times10^{-5}$.

According to the curvaton paradigm, this $\mathcal{P}_\zeta^{\frac{1}{2}} \ll 10^{-5}$ and the observed $\mathcal{P}_\zeta^{1/2}$ is generated later from the perturbation of some 'curvaton' field.

# The spectral index

For a given scale $k/a$, we get more accurate result if we replace $H_*$ and $V'$ by the values at horizon exit

$$\mathcal{P}_{\mathcal{R}}^{\frac{1}{2}}(k) = \frac{3}{2\pi} \frac{H^3}{V} \Big|_{k=aH}$$

Using slow-roll and flatness conditions,

$$n = 1 + \cancel{2\epsilon}\, 2\eta - 6\epsilon$$

[recall $n-1 \equiv d \ln \mathcal{P} / d \ln k$]

# Creation of gravitational waves from the vacuum fluctuation

The amplitude of gravitational waves has the classical and quantum dynamics of a massless scalar field. During inflation before horizon entry, assume no gravitational waves (vacuum state). After horizon exit the amplitude of gravitational waves becomes classical, and actual oscillation starts after horizon entry, affecting the low multipoles of the CMB anisotropy to some extent. But ratio $\sqrt{r}$ given by

$$r = \frac{V_*^{1/4}}{3.3\times 10^{16}\,\mathrm{GeV}} \quad (\doteq 16\,\epsilon_* \text{ for slow-roll inflation})$$

which is probably negligible.

Summary of

# Slow-roll inflation

Flat potential: $\epsilon \ll 1, \quad |\eta| \ll 1$

$$\epsilon \equiv \frac{1}{2} M_P^2 \left(\frac{V'}{V}\right)^2 \qquad \left(\frac{M_P}{M_{EW}}\right)^2 \simeq 10^{32}$$

$$\eta \equiv M_P^2 \frac{V''}{V}$$

**The predictions (single-field inflation)** (inflaton paradigm).

$$\mathcal{P}_\zeta^{\frac{1}{2}} = 2.0 \times 10^{-5} \left( \frac{(V_*/\epsilon_*)^{\frac{1}{4}}}{6.6 \times 10^{16}\,\mathrm{GeV}} \right)^2$$

$$n - 1 = 2\eta_* - 6\epsilon_* \boxed{\simeq 2\eta_*} \quad \text{(typically)}$$

$$r = \left( \frac{V_*^{\frac{1}{4}}}{3.3 \times 10^{16}\,\mathrm{GeV}} \right)^4$$

$$= 16\epsilon_* \simeq 0$$

$$N = \frac{1}{\sqrt{2} M_P} \int_{\phi_{\rm end}}^{\phi_*} \frac{d\phi}{\sqrt{\epsilon(\phi)}} \simeq 50$$

Note: to get measurable $\epsilon_*$ would need $(\phi_* - \phi_{\rm end}) \sim M_P$. [DHL 1997]

## Condition for the validity of the inflaton paradigm

When considering a given cosmological scale, $\frac{k}{a}$ outside the horizon, we should invoke a smoothed Universe where all shorter-scale modes ~~are~~ are dropped. Then there are no causal processes, and at each comoving position $\underline{x}$ the Universe looks like some unperturbed ~~Universe~~ 'separate universe'. In general these are not identical. Rather, starting at some time $\tilde{t}$ a few Hubble times after inflation, the future evolution is determined by the values of all of the light fields $\{\phi(\underline{x}, \tilde{t}),\ \cdots$ OTHER LIGHT FIELDS$\}$

(Of course the heavy fields, fixed during inflation, also ~~do~~ may play a role in determining the ~~exact~~ evolution, but they are unperturbed so we don't need to consider them.)

~~A~~ The <u>inflaton paradigm</u> will be valid if only the inflaton $\phi(\underline{x}, \tilde{t})$ ~~is~~ plays a role. Why? ~~It~~ Because then $\rho(\underline{x}, t)$ and $P(\underline{x}, t)$ at future times are both functions of $\phi(\underline{x}, \tilde{t})$ only, hence $P$ is a function of $\rho$ only leading to conservation of $\zeta$ as we noted earlier.

As a bonus from the inflaton paradigm, we learn that the primordial density perturbation (present a bit before horizon entry, after nucleosynthesis) is also adiabatic $\{S_i = 0\}$: the ~~$n_i$~~ too $n_i(\underline{x}, t)$ too depend only on $\phi(x, \tilde{t})$ hence only on $\rho(\underline{x}, t)$.

In contrast, the <u>curvaton paradigm</u> invokes a ~~second field~~ curvaton field $\sigma(\underline{x}, t)$, and supposes that $\zeta$ is negligible during inflation but generated later by $\delta\sigma$. Now there may be <u>isocurvature</u> density perturbations $S_i$.

## General comments about inflation model-building

Supersymmetry makes it **possible but not easy**.

Usually inflate near a maximum. NOT DIFFICULT for inflaton to get there.

We DON'T need fine tuning.

We CAN use extensions of the MSSM which already exist for other purposes.

We DON'T need fancy ideas like branes.

Many models give $1 - n \gtrsim 0.01$, allowing observation to **rule them out**.

In this talk, we just have time for one paradigm, 'Modular Inflation'.

## Modular Inflation

Working definition of modulus:
A field whose potential has width $\sim M_{\rm P}$,
$V(\phi) = \Lambda^4 f(\phi/M_{\rm P})$

**The simplest model**

[Binetruy & Gaillard 1986]

$V(\phi) = M_{\rm S}^4 - \frac{1}{2} m_\phi^2 \phi^2 + \cdots$
$M_{\rm S} \sim 10^{10}$ GeV (**vacuum** SUSY breaking scale).
$m_\phi^2 \sim M_{\rm S}^4 / M_{\rm P}^2$ (soft mass) so $|\eta| \sim 1$.

**Optimistically** $|\eta| \sim 10^{-2}$ giving $|n-1| \sim 10^{-2}$.

BUT inflaton perturbation gives only
$\mathcal{P}_\zeta^{\frac{1}{2}} \sim 10^{-15}$ (cf. observation: $\mathcal{P}_\zeta^{\frac{1}{2}} \sim 10^{-5}$).

**What to do?**

**EITHER** invoke a curvaton (but the low scale might make this tricky; work in progress K. Dimopoulos, Y. Rodriguez-Garcia, DHL)

**OR** fix modular inflation so that inflaton gives correct $\mathcal{P}_\zeta$: three possibilities.

a) Raise $M_S$ to $10^{15}$ GeV

Eg. (Banks 1999) use a 'bulk modulus' of Horava-Witten $M$-theory.

(See also Adams, Ross & Sarkar 1997)

**b) Go to 2-field inflaton**

[Kodata & Stewart 2003]

(i) Remember that $\phi$ is complex.

(ii) Assume unsuppressed couplings at $\phi = 0$ ('point of enhanced symmetry'). Assume running mass gives crater potential.

(iii) Use both magnitude and phase of $\phi$.

**Result**: a two-field inflation model which gives correct $\mathcal{P}_\zeta$ if we live in the right part of the Universe.

## c) Go to Hybrid Modular Inflation

[Linde 1991; Randall, Soljacic & Guth, 1996]

$$\begin{aligned} V &= M_\mathrm{S}^4 + \frac{1}{2}m_\phi^2\phi^2 + \lambda\phi^2\chi^2 - \frac{1}{2}m_\chi^2\chi^2 + \cdots \\ M_\mathrm{S} &\sim 10^{10}\,\mathrm{GeV} \\ m_\phi^2 &\sim 10^{-2}M_\mathrm{S}^4/M_\mathrm{P}^2 \\ m_\chi^2 &\sim 10 M_\mathrm{S}^4/M_\mathrm{P}^2 \\ \lambda &\sim 10^{-3} \end{aligned}$$

These 'natural' parameters give correct $\mathcal{P}_\zeta$.

**Running mass variant [Stewart, 1997]**

Just $m_\phi^2 \sim M_\mathrm{S}^4/M_\mathrm{P}^2$ at Planck scale.

Running generates $m_\phi^2 \sim 0$ during inflation.

Insensitive to $\lambda$, Gives correct $\mathcal{P}_\zeta$.

BUT gives running spectral index; dangerous?
[L. Covi, A. Melchiorri & DHL in progress]

## Summary

**The primordial density perturbation originates as the quantum fluctuation of some scalar field. An astonishing extension of quantum physics. (Cf. white dwarf)**

**This field might be the INFLATON or some CURVATON.**

**Observation can rule out models.**

**In 2008 (??) PLANCK and LHC might make or break our subject.**

# Inflation problems

1. Going back in time, for as long as the energy density of the Universe is dominated by particles, the ratio $r \equiv (aH)/(a_0H_0)$ increases.
a) Explain why this gives rise to the *horizon* and *flatness* problems.
b) Estimate $r$ at the beginning of the matter–dominated epoch, using the approximation $a \propto t^{\frac{2}{3}}$.
c) Estimate $r$ at the epoch $k_{\mathrm{B}}T \sim 10\,\mathrm{MeV}$ using the approximation $T \propto 1/a$ together with the radiation–domination expression $a \propto t^{\frac{1}{2}}$.

2. The horizon and flatness problems can both be solved if there is an era of cosmological inflation during which the energy density is constant. Assuming that reheating after inflation is instantaneous with reheat temperature given by $k_{\mathrm{B}}T \sim 10\,\mathrm{MeV}$, estimate the minimum number of Hubble times of inflation that is needed to solve the horizon problem. Do the same with $k_{\mathrm{B}}T \sim 10^{16}\,\mathrm{GeV}$.

3. Use the definition of the spectrum $\mathcal{P}_g(k,t)$ (of a generic perturbation $g$) to show that $\langle|g(\mathbf{x},t)|^2\rangle$ is actually independent of the comoving position $\mathbf{x}$.

4. Show that the conditions $\epsilon \ll 1$ and $|\eta| \ll 1$ are necessary for the slow-roll approximation $\dot{\phi} = -V'/3H$.

5. Work out the most general potential $V(\phi)$ which gives exactly $n = 1$ in the slow-roll approximation. In which regime of $\phi$ does this potential in fact lead to slow-roll?

SOLUTIONS TO INFLATION PROBLEMS

1. b) $a \propto t^{2/3}$, $H \propto \frac{1}{t}$, $aH \propto t^{-1/3} \propto a^{-\frac{1}{2}}$

M.D. begins at $a \sim 10^4$, so $r \sim 10^2$

c) M.D. begins at ~~$a \propto$~~ $T \sim 1\,eV$. Before that,

$a \propto t^{\frac{1}{2}}$, $H \propto \frac{1}{t}$, $aH \propto t^{-\frac{1}{2}} \propto a^{-1} \propto T$

So $r \sim 10^2 \times \frac{10\,MeV}{1\,eV} \sim 10^9$

2. Need $(aH)_{beg} \sim 1$ to solve horizon problem.

$$\begin{array}{r} 56 \\ 11 \\ \hline 67 \end{array}$$

Hence $N \sim \ln 10^9 \sim 22$ if $T_{reh} \sim 10\,MeV$

But if $T_{reh} \sim 10^{16}\,GeV$,

$r \sim 10^2 \times \frac{10^{16}\,GeV}{1\,eV} \sim 10^{27}$ and $N \sim \ln 10^{27} \sim 67$

3. $g(\underline{x}) = \text{const} \int g(\underline{k})\, e^{i\underline{k}\cdot\underline{x}}$

$\langle |g(\underline{x})|^2 \rangle = \text{const} \int \langle g^*(\underline{k})\, g(\underline{k}') \rangle\, e^{i(\underline{k}-\underline{k}')\cdot\underline{x}}\, d^3k\, d^3k$

and $\langle g^*(\underline{k})\, g(\underline{k}') \rangle = \text{const}\; \mathcal{P}(k)\; \delta^3(\underline{k}-\underline{k}')$

hence $\langle |g(\underline{x})|^2 \rangle = \text{const} \int \mathcal{P}(k)\, d^3k$, indep of $\underline{x}$

~~4. $n \pm 2\gamma - 6\epsilon = 0$ requires $\mathcal{P}$~~

4. Want to show that

$$3H\dot{\phi} \simeq -V' \qquad (1)$$

and $$\dot{\phi}^2 \ll V \qquad (2)$$

imply $\epsilon \ll 1$ and $|\eta| \ll 1$.

Using (1) for $\dot{\phi}$, (2) becomes

$$\boxed{\epsilon \ll \frac{1}{2}}.$$

Comparing (1) with the exact eqn

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

gives $$\ddot{\phi} \ll 3H\dot{\phi},$$

and using (1) for $\dot{\phi}$ this becomes

$$|\epsilon - \eta| \ll 1$$

which gives $|\eta| \ll 1$ since $\epsilon \ll 1$.

(We need to remember that

$$3H^2 M_p^2 = V$$

and that $\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi}$ )

5. Remember $P_\zeta^{\frac{1}{2}}(k) = \text{const.} \left. \frac{V^3}{V'} \right|_{k=aH}$

and the question supposes that $P_\zeta$ is independent of $k$. This requires $V^3 \propto V'^2$,

$$\frac{d\phi}{dV} \propto V^{-3/2}$$

$$\therefore \boxed{V \propto 1/\phi^2}$$

To find out in what regime it inflates, evaluate $\epsilon$ and $\eta$:

$$\epsilon \equiv \frac{1}{2} M_p^2 \left( \frac{V'}{V} \right)^2 = 2 \frac{M_p^2}{\phi^2}$$

$$\eta \equiv M_p^2 V''/V = 6 \frac{M_p^2}{\phi^2}$$

so $V$ inflates ~~when~~ if

$$\phi \gg M_p .$$