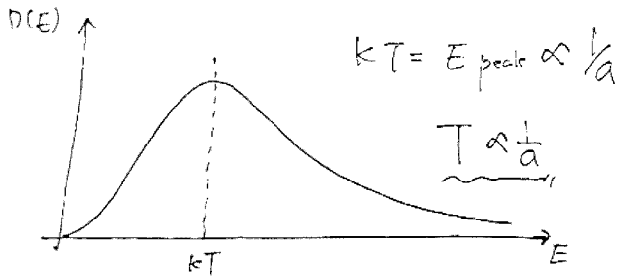


Energy Distribution of  $\gamma$ ; at low T, no interaction  
 $L[f]=0$

1

$$\frac{\partial f}{\partial t} - 2aEf' = 0$$

This is satisfied by  $f(aE)$  ( $f(E) = \frac{2}{e^{E/kT} - 1}$ )



parametrize deviations from this spectrum:

$$(a) \quad f(E) = \frac{1}{e^{(E-\mu)/kT} - 1}$$

ex) decay of some particle X into  $\gamma$  (at a time when thermal equilibrium can't be maintained)

$$|M| < 9 \times 10^{-5}$$

(b) The interaction of the CMB with a non-relativistic gas

$$y = \int_0^l \frac{\sigma_e n_e}{T} \frac{kT_{\text{CMB}}}{m_e c^2} dl \rightarrow$$

Thomson  
S.C.S

$$y = \frac{k\omega}{kT_{\text{CMB}}}$$

Ex: Sunyaev-Zeldovich effect.

$$\frac{\delta T}{T} = -y \frac{\alpha e^2}{e^x - 1} \left[ 4 - x \coth\left(\frac{x}{2}\right) \right]$$

$$|y| < 1.2 \times 10^{-5}$$

## Exercise (a)

Observer with velocity  $\vec{v}$  relative to a source emitting a photon with proper momentum  $E = p \hat{n}$

Prove:  $E' = \gamma E (1 - \hat{n} \cdot \vec{v})$   
 $T' = ? \quad \left( \frac{\Delta T}{T} \sim 10^{-3} \right) \quad \underline{v = ?}$

(b)

We modeled the surface of the last scattering:

$$D = t_{*}(-1, \hat{n}) \equiv (t, \vec{D})$$

Now angular positions in boosted frame related to the one at rest?

$$\vec{D}' = \vec{D} + [(\gamma-1)\hat{\beta} \cdot \vec{D} + \gamma t \beta] \hat{\beta}$$

Characterizing radiation

$$\vec{E} = \vec{E} e^{i(\omega t + k z)}$$

$$\vec{E} = \begin{bmatrix} a_x e^{i\phi_x} \\ a_y e^{i\phi_y} \end{bmatrix} \text{ Polarization}$$

(a)  $\phi_x = \phi_y$  : linearly polarized wave.

$a_x = a_y$  ;  $\phi_x = \phi_y + \frac{\pi}{2}$  : circularly polarized.

This is not good for rotation !!

$$\vec{E}' = R(\theta) \vec{E} \quad \text{let } \vec{E}' = \begin{bmatrix} a'_x e^{i\phi'_x} \\ a'_y e^{i\phi'_y} \end{bmatrix}$$

$$a(x)^2 = a_x^2 \cos^2 \theta + a_y^2 \sin^2 \theta + \sin^2 \theta a_x a_y \cos(\phi_x - \phi_y)$$

→ bad coordinate !!

Construct ~~the~~ Intensity tensor:

$$I_{ab} = E_a^* E_b = \begin{bmatrix} a_x^2 & a_x a_y e^{i(\phi_x - \phi_y)} \\ a_x a_y e^{-i(\phi_x - \phi_y)} & a_y^2 \end{bmatrix}$$

Hermitian  $I_{ab} = I_{ba}^*$

We can decompose it into 4 parts

$$I_{ab} = I \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + P_{ab} + \underbrace{V \frac{1}{2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}}_?$$

$$P_{ab} = Q \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + U \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$[I, Q, U, V]$  are the stoke's parameters

$$I = a_x^2 + a_y^2 \text{ (total Intensity)}$$

$$Q = a_x^2 - a_y^2 \text{ (linear polarization along } xy \text{ axis)}$$

$$U = 2 a_x a_y \cos(\phi_x - \phi_y) \text{ (linear polarization at } 45^\circ \text{ to } xy)$$

$$V = 2 a_x a_y \sin(\phi_x - \phi_y) \text{ (} \overset{\text{Circular}}{\text{pol.}})$$

What happens under rotations?

$$U \text{ \& } V \equiv \text{invariant}$$

$$\left. \begin{aligned} Q' &= Q \cos 2\theta + U \sin 2\theta \\ U' &= U \cos 2\theta + Q \sin 2\theta \end{aligned} \right\} P_{ab} \text{ transforms as a spin } 2\text{-field.}$$

Sometimes it's useful to represent  $P_{ab}$  as a vector

$$\vec{P} = \begin{bmatrix} Q \\ U \end{bmatrix} = \begin{bmatrix} P \cos 2\theta \\ P \sin 2\theta \end{bmatrix}$$

$$I_{ab}(\vec{r}, t)$$

We can decompose into  $I(\vec{r}), P_{ab}(\vec{r}), V(\vec{r})$

$$P_{ab}(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k P_{ab}(\vec{k}) e^{i\vec{k}\cdot\vec{r}}$$

let us define a new basis:

$$E_{ab}(\vec{k}, \vec{r}) = \frac{1}{k^2} (k_a k_b - \frac{1}{2} k^2 \delta_{ab}) e^{i\vec{k}\cdot\vec{r}}$$

$$B_{ab}(\vec{k}, \vec{r}) = \frac{1}{k^2} (k_a k_b \epsilon_{bc}^c + k_b k_c \epsilon_{ca}^c) e^{i\vec{k}\cdot\vec{r}}$$

⚡ completely antisym. tensor

$$P_{ab}(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k [E(k) E_{ab}(\vec{k}, \vec{r}) + B(k) B_{ab}(\vec{k}, \vec{r})]$$

$E(k)$ : electrical part of  $P_{ab}$  ( $\vec{E}$  is even under parity)  
transf.

$$k_a k_a E_{ab} = E_{ab} k_a k_b = 0$$

$k \times E$  (curl free)

$B(k)$ : magnetic part of  $P_{ab}$

$$k_a B_{ab} = k_a B_{ba} = 0 \quad (\text{div. free})$$

$B$  is odd.

The variables (the physics)

$$\frac{\Delta T}{T} \sim 10^{-5} \Rightarrow \text{universe is quasi-homogeneous}$$

we can use linear perturbation theory around a homogeneous B.G.

homogeneous :  $a(\eta), P_x(\eta), f_r(E, \eta), [= f_r(aE)]$

Inhomogeneous :  $\Phi, \Psi$  ,  $\delta P_x, \delta P_x$  ,  $\delta f_r$   
 (metric perturbations) ↓ density perturbation

$\delta(\eta, \vec{x})$

\* perturbation variables depend on choice of coordinate system.

gauge choice

one can pick different coordinate system:

$$(\eta, \vec{x}) \rightsquigarrow (\eta', \vec{x}')$$

In an unperturbed space, there is a preferred coordinate system.

- motion of comoving observers: zero momentum
  - free falling (isotropic)
  - energy density is constant (homogeneous)
- } are equivalent
- $\rho(\eta)$

If spacetime is perturbed.



$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x^\mu)$$

$$S'(x^\mu) = S(x^\mu) + \epsilon^\alpha \nabla_\alpha S$$

$$P' = P - \epsilon^0 \dot{P}$$

Local changes of coordinates can lead to local change of energy density.

Solutions

- Can construct gauge invariant variable
- gauge ambiguities disappear on sub-horizon scale.
- observable quantities are gauge invariant.

(C. Peelman + E. Bertsch) ApJ, 455, 12 (1995)

spatial variations of our physical quantities

Metric: 
$$g_{00}(\eta, \vec{x}) = a^2(\eta) (1 + 2\Phi(\eta, \vec{x}))$$

$$g_{ij}(\eta, \vec{x}) = -a^2(\eta) (1 - 2\Psi(\eta, \vec{x})) \delta_{ij}$$
 (Note:  $\Phi$  is labeled as Newtonian potential)

Matter: 
$$P_x(\eta, \vec{x}) = \bar{P}_x(\eta) (1 + \delta_x(\eta, \vec{x}))$$

$$\dot{P}_x(\eta, \vec{x}) = \dot{\bar{P}}_x(\eta) + \delta \dot{P}_x(\eta, \vec{x}) = \omega \bar{P}_x(\eta) + c_s^2 \delta \dot{P}_x = \bar{P}_x(\eta) (\omega + c_s^2 \delta_x(\eta, \vec{x}))$$
 (Note:  $c_s^2 = \frac{\partial P}{\partial \rho}$  is labeled as speed of sound)

Radiation: (describe in terms of phase space distn.)  
 $f_I(\eta, \vec{p}, \vec{x}), f_Q(\eta, \vec{p}, \vec{x}), f_V, \dots$

At 0-th order (the homogeneous terms)  $f_{0Q} = f_{0V} = f_{0U} = 0$

$$f_{0I}(\eta, \vec{p}, \vec{x}) = \frac{2}{\exp[\rho c / kT] - 1} = f_{0I}(\eta, \rho)$$

Define brightness function:

The intensity,  $I$  (power per unit area perpendicular to  $\vec{p}$ ) in a given range  $d^3p$ .

$$\frac{\delta I}{I} = \Delta \leftarrow \text{Brightness}$$

Note: for blackbody:

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$$I \propto T^4 \Rightarrow \delta I \propto 4T^3 \delta T$$

$$4 \frac{\delta T}{T} = \frac{\delta I}{I}$$

• assume (for now)  $\Delta$  is indep. on  $p$ .

$$f_I(n, \vec{p}, \vec{x}) = \frac{2}{\exp\left(\frac{pc}{k_B(\tau + \delta T)}\right) - 1} = f_{0I} + \delta f_{0I}$$

$$\delta T(n, \vec{p}, \vec{x})$$

$$f_I(x) \approx \frac{1}{e^x - 1} \quad x = \frac{pc}{k_B(\tau + \delta T)}$$

$$\delta f_I = \frac{\partial f_I}{\partial x} \frac{dx}{dT} \delta T = \frac{\partial f_{0I}}{\partial x} \left[ -\frac{k_B pc}{(k_B T_0)^2} \right] \delta T$$

$$= \frac{\partial f_{0I}}{\partial x} \left( -\frac{pc}{k_B T_0} \right) \frac{\delta T}{T_0} = -p \frac{\partial f_{0I}}{\partial p} \frac{\delta T}{T_0} = -\frac{p}{4} \frac{df_I}{dp} \Delta I$$

$$-p \frac{\partial f_{0I}}{\partial p}$$

$$\Delta I = \left( -\frac{1}{4} \frac{df_I}{dp} \right)^{-1} \delta f_I$$

$$\Delta X = \left( -\frac{1}{4} \frac{df_I}{dp} \right)^{-1} \delta f_I \quad X = I, Q, U, V$$

$$\vec{I} = \frac{\beta_V}{4\pi} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta_V \begin{pmatrix} \Delta I \\ \Delta Q \\ \Delta U \\ \Delta V \end{pmatrix}$$

(Stokes vector)

## The equations

Einstein's equation:

$$\delta G^a_b = 8\pi G \delta T^a_b$$

example)

$$k^2 \Phi + 3 \frac{\dot{a}}{a} (\dot{\Phi} + \frac{\dot{a}}{a} \Phi) = -4\pi G a^2 \sum_k \delta P_x$$

$$k^2 (\dot{\Phi} + \frac{\dot{a}}{a} \Phi) = 4\pi G a^2 \sum_x (P_x + P_x) \Theta_x \quad \hookrightarrow \Theta_x = \vec{v} \cdot \vec{v}_x$$

Energy momentum conservation:

for example, (CDM)

$$\begin{cases} \dot{\delta}_c = -(\Theta_c - 3\dot{\Phi}) \\ \dot{\Theta}_c = -\frac{\dot{a}}{a} \Theta_c + k^2 \Phi \end{cases}$$

Let's not forget Baryons

Boltzmann Equation

$$\frac{d\vec{I}}{d\eta} = \underbrace{\sigma_T n_e a}_{\text{Liouville term}} [\vec{I} - \vec{I}_c] \quad \rightarrow \quad I_s(\hat{n}, \vec{x}) = \frac{1}{4\pi} \int d^n \hat{n}' P(\hat{n}, \hat{n}') \vec{I}(\hat{n}', \vec{x})$$

↑  
Scattering matrix

Liouville term

$$\frac{df_x}{d\eta} = \frac{\partial f_x}{\partial \eta} + \frac{\partial f_x}{\partial x^i} \frac{dx^i}{d\eta} + \frac{\partial f_x}{\partial p^i} \frac{dp^i}{d\eta}$$

↓ linear order

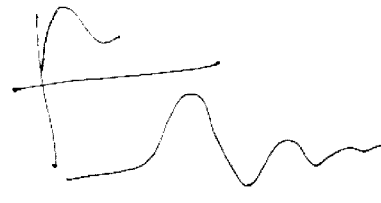
↪  $\frac{\partial f_x}{\partial \hat{n}} \frac{d\hat{n}}{d\eta} + \frac{\partial f_x}{\partial \mathcal{E}} \frac{d\mathcal{E}}{d\eta}$  (moving)

↓  $\mathcal{O}(2)$ , ignore!!

$$\frac{\partial \delta f_x}{\partial \eta} + \hat{n} \cdot \vec{\nabla}(\delta f_x) + ( ? )$$

$$dt = a d\eta$$

$$d\vec{r} = a d\vec{x}$$



comoving	physical
$(\eta, \vec{x})$	$(t, \vec{r})$
↓	↓
$(E, \vec{\delta})$	$(E, \vec{P})$

$$\left. \begin{aligned} E &= a\bar{E} \\ \vec{\delta} &= a\vec{P} \end{aligned} \right\} \leftarrow \text{upper-turbled spacetime.}$$

$$\left. \begin{aligned} P^0 &= g_{00}^{1/2} E = a(1+\Phi) E \\ P^i &= g_{ii}^{1/2} P^i = a(1-\Psi) P^i \end{aligned} \right\} \leftarrow \text{perturbed spacetime.}$$

geodesic eq.

$$\frac{dP^\mu}{d\eta} = g^{\mu\nu} \left( \frac{1}{2} g_{\alpha\beta,\nu} - g_{\nu\alpha,\beta} \right) \frac{P^\alpha P^\beta}{P^0}$$

Exercise derive  $\left( \frac{d\delta}{d\eta} \right)$

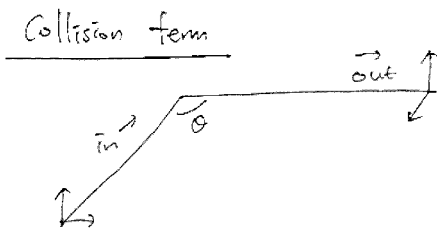
Recall :  $\vec{n} = \frac{\vec{\delta}}{\delta}$

$$\frac{1}{\delta} \frac{d\delta}{d\eta} = \frac{\partial \psi}{\partial \eta} - \vec{n} \cdot \nabla \Phi$$

$$\frac{d \delta f_x}{d\eta} - \frac{\partial \delta f_x}{\partial \eta} + \vec{n} \cdot \nabla (\delta f_x) + \delta \frac{\partial f_0}{\partial \eta} (\psi - \vec{n} \cdot \nabla \Phi)$$

Multiply all by  $\left( -\frac{1}{4} \frac{\partial f_0}{\partial \ln \delta} \right)^{-1} : \mu = \vec{n} \cdot \hat{k}$

$$\dot{\Delta}_I + i k_\mu \Delta_I + 4(\dot{\Phi} - i k_\mu \Phi) : \text{Liouville term}$$



$$\vec{U}_{in} = (0, \sin\theta, \cos\theta)$$

$$\vec{U}_{out} = (0, 0, 1)$$

$$\vec{E}_{(in)} \perp \vec{U}_{(in)}$$

$$\vec{E}_{(out)} \perp \vec{U}_{(out)}$$

$$E_{(in)x} \parallel E_{(out)x}$$

$$E_{(in)y} - E_{(out)y} = \cos\theta$$

Thomson Scattering:  $\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{8\pi} |\vec{E}_{in} \cdot \vec{E}_{out}|^2$

Interpretation:

the prob. of

$$\left( \begin{array}{l} E_{(in)x} \rightarrow E_{(out)x} \quad \text{is } \frac{3\sigma_T}{8\pi} \\ E_{(in)y} \rightarrow E_{(out)y} \quad \text{is } \frac{3\sigma_T}{8\pi} \cos^2\theta \end{array} \right.$$

$$\vec{I}_{(in)} = (I_{in}, 0, 0, 0)$$

$$\vec{I}_{(out)} = \frac{3\sigma_T}{8\pi} (1 + \cos^2\theta, \sin^2\theta, 0, 0)$$

Note:

$$1 + \cos^2\theta = 2 \left( 1 + \frac{1}{2} P_2(\cos\theta) \right)$$

Consider the total scattered intensity.  $I_{(in)}(\theta, \phi)$

$$I_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega (1 + \cos^2\theta) I_{(in)}(\theta, \phi)$$

$$Q_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega \sin^2\theta \cos 2\phi I_{(in)}(\theta, \phi)$$

$$L_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega \sin^2\theta \sin 2\phi I_{(in)}(\theta, \phi)$$

$$I_{(in)}(\theta, \varphi) = \sum a_{lm}^{I_{(in)}} Y_{lm}(\theta, \varphi)$$

$$I_{(out)} = \frac{3\sigma_T}{16\pi} \left[ \frac{8\sqrt{\pi}}{3} a_{00} + \frac{4}{3} \sqrt{\frac{\pi}{3}} a_{20} \right]$$

$$Q_{(out)} = \frac{3\sigma_T}{8\pi} \left[ a_{22}^{I_{(in)}} + a_{2,-2}^{I_{(in)}} \right] \sqrt{\frac{2\pi}{15}} \left\{ \begin{array}{l} \text{polarization is generated by the} \\ \text{quadrupole anisotropy of the incoming} \\ \text{Intensity.} \end{array} \right.$$

$$U_{(out)} = -\frac{13\sigma_T}{8\pi} \sqrt{\frac{2\pi}{15}} \left( a_{22}^{I_{(in)}} - a_{2,-2}^{I_{(in)}} \right)$$

The full Boltzmann Equations are

peculiar velocity of the scatterers

$$\dot{\Delta}_I + ik_\mu \Delta_I + 4[\dot{\Psi} - ik_\mu \Phi] = -\sigma_T n_e a [\Delta_I - \Delta_{I_0} + 4\mu \Theta_B - \frac{1}{2} P_2(\mu) (\Delta_{I_2} + \Delta_{Q_2} - \Delta_{Q_0})]$$

$$\dot{\Delta}_Q + ik_\mu \Delta_Q = -\sigma_T n_e a [\Delta_Q - \frac{1}{2} (1 - P_2(\mu)) (\Delta_{I_2} + \Delta_{Q_2} + \Delta_{Q_0})]$$

NOTE

$$\vec{k} \cdot \hat{m} = \mu$$

$$\Delta(\eta, \hat{m}, \vec{k}) \Rightarrow \Delta_l(\eta, k) = \frac{1}{2} \int_{-1}^1 d\mu \Delta(\eta, k, \mu) P_l(\mu)$$

Baryon

$$\dot{\Theta}_B = -\Theta_B + 3\dot{\Psi}$$

$$\dot{H}_B = -\frac{\dot{a}}{a} H_B + k^2 \Phi + \frac{4}{3} \frac{\rho_Y}{\rho_B} a n_e \sigma_T (\Theta_Y \cdot \Theta_B)$$

$$H_Y = \frac{3}{4} ik \Delta_{I1} \quad \text{exchange of momentum betw. B and Y.}$$

## Initial Conditions

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Consider two fluids

- non-relativistic fluid :  $(E \propto m), p=0$
- relativistic fluid :  $E \propto U, p = \frac{1}{3} E$

\* Adiabatic (isotropic)

$$\frac{n_x}{n_r} = \text{cte} \Rightarrow \delta \left( \frac{n_x}{n_r} \right) = 0 \Rightarrow \frac{\delta n_x}{n_x} = \frac{\delta n_r}{n_r} //$$

$$p_r \propto T^4, \quad m_r \propto T^3$$

$$\Rightarrow p_r \propto n_r^{4/3}$$

$$\Rightarrow \frac{\delta p_r}{p_r} = \frac{4}{3} \frac{\delta n_r}{n_r}$$

$$p_x \propto n_x \Rightarrow \frac{\delta p_x}{p_x} = \frac{\delta n_x}{n_x}$$

$$\therefore \text{Adiabatic condition : } \boxed{\frac{\delta p_x}{p_x} - \frac{3}{4} \frac{\delta p_r}{p_r} = 0}$$
$$\delta x - \frac{3}{4} \Delta_{I_0} = 0$$

\* Isocurvature : overall density perturbation is zero.

$$p_T = p_x + p_r$$

$$\Rightarrow \delta p_T = 0 = \delta p_x + \delta p_r = p_x \frac{\delta p_x}{p_x} + p_r \frac{\delta p_r}{p_r} = 0$$

$$\boxed{\Delta_{I_0} = -\frac{p_x}{p_r} \delta x}$$

(early 70's ... isothermal)

Solutions

$$\tau = \int_{\eta}^{\eta_0} \sigma_{\tau} n_e a \, d\eta$$

$$\dot{\Delta}_{\text{I}} + ik_{\mu} \Delta_{\text{I}} - \dot{\tau} \Delta_{\text{I}} = -4 [\dot{\Psi} - ik_{\mu} \Phi] - i [\Delta_{\text{I}0} - 4\mu \Theta_{\text{B}}]$$

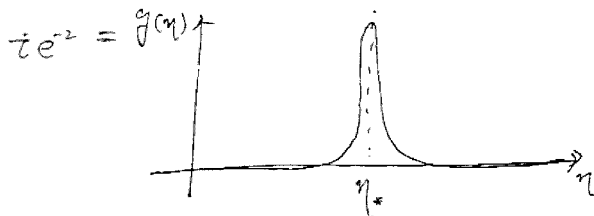
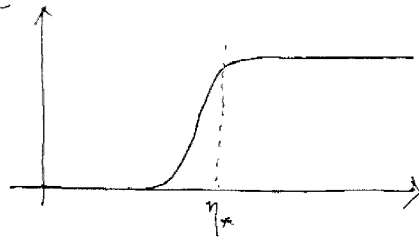
Rewrite w/ integrating factors

$$\frac{d}{d\eta} [e^{ik_{\mu}\eta} e^{-\tau} \Delta_{\text{I}}] = -4 [\dot{\Psi} - ik_{\mu} \Phi] e^{ik_{\mu}\eta} e^{-\tau}$$

$$\Delta_{\text{I}} = -4 \int_0^{\eta_0} d\eta (\dot{\Psi} - ik_{\mu} \Phi) e^{-\tau} e^{ik_{\mu}(\eta-\eta_0)} - \int_0^{\eta_0} d\eta i [\Delta_{\text{I}0} - 4\mu \Theta_{\text{B}}] e^{-\tau} e^{ik_{\mu}(\eta-\eta_0)}$$

↓ Integrate by part.

$$-4 \int_0^{\eta_0} d\eta (\dot{\Psi} + \dot{\Phi}) e^{-\tau} e^{ik_{\mu}(\eta-\eta_0)} - 4 \int_0^{\eta_0} d\eta \Phi i e^{-\tau} e^{ik_{\mu}(\eta-\eta_0)}$$



Approximation using a step function +  $\delta$  function

$$\Delta_{\text{I}} = -4 \int_{\eta_*}^{\eta_0} d\eta (\dot{\Psi} + \dot{\Phi}) e^{ik_{\mu}(\eta-\eta_0)} + [\Delta_{\text{I}0} + 4\dot{\Phi} - 4\mu \Theta_{\text{B}}] \Big|_{\eta_*}$$

Recall that:

$$i) \frac{1}{4} \Delta I = \frac{\delta T}{T}$$

ii) Fourier transform back

$$iii) \Delta I_0 = \delta r$$

$$iv) d_x = \eta_0 - \eta_x$$

$$\frac{\delta T}{T}(\hat{n}) = \underbrace{\frac{1}{4} \delta r(\eta_x, d_x \hat{n})}_{\text{intrinsic term}} - \underbrace{\hat{n} \cdot \vec{V}_B(\eta_x, d_x \hat{n})}_{\text{Doppler term}} + \underbrace{\Phi(\eta_x, d_x \hat{n})}_{\text{Sachs-Wolfe}} - \underbrace{\int_{\eta_x}^{\eta_0} d\eta (\dot{\Phi} + \dot{\Psi})(\eta, \eta \hat{n})}_{\text{integrated Sachs-Wolfe term}}$$

(Rees-Sciama effect)

$$\dot{z} = a \sigma_T n_e = a \sigma_T x_e n$$

three regimes:

}	strong coupling: $\frac{\dot{z}}{a} \gg 1$	MFP = 0	fully ionized
	weak coupling: $\frac{\dot{z}}{a} \approx 1$	MFP = finite	recombination
	free streaming: $\frac{\dot{z}}{a} \ll 1$	MFP = $\infty$	neutral

# Strong Coupling

$$\Delta_l = \frac{1}{2} \int_{-1}^1 d\mu P_l(\mu) \Delta(k, \mu)$$

$$\left. \begin{aligned} \delta_r = \Delta_{I_0} \\ \Theta_r = -\frac{3}{4} k \Delta_{I_1} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} &\text{Boltz. eq. (take 0th \& 1st moment of B.E.)} \\ &\dot{\delta}_r + \frac{4}{3} \Theta_r = 4\dot{\Phi} \\ &\dot{\Theta}_r = \frac{1}{4} k^2 \delta_r - ik^2 \Phi + \dot{\tau} (\Theta_B - \Theta_r) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \dot{\delta}_B &= -\Theta_B + 3\dot{\Phi} \\ \dot{\Theta}_B &= -\frac{\dot{\tau}}{\tau} \Theta_B + R^{-1} \dot{\tau} (\Theta_r - \Theta_B) + k^2 \Phi \end{aligned} \right.$$

$R = \frac{3 \rho_B}{4 \rho_r}$

for all  $l \geq 2$ ,  $\Delta_{I_l} = \frac{k}{2l+1} [l \Delta_{I_{l-1}} - (l+1) \Delta_{I_{l+1}}] + \dot{\tau} \Delta_{I_l}$

tight coupling ~~τ > 1~~  
 $\tau \gg 1 \Rightarrow \Delta_{I_l} = 0 \quad (l \geq 2)$

$\dot{\tau} \gg 1 \Rightarrow \Theta_B = \Theta_r$

$$\ddot{\delta}_r - \frac{4}{3} \tau \left( \frac{1}{1+R} \right) \dot{\delta}_r + \frac{k^2 R^{-1}}{3(1+R)} \delta_r + \frac{4}{3} k^2 \Phi = 4\ddot{\Phi} \quad (\text{exercise})$$

Define  $\bar{\delta}_r = \delta_r - 4\Phi$

$$\ddot{\bar{\delta}}_r + \frac{\dot{R}}{1+R} \dot{\bar{\delta}}_r + \frac{k^2}{3(1+R)} \bar{\delta}_r = -\frac{4}{3} k^2 \left( \frac{1}{1+R} \Phi - \Phi \right)$$

→ forced harmonic oscillator

Define the sound speed of this baryon/photon fluid.

$$c_s^2 \equiv \frac{1}{3(1+R)} \begin{cases} \frac{1}{3} & \rho_B \ll \rho_r \\ 0 & \rho_B \gg \rho_r \end{cases}$$

Define a sound horizon


$$r_s(\eta) = \int_0^\eta c_s(\eta') d\eta'$$

We can solve the eq.

$$\begin{aligned} \text{a) WKB} & \quad \bar{\delta}_r = A(\eta) e^{ikr_s} \\ \text{b) neglect source term} & \quad A(\eta) = \left( \frac{\rho + \dot{\rho}}{c_s^2 \rho^2 a^2} \right)^{1/2} \end{aligned} \quad \begin{aligned} \rho &= \rho_s + \rho_r \\ \dot{\rho} &= \dot{\rho}_r \end{aligned}$$

~~scribble~~

$$\Delta_{\mathcal{I}_0} = 4\Phi + A \cos(kr_s) + B \sin(kr_s)$$

Adiabatic i.c. ;  $\Delta_{\mathcal{I}_0} \neq 0$   
 $\dot{\Delta}_{\mathcal{I}_0} = 0$    $= 0$   $\Rightarrow \cos(kr_s)$

isocurvature i.c. ;  $\Delta_{\mathcal{I}_0} = 0$   
 $\dot{\Delta}_{\mathcal{I}_0} \neq 0$   $\Rightarrow \sin(kr_s)$

initial condition  $\rightarrow$  different modes are stimulated.

— Solve the equation on superhorizon scales  $(k\eta) \ll 1$

•  $\Phi$  is constant on superhorizon scales.

• Matter dominant era:

$$3\mathcal{H}^2 \Phi = -\frac{3}{2}\mathcal{H}^2 \delta \Rightarrow \bar{\Phi} = -\frac{1}{2}\delta$$

• Adiabatic :  $\delta - \frac{3}{4}\Delta_{\mathcal{I}_0} = 0 \rightarrow \Delta_{\mathcal{I}_0} = +\frac{4}{3}\delta = -\frac{8}{3}\bar{\Phi}$

•  $\frac{\delta T}{T}(\eta) = \frac{1}{4}\delta_r + \bar{\Phi} = -\frac{2}{3}\bar{\Phi} + \bar{\Phi} = \frac{1}{3}\bar{\Phi}$

Sachs - Wolfe formula

$$\frac{\delta T}{T}(\hat{n}) = \frac{1}{3} \Phi(\eta_*, d_* \times \hat{n})$$

Weak coupling ~~regime~~ regime

$$\frac{\dot{c}}{(\frac{\dot{a}}{a})} \sim 1$$

Define  $\eta_c = \frac{1}{a \sigma_T n_e}$  ,  $\dot{c} = \frac{1}{\eta_c}$   
 $\propto$  neglect metric terms

$$\dot{\Delta}_I + i k_\mu \Delta_I + \frac{1}{\eta_c} \Delta_I = \frac{1}{\eta_c} [\Delta_{I_0} - 4\mu \Theta_B] \quad \text{B.E.}$$

$$\zeta = \Delta_{I_0} - 4\mu \Theta_B$$

Solve iteratively in powers of  $\eta_c$

$$\Delta_I = \zeta - \eta_c \dot{\Delta}_I - \eta_c i k_\mu \Delta_I$$

replace to get next order

$$\Delta_I = \zeta - \eta_c (\dot{\zeta} + i k_\mu \zeta) + \eta_c^2 (\ddot{\zeta} + i k_\mu \dot{\zeta} - k^2 \mu^2 \zeta)$$

We can rewrite the equation :

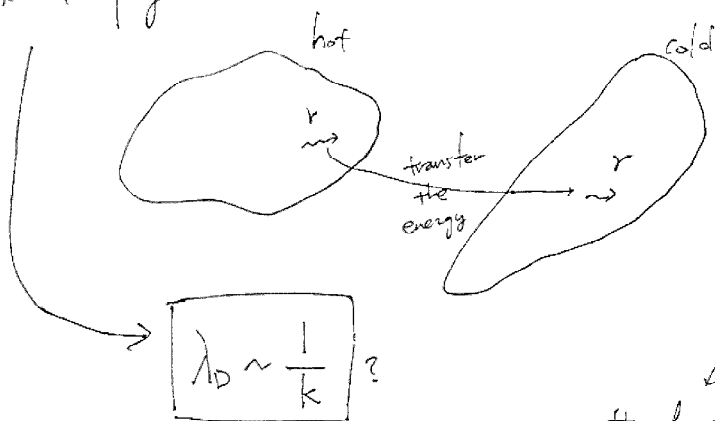
$$\begin{cases} \Delta_{I_0} = \dots \\ \Theta_B = \dots \end{cases}$$

Solve : we want the dispersion equation ( $e^{i\omega}$ )  
(relation)

$$\omega = \pm \frac{k}{\sqrt{3(1+R)}} + \underbrace{\frac{2k^2\eta_c}{6} \left( 1 - \frac{6}{5(1+R)} + \frac{1}{(1+R)^2} \right)}_{\text{damping term}}$$

$c_s k$

Silk damping

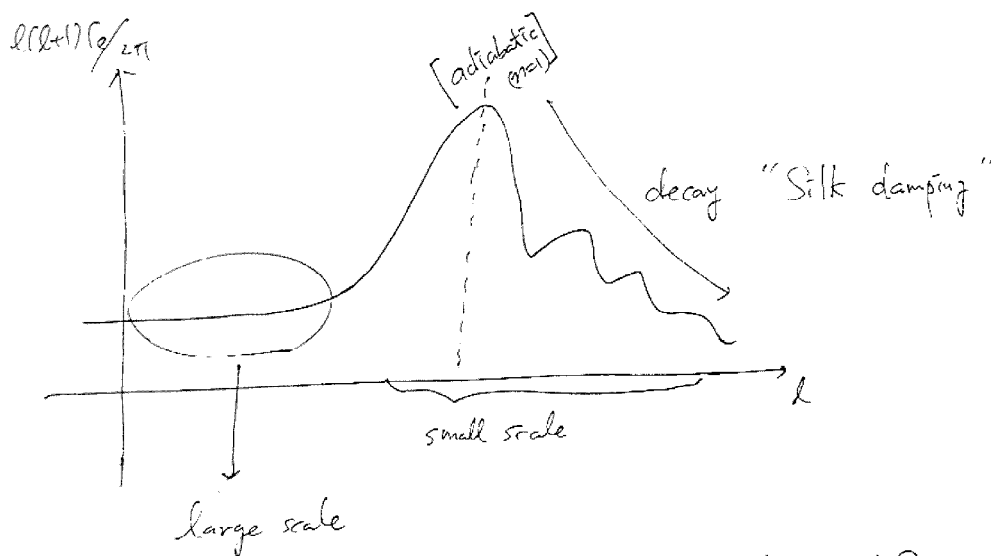


$$\eta_c = \frac{1}{a \sigma_T n_e} \quad \text{MFP}$$

$$\lambda_D = \sqrt{N} \eta_c$$

$$N = \frac{\eta}{\eta_c} \Rightarrow \lambda_D = (\eta \eta_c)^{1/2}$$

# of scattering



(Sachs-wolfe)  $\rightarrow$  gravitational potential at LS.

Putting it all together

Solve for  $\Delta$  ( $= \frac{\delta T}{T} \leftrightarrow \Delta_{lm}$ )

$$\langle a_{\ell m}^* a_{\ell' m'} \rangle = C_{\ell}^{-1} \underbrace{\delta_{\ell \ell'} \delta_{m m'}}_{\text{statistical isotropy}}$$

$$\langle \delta^*(\vec{k}') \delta(\vec{k}) \rangle = P(k) \delta^3(\vec{k} - \vec{k}')$$

$$\boxed{C_{\ell}^{-1} = \frac{1}{8\pi} \int k^2 dk |\Delta_{\ell}(k, \eta_0)|^2}$$

relation btw. what we solved & what we will plot.

## Polarization

• polarizations only generated at weak coupling

$$a) \Delta_{I2} = -\frac{1}{3} \Delta_{Q0} = -\frac{2}{15} ik \eta_0 \Delta_I$$

↑ finite

• Polarization is generated at recombination through the quadrupole

b) scalar perturbations ( $\delta, \Phi, \Psi$ ) will only generate E type polarization. (even)  
in parity

c) Tensor perturbation  $\Rightarrow$  both E+B types  
(gravity waves)

d) typically, E mode polarization is  $\sim 10\% \frac{\delta T}{T}$

What can we learn from the  $C_l$

i) test the initial conditions

adiabatic :  $\cos^2(kr_s) \sim C_l$

isocurvature :  $\sin^2(kr_s) \sim C_l$

$$r_s k = n\pi \quad (\text{for adiabatic})$$

$$r_s k = (n + \frac{1}{2})\pi \quad (\text{for iso.})$$

adiabatic : first peak at  $l \sim 220 \leftarrow \text{data}$

isocurvature : first peak at  $l \sim 330$

ii) The spectral index

$$\frac{\delta T}{T}(\hat{n}) = \frac{1}{3} \Phi(dx \hat{n})$$

$$k^3 \Phi \propto k^{n-1}$$

The  $C_l$  depend on  $n$

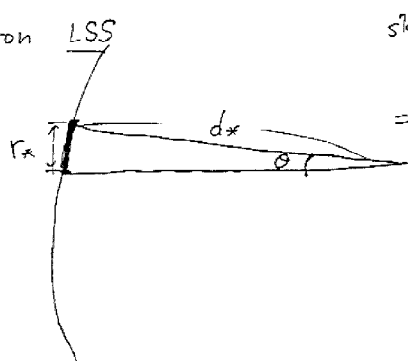
$$C_l \propto \frac{\Gamma[l + \frac{n-1}{2}] \Gamma[\frac{9-n}{2}]}{\Gamma[l + \frac{5-n}{2}] \Gamma[\frac{3+n}{2}]} \quad \left[ \Gamma \text{ is the gamma function} \right]$$

$n=1$  is scale invariant  $C_l \propto \frac{1}{l^2}$

iii) Geometry

$r_* = r_s(\eta_*)$  : sound horizon LSS

$$dx = \eta_0 - \eta_*$$

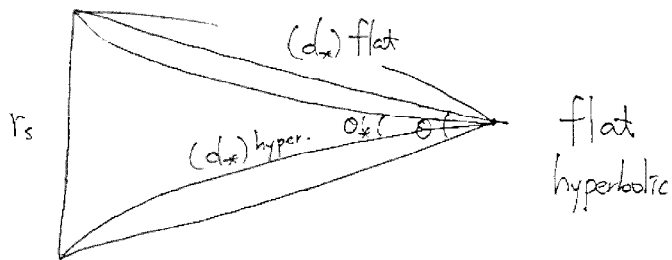


Flat universe

$$\sin \frac{\Theta}{2} = \frac{r_*}{2dx}$$

$$\Rightarrow \Theta = 2 \arcsin \left( \frac{r_*}{2dx} \right)$$

Suppose the universe is hyperbolic



$$\theta'_* = 2 \arcsin \left[ \frac{\sqrt{\frac{\cosh\left(\frac{r_s}{R_c}\right) - 1}{2}}}{\sinh\left(\frac{d_*}{R_c}\right)} \right]$$

$R_c$  is the curvature of space.