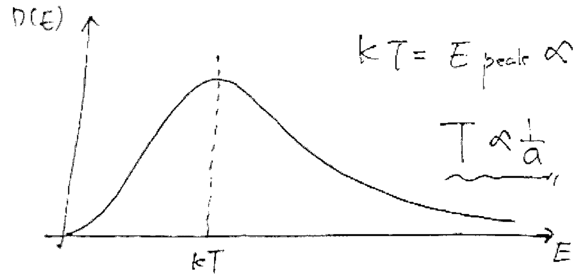
Energy Distribution of γ; at low T, no interaction

$L[f]=0$

$$\frac{\partial f}{\partial \eta} - 2\dot{a}Ef' = 0$$

This is satisfied by $f(aE)$ $\left(f(E) = \frac{2}{e^{E/kT}-1} \right)$



$kT = E_{peak} \propto 1/a$

$T \propto \frac{1}{a}$

parametrize deviations from this spectrum:

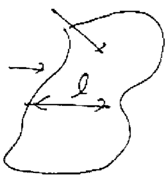
(a) $$f(E) = \frac{1}{e^{(E-\mu)/kT}-1}$$

ex) decay of some particle X into γ (at a time when thermal equilibrium can't be maintained)

$$|\mu| < 9 \times 10^{-5}$$

(b) The interaction of the CMB with a non-relativistic gas

$$y = \int_0^l \underset{\substack{\downarrow \\ \text{Thomson} \\ \text{S.C.S}}}{\sigma_e} n_e \frac{kT_{CMB}}{m_e c^2} dl$$



$$x = \frac{\hbar\omega}{kT_{CMB}}$$

Ex: Sunyaev-Zeldovich effect.

$$\frac{\delta T}{T} = -y \frac{x e^x}{e^x - 1}\left[4 - x \coth\left(\frac{x}{2}\right)\right]$$

$$|y| < 1.2 \times 10^{-5}$$

Exercise (a)

Observer with velocity $\vec{v}$ relative to a source emitting a photon with proper momentum $E = p\hat{n}$

Prove ; $E' = \gamma E(1 - \hat{n}\cdot\vec{v})$

$T' = ?$ $\left(\frac{\Delta T}{T} \sim 10^{-3}\right)$ $\underline{v = ?}$

(b)

We modeled the surface of the last scattering ;

$$D = t_*(-1, \hat{n}) \equiv (t, \vec{D})$$

How angular positions in ~~boosted~~ frame related to the one at rest ?

$$\vec{D}' = \vec{D} + [(\gamma - 1)\hat{\beta}\cdot\vec{D} + \gamma t\beta]\hat{\beta}$$

Characterizing radiation

$$\vec{E} = \vec{\varepsilon}e^{i(\omega t + kz)}$$

$$\vec{\varepsilon} = \begin{bmatrix} a_x e^{i\phi_x} \\ a_y e^{i\phi_y} \end{bmatrix} \quad \text{Polarization}$$

ex) $\phi_x = \phi_y$: linearly polarized wave.

$a_x = a_y$; $\phi_x = \phi_y + \frac{\pi}{2}$: circularly polarized.

This is not good for rotation !!

$$\vec{E}' = R(\theta)\vec{E} \qquad \text{let } \vec{E}' = \begin{bmatrix} a_x' e^{i\phi_x'} \\ a_y' e^{i\phi_y'} \end{bmatrix}$$

$$a(x)'^2 = a_x^2\cos^2\theta + a_y^2\sin^2\theta + \sin 2\theta\, a_x a_y \cos(\phi_x - \phi_y)$$

$\longrightarrow$ bad coordinate!!

Construct the intensity tensor :

$$I_{ab} = E_a^* E_b = \begin{bmatrix} a_x^2 & a_x a_y e^{i(\phi_x - \phi_y)} \\ a_x a_y e^{-i(\phi_x - \phi_y)} & a_y^2 \end{bmatrix}$$

Hermitian $I_{ab} = I_{ba}^*$

We can decompose it into 4-parts

$$I_{ab} = I \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + P_{ab} + \underbrace{V \frac{1}{2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}}_{?}$$

$$P_{ab} = Q \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + U \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$[I, Q, U, V]$ are the stoke's parameters

$I = a_x^2 + a_y^2$ (total intensity)

$Q = a_x^2 - a_y^2$ (linear polarization along xy axis)

$U = 2 a_x a_y \cos(\phi_x - \phi_y)$ (linear polarization at 45° to xy)

$V = 2 a_x a_y \sin(\phi_x - \phi_y)$ (circular pol.)

What happens under rotations?

U & $V \equiv$ invariant

$Q' = Q \cos 2\theta + U \sin 2\theta$

$U' = U \cos 2\theta + Q \sin 2\theta$

P_{ab} transforms as a spin 2-field.

Sometimes its useful to represent P_{ab} as a vector

$$\vec{P} = \begin{bmatrix} Q \\ U \end{bmatrix} = \begin{bmatrix} P \cos 2\theta \\ P \sin 2\theta \end{bmatrix}$$

$I_{ab}(\vec{r}, t)$

We can decompose into $I(\vec{r})$, $P_{ab}(\vec{r})$, $V(\vec{r})$

$$P_{ab}(\vec{r}) = \frac{1}{(2\pi)^2}\int d^2k\, P_{ab}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

let us define a new basis ;

$$\mathcal{E}_{ab}(\vec{k}, \vec{r}) = \frac{1}{k^2}\left(k_a k_b - \frac{1}{2}k^2 \delta_{ab}\right) e^{ikr}$$

$$\mathcal{B}_{ab}(\vec{k}, \vec{r}) = \frac{-1}{k^2}\left(k_a k_c \epsilon^c_b + k_b k_c \epsilon^c_a\right) e^{ikr}$$

completely antisym. tensor

$$P_{ab}(\vec{r}) = \frac{1}{(2\pi)^2}\int d^2k \left[E(k)\, \mathcal{E}_{ab}(k, \vec{r}) + B(k)\, \mathcal{B}_{ab}(\vec{k}, \vec{r}) \right]$$

$E(k)$: electrical part of P_{ab} ($\vec{\mathcal{E}}$ is even under parity transf.)

$$\epsilon_{ac} k_a \mathcal{E}_{cb} = \epsilon_{ab} k_a \mathcal{E}_{cb} = 0$$

$k \times \mathcal{E}$ (curl free)

$B(k)$: magnetic part of P_{ab}

$$k_a \mathcal{B}_{ab} = k_a \mathcal{B}_{ba} = 0 \quad \text{(div. free)}$$

B : is odd.

The variables (the physics)

$\frac{\Delta T}{T} \sim 10^{-5} \Rightarrow$ universe is quasi-homogeneous

we can use linear perturbation theory around a homogeneous B.G.

homogeneous ; $a(\eta)$, $\rho_x(\eta)$, $f_r(E,\eta)$, $[= f_r(aE)]$

Inhomogeneous ; $\underbrace{\Phi, \Psi}_{\text{metric perturbations}}$, $\underbrace{\delta\rho_x, \delta P_x}_{\text{density perturbation}}$, δf_r

$\boxed{\delta(\eta, \vec{x})}$

※ perturbation variables depend on choice of coordinate system.

gauge choice

one can pick different coordinate system:

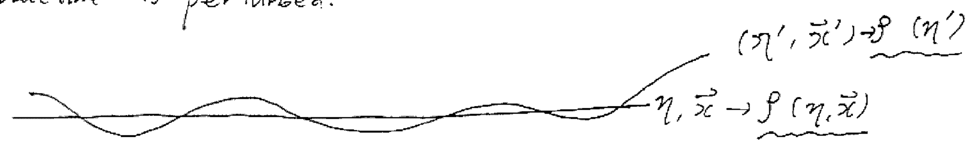
$$(\eta, \vec{x}) \rightsquigarrow (\eta', x')$$

In an unperturbed space, there is a prefered coordinate system.

- motion of comoving observers ; zero momentum
- free falling (isotropic)
- energy density is constant (homogeneous) $\rho(\eta)$

} are equivalent

If spacetime is perturbed.



$x^\mu \to x^\mu + \varepsilon^\mu(x^\mu)$

$$S'(x^\mu) = S(x^\mu) - \varepsilon^\alpha \nabla_\alpha S$$

$$\rho' = \rho - \varepsilon^0 \dot{\rho}$$

Local changes of coordinates can lead to local change of energy density.

Solutions

- Can construct gauge invariant variable
- gauge ambiguities disappear on sub-horizon scale.
- observable quantities are gauge invariant.

(C. Pelma + E. Berts.) ApJ. 455, 12 (95)

spacial variations of our physical quantities

Metric ;

$$g_{00}(\eta,\vec{x}) = a^2(\eta)\,(1+2\underbrace{\Phi}_{\text{newtonian potential}}(\eta,\vec{x}))$$

$$g_{ij}(\eta,\vec{x}) = -a^2(\eta)\,(1-2\Psi(\eta,\vec{x}))\,\delta_{ij}$$

Matter :

$$\rho_x(\eta,\vec{x}) = \bar{\rho}_x(\eta)\,(1+\delta_x(\eta,\vec{x}))$$

speed of sound $c_s^2 = \frac{\partial P}{\partial \rho}$

$$P_x(\eta,\vec{x}) = \bar{P}_x(\eta) + \delta P_x(\eta,\vec{x}) = \omega\bar{\rho}_x(\eta) + c_s^2\,\delta\rho_x = \bar{\rho}_x(\eta)\,(\omega + c_s^2\delta_x(\eta,\vec{x}))$$

Radiation ; (describe in terms of phase space distn.)

$$f_I(\eta,\vec{p},\vec{x}),\quad f_Q(\eta,\vec{p},\vec{x}),\quad f_V, \cdots$$

At 0-th order (the homogeneous terms) $f_{0Q} = f_{0V} = f_{0U} = 0$

$$f_{0I}(\eta,\vec{p},\vec{x}) = \frac{2}{\exp[pc/kT]-1} = f_{0I}(\eta,p)$$

Define brightness function :

The intensity, I (power per unit area perpendicular to $\vec{p}$) in a given range d^3p.

$$\frac{\delta I}{I} = \Delta \quad \leftarrow \text{Brightness}$$

Note: for blackbody:

$$I \propto T^4 \Rightarrow \delta I \propto 4T^3 \delta T$$

$$4\frac{\delta T}{T} = \frac{\delta I}{I}$$

- assume (for now) Δ is indep. on $\underline{p}$.

$$f_I(\eta, \vec{p}, \vec{x}) = \frac{2}{\exp\left(\frac{pc}{k_B(T_0+\underset{\substack{\uparrow \\ \delta T(\eta, \hat{p}, \vec{x})}}{\delta T})}\right) - 1} = f_{0I} + \delta f_{0I}$$

$$f_I(x) \simeq \frac{1}{e^x - 1} \qquad x = pc/k(T_0+\delta T)$$

$$\delta f_I = \frac{\partial f_{0I}}{\partial x} \left.\frac{dx}{d\delta T}\right|_{\delta T=0} \delta T = \frac{\partial f_{0I}}{\partial x}\left[-\frac{k_B pc}{(k_B T_0)^2}\right]\delta T$$

$$= \underbrace{\frac{\partial f_{0I}}{\partial x}\left(-\frac{pc}{k_B T_0}\right)}_{-p\frac{\partial f_{0I}}{\partial p}} \frac{\delta T}{T_0} = -p\frac{\partial f_{0I}}{\partial p}\frac{\delta T}{T} = -\frac{p}{4}\frac{df_{0I}}{dp}\Delta_I$$

$$\Delta_I = \left(-\frac{1}{4}\frac{d f_{0I}}{d\ln p}\right)^{-1}\delta f_I \qquad \boxed{\Delta_X = \left(-\frac{1}{4}\frac{d f_{0I}}{d\ln p}\right)^{-1}\delta f_X} \quad X = I, Q, U, V$$

$$\vec{I} = \frac{\rho_\gamma}{4\pi}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \rho_\gamma\begin{pmatrix}\Delta_I\\\Delta_Q\\\Delta_U\\\Delta_V\end{pmatrix}$$

(Stokes vector)

The equations

Einstein's equation ;

$$\delta G^a{}_b = 8\pi G \, \delta T^a{}_b$$

example)

$$k^2 \Psi + 3\frac{\dot a}{a}\left(\dot\Psi + \frac{\dot a}{a}\Phi\right) = -4\pi G a^2 \sum_k \delta\rho_x$$

$$k^2\left(\dot\Phi + \frac{\dot a}{a}\Phi\right) = 4\pi G a^2 \sum_x (\rho_x + P_x)\Theta_x$$

$\hookrightarrow \Theta_x = \vec\nabla\cdot\vec V_x$

Energy momentum conservation ;

for example, CCDM]

$$\begin{cases} \dot\delta_c = -(\Theta_c - 3\dot\Phi) \\ \dot\Theta_c = -\frac{\dot a}{a}\Theta_c + k^2\Phi \end{cases}$$

$\boxed{\text{Let's not forget Baryons}}$

Boltzmann Equation

$$\underbrace{\frac{d\vec I}{d\eta}}_{\text{Liouville term}} = \underbrace{\sigma_T n_e a \left[\vec I - \vec I_s\right]}_{\text{collision term}}$$

$\longrightarrow I_s(\hat n, \vec x) = \frac{1}{4\pi}\int d^2\hat n' \, \underbrace{\mathbb{P}(\hat n, \hat n')}_{\text{Scattering matrix}} \vec I(\hat n', \vec x)$

Liouville term

$$\frac{df_x}{d\eta} = \frac{\partial f_x}{\partial\eta} + \frac{\partial f_x}{\partial x^i}\frac{dx^i}{d\eta} + \underline{\frac{\partial f_x}{\partial p^i}\frac{dp^i}{d\eta}}$$

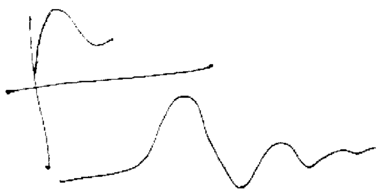
$$\hookrightarrow \underbrace{\frac{\partial f_x}{\partial\hat n}\frac{d\hat n}{d\eta}}_{\mathcal{O}(2),\ \text{ignore!!}} + \underline{\underline{\frac{\partial f_x}{\partial q}\frac{dq}{d\eta}}}$$

comoving

linear order.

$$\frac{\partial\,\delta f_x}{\partial\eta} + \hat n\cdot\vec\nabla(\delta f_x) + (\ \ ?\ \ )$$

$dt = a\,d\eta$
$d\vec{r} = a\,d\vec{x}$



comoving	physical
$(\eta, \vec{x})$	$(t, \vec{r})$
$\downarrow$	$\downarrow$
$(\varepsilon, \vec{q})$	$(E, \vec{p})$

$$\left.\begin{array}{l} \varepsilon = aE \\ \vec{q} = a\vec{p} \end{array}\right) \leftarrow \text{unperturbed spacetime.}$$

$$\left.\begin{array}{l} \mathbb{P}^0 = g_{00}^{1/2} E = a(1+\Phi) E \\ \mathbb{P}^i = g_{ii}^{1/2} p^i = a(1-\psi) p^i \end{array}\right\} \leftarrow \text{perturbed spacetime.}$$

Geodesic eq.

$$\frac{d\mathbb{P}^\mu}{d\eta} = g^{\mu\nu}\left(\frac{1}{2} g_{\alpha\beta,\nu} - g_{\nu\alpha,\beta}\right)\frac{\mathbb{P}^\alpha \mathbb{P}^\beta}{\mathbb{P}^0}$$

Exercise derive $\dfrac{dq}{d\eta}$

Recall : $\hat{n} = \dfrac{\vec{q}}{q}$

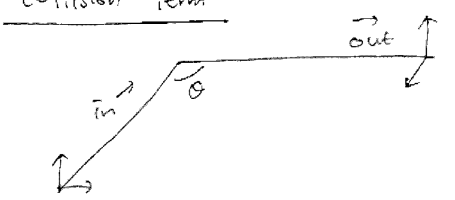
$$\frac{1}{q}\frac{dq}{d\eta} = \frac{\partial\psi}{\partial\eta} - \hat{n}\cdot\nabla\Phi$$

$$\frac{d\,\delta f_x}{d\eta} - \frac{\partial \delta f_x}{\partial\eta} + \hat{n}\cdot\nabla(\delta f_x) + q\frac{\partial f_0}{\partial q}(\dot{\psi} - \hat{n}\cdot\nabla\Phi)$$

Multiply all by $\left(-\frac{1}{4}\frac{\partial f_0}{\partial \ln q}\right)^{-1}$ $\quad : \mu = \hat{n}\cdot\hat{k}$

$$\dot{\Delta}_I + ik\mu\Delta_I + 4(\dot{\Psi} - ik\mu\Phi) \quad : \text{Liouville term}$$

Collision term



$$\vec{U}_{in} = (0, \sin\theta, \cos\theta)$$

$$\vec{U}_{out} = (0, 0, 1)$$

$$\vec{\mathcal{E}}_{(in)} \perp \vec{U}_{(in)}$$

$$\vec{\mathcal{E}}_{(out)} \perp \vec{U}_{(out)}$$

$$\mathcal{E}_{(in)x} \parallel E_{(out)x}$$

$$\mathcal{E}_{(in)y} \cdot \mathcal{E}_{out(y)} = \cos\theta$$

Thomson Scattering ; $\dfrac{d\sigma}{d\Omega} = \dfrac{3\sigma_T}{8\pi} \left| \mathcal{E}_{in} \cdot \mathcal{E}_{out} \right|^2$

Interpretation :

the prob. of

$$\left(\begin{array}{ll} \mathcal{E}_{(in)x} \to \mathcal{E}_{(out)x} & \text{is } \dfrac{3\sigma_T}{8\pi} \\ \mathcal{E}_{(in)y} \to \mathcal{E}_{(out)y} & \text{is } \dfrac{3\sigma_T}{8\pi} \cos^2\theta \end{array} \right.$$

$$\vec{I}_{(in)} = (I_{in}, 0, 0, 0)$$

$$\vec{I}_{(out)} = \frac{3\sigma_T}{8\pi} (1 + \cos^2\theta, \sin^2\theta, 0, 0)$$

Note :

$$1 + \cos^2\theta = 2\left(1 + \tfrac{1}{2} P_2(\cos\theta)\right)$$

Consider the total scattered intensity. $I_{(in)}(\theta, \varphi)$

$$I_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega \, (1 + \cos^2\theta) \, I_{(in)}(\theta, \varphi)$$

$$Q_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega \, \sin^2\theta \cos 2\phi \, I_{(in)}(\theta, \phi)$$

$$U_{(out)} = \frac{3\sigma_T}{8\pi} \int d\Omega \, \sin^2\theta \sin 2\phi \, I_{(in)}(\theta, \phi)$$

$$I_{(in)}(\theta,\varphi) = \Sigma\, a_{lm}^{I(in)}\, Y_{lm}(\theta,\varphi)$$

$$I_{(out)} = \frac{3\sigma_T}{16\pi}\left[\frac{8\sqrt{\pi}}{3} a_{00} + \frac{4}{3}\sqrt{\frac{\pi}{3}}\, a_{20}\right]$$

$$Q_{(out)} = \frac{3\sigma_T}{8\pi}\left[a_{22}^{I(in)} + a_{2,-2}^{I(in)}\right)\sqrt{\frac{2\pi}{15}}$$

polarization is generated by the quadropole anisotropy of the incoming Intensity.

$$U_{(out)} = -\frac{i\,3\sigma_T}{8\pi}\sqrt{\frac{2\pi}{15}}\left(a_{2,2}^{I(in)} - a_{2,-2}^{I(in)}\right)$$

The full Boltzmann Equations are

peculiar velocity of the scatterers

$$\dot{\Delta}_I + ik\mu\Delta_I + 4[\dot{\Phi} - ik\mu\Phi] = -\sigma_T n_e a\left[\Delta_I - \Delta_{I_0} + 4\mu\Theta_B - \frac{1}{2}P_2(\mu)\left(\Delta_{I_2} + \Delta_{Q_2} - \Delta_{Q_0}\right)\right]$$

$$\dot{\Delta}_Q + ik\mu\Delta_Q = -\sigma_T n_e a\left[\Delta_Q - \frac{1}{2}(1 - P_2(\mu))(\Delta_{I_2} + \Delta_{Q_2} \mp \Delta_{Q_0})\right]$$

NOTE

$$\vec{k}\cdot\hat{n} = \mu$$

$$\Delta(\eta,\hat{n},\vec{k}) \Rightarrow \Delta_l(\eta,k) = \frac{1}{2}\int_{-1}^{1} d\mu\, \Delta(\eta,k,\mu)\, P_l(\mu)$$

Baryon

$$\dot{\delta}_B = -\Theta_B + 3\dot{\Phi}$$

$$\dot{\Theta}_B = -\frac{\dot{a}}{a}\Theta_B + k^2\Phi + \underbrace{\frac{4}{3}\frac{\rho_\gamma}{\rho_B}\, a n_e \sigma_T(\Theta_\gamma \cdot \Theta_B)}$$

$$\Theta_\gamma = \frac{3}{4}\, ik\Delta_{I_1}$$

exchange of momentum btw. B and γ.

Initial Conditions

Consider two fluids

- non-relativistic fluid : $(E \propto m)$, $p=0$
- relativistic fluid : $E \propto \omega$, $p = \frac{\rho}{3}$

* Adiabatic (isotropic)

$$\frac{n_x}{n_\gamma} = cte \Rightarrow \delta\left(\frac{n_x}{n_\gamma}\right) = 0 \Rightarrow \underline{\frac{\delta n_x}{n_x} = \frac{\delta n_\gamma}{n_\gamma}}$$

$\rho_\gamma \propto T^4$, $n_\gamma \propto T^3$

$$\Rightarrow \rho_\gamma \propto n_\gamma^{4/3}$$

$$\Rightarrow \frac{\delta\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta n_\gamma}{n_\gamma}$$

$$\rho_x \propto n_x \Rightarrow \frac{\delta\rho_x}{\rho_x} = \frac{\delta n_x}{n_x}$$

$\therefore$ Adiabatic condition : $\boxed{\frac{\delta\rho_x}{\rho_x} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma} = 0}$

$$\delta_x - \frac{3}{4}\Delta_{I_0} = 0$$

* Isocurvature : overall density perturbation is zero.

$$\rho_T = \rho_x + \rho_\gamma$$

$$\Rightarrow \delta\rho_T = 0 = \delta\rho_x + \delta\rho_\gamma = \rho_x\frac{\delta\rho_x}{\rho_x} + \rho_\gamma\frac{\delta\rho_\gamma}{\rho_\gamma} = 0$$

$$\boxed{\Delta_{I_0} = -\frac{\rho_x}{\rho_\gamma}\delta_x}$$

(early 70's ... isothermal)

Solutions

$$\tau = \int_{\eta}^{\eta_0} \sigma_T n_e a \, d\eta$$

$$\dot{\Delta}_I + ik\mu \Delta_I - \dot{\tau} \Delta_I = -4 [\dot{\Psi} - ik\mu \Phi] - \dot{\tau} [\Delta_{I0} - 4\mu \Theta_B]$$

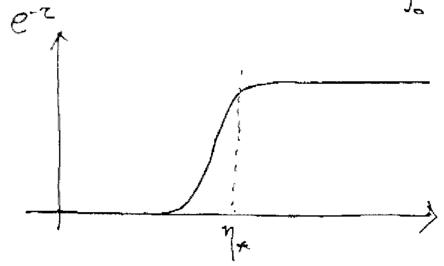
Rewrite w/ integrating factors

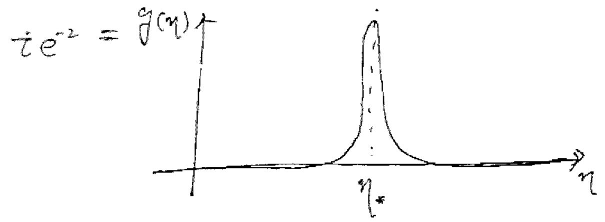
$$\frac{d}{d\eta} [e^{ik\mu\eta} e^{-\tau} \Delta_I] = -4 [\dot{\Phi} - ik\mu \Phi] e^{ik\mu\eta} e^{-\tau}$$
$$- \dot{\tau} [\Delta_{I0} - 4\mu \Theta_B] e^{ik\mu(\eta-\eta_0)} e^{-\tau}$$

$$\Delta_I = -4 \int_0^{\eta_0} d\eta \, (\dot{\Phi} - ik\mu \Phi) e^{-\tau} e^{ik\mu(\eta-\eta_0)} - \int_0^{\eta_0} d\eta \, \dot{\tau} [\Delta_{I0} - 4\mu \Theta_B] e^{-\tau} e^{ik\mu(\eta-\eta_0)}$$

$\downarrow$ Integrate by part.

$$-4 \int_0^{\eta_0} d\eta \, (\dot{\Psi} + \dot{\Phi}) e^{-\tau} e^{ik\mu(\eta-\eta_0)} - 4 \int_0^{\eta_0} d\eta \, \Phi \dot{\tau} e^{-\tau} e^{ik\mu(\eta-\eta_0)}$$





Approximation using a step function + δ function

$$\Delta_I = -4 \int_{\eta_*}^{\eta_0} d\eta \, (\dot{\Psi} + \dot{\Phi}) e^{ik\mu(\eta-\eta_0)} + [\Delta_{I0} + 4\Phi - 4\mu \Theta_B] \Big|_{\eta_*} e^{ik\mu(\eta_0-\eta_*)}$$

Recall that :

i) $\frac{1}{4}\Delta_I = \frac{\delta T}{T}$

ii) Fourier transform back

iii) $\Delta_{I_0} = \delta_\gamma$

iv) $d_* = \eta_0 - \eta_*$

(denser regions of radiation look hotter)

intrinsic term — dopler term — Sachs-Wolfe

$$\frac{\delta T}{T}(\hat{n}) = \underbrace{\frac{1}{4}\delta_\gamma(\eta_*, d_*\hat{n})} - \underbrace{\hat{n}\cdot\vec{V}_B(\eta_*, d_*\hat{n})} + \Phi(\eta_*, d_*\hat{n})$$

$$- \underbrace{\int_{\eta_*}^{\eta_0} d\eta\,(\dot{\Phi} + \dot{\Psi})(\eta, \eta\hat{n})}_{\text{Integrated Sachs-Wolfe term}}$$

(Rees - Sciama effect)

$\dot{\tau} = a\sigma_T n_e = a\sigma_T x_e n$

three regimes :

- strong coupling : $\frac{\dot{\tau}}{\frac{\dot{a}}{a}} \gg 1$ MFP $= 0$ fully ionized
- weak coupling : $\frac{\dot{\tau}}{\frac{\dot{a}}{a}} \approx 1$ MFP = finite recombination
- free streaming : $\frac{\dot{\tau}}{\frac{\dot{a}}{a}} \ll 1$ MFP $= \infty$ neutral

Strong Coupling

$$\Delta_\ell = \frac{1}{2}\int_{-1}^{1} d\mu\, P_\ell(\mu)\,\Delta(k,\mu)$$

$$\left.\begin{array}{l} \delta_\gamma = \Delta_{I0} \\ \Theta_\gamma = i\frac{3}{4}k\Delta_{I1} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{Boltz. eq (take 0th \& 1st moment of B.E.)} \\ \dot{\delta}_\gamma + \frac{4}{3}\Theta_\gamma = 4\dot{\Phi} \\ \dot{\Theta}_\gamma = \frac{k^2}{4}\delta_\gamma - ik^2\Phi + \dot{\tau}(\Theta_B - \Theta_\gamma) \end{array}\right.$$

$$\left\{\begin{array}{l} \dot{\delta}_B = -\Theta_B + 3\dot{\Phi} \\ \dot{\Theta}_B = -\frac{\dot{a}}{a}\Theta_B + R^{-1}\dot{\tau}(\Theta_\gamma - \Theta_B) + k^2\Phi \end{array}\right.$$

$$\boxed{R = \frac{3}{4}\frac{\rho_B}{\rho_\gamma}}$$

for all $\ell \geq 2$, $\dot{\Delta}_{I\ell} = \frac{k}{2\ell+1}\left[\ell\Delta_{I\ell-1} - (\ell+1)\Delta_{I\ell+1}\right] + \dot{\tau}\Delta_{I\ell}$

tight coupling $\Longrightarrow$ $\tau \gg 1 \Rightarrow \Delta_{I\ell} = 0 \quad (\ell \geq 2)$

$$\dot{\tau} \gg 1 \Rightarrow \Theta_B = \Theta_\gamma$$

$$\ddot{\delta}_\gamma - \frac{4}{3}\mathcal{H}\frac{1}{1+R^{-1}}\dot{\delta}_\gamma + \frac{k^2R^{-1}}{3(1+R^{-1})}\delta_\gamma + \frac{4}{3}k^2\Phi = 4\ddot{\Phi} \quad \text{(exercise)}$$

Define $\bar{\delta}_\gamma = \delta_\gamma - 4\Phi$

$$\ddot{\bar{\delta}}_\gamma + \frac{\dot{R}}{1+R}\dot{\bar{\delta}}_\gamma + \frac{k^2}{3(1+R)}\bar{\delta}_\gamma = -\frac{4}{3}k^2\left(\frac{1}{1+R}\Phi - \Phi\right)$$

$\rightarrow$ forced Harmonic oscillator

Define the sound speed of this Baryon/photon fluid.

$$c_s^2 \equiv \frac{1}{3(1+R)} \begin{array}{l} \nearrow \frac{1}{3} \quad \rho_B \ll \rho_\gamma \\ \searrow 0 \quad \rho_B \gg \rho_\gamma \end{array}$$

Define a sound Horizon

$$r_s(\eta) = \int_0^\eta c_s(\eta')\,d\eta'$$

We can solve the eq.

a) WKB $\qquad \bar{\delta}_\gamma = A(\eta)\, e^{ikr_s}$ $\qquad \rho = \rho_B + \rho_r$

b) neglect source term $\qquad A(\eta) = \left(\frac{\rho + P}{c_s^2 \rho a^4}\right)^{1/2}$ $\qquad P = P_r$

$$\Delta_{I_0} = 4\Psi + A\cos(kr_s) + B\sin(kr_s)$$

Adiabatic i.c. ; $\left.\begin{array}{l}\Delta_{I_0} \neq 0 \\ \dot{\Delta}_{I_0} = 0\end{array}\right\} \Rightarrow \cos(kr_s)$

Isocurvature i.c. ; $\left.\begin{array}{l}\Delta_{I_0} = 0 \\ \dot{\Delta}_{I_0} \neq 0\end{array}\right\} \Rightarrow \sin(kr_s)$

initial condition $\longrightarrow$ different modes are stimulated.

— Solve the equation on superhorizon scales $(k\eta) \ll 1$

- Φ is constant on superhorizon scales.
- Matter dominant era ;

$$3\mathcal{H}^2\Phi = -\frac{3}{2}\mathcal{H}^2\delta \Rightarrow \Phi = -\frac{1}{2}\delta$$

- Adiabatic ; $\delta - \frac{3}{4}\Delta_{I_0} = 0 \rightarrow \Delta_{I_0} = +\frac{4}{3}\delta = -\frac{8}{3}\Phi$
- $\frac{\delta T}{T}(\hat{n}) = \frac{1}{4}\delta_\gamma + \Phi = -\frac{2}{3}\Phi + \Phi = \frac{\Phi}{3}$

$$\boxed{\text{Sachs - Wolfe formula} \quad \frac{\delta T}{T}(\hat{n}) = \frac{1}{3}\Phi(\eta_*, d_* x \hat{n})}$$

Weak coupling ~~region~~ regime

$$\frac{\dot{\tau}}{\left(\frac{\dot{a}}{a}\right)} \sim 1$$

Define $\eta_c = \dfrac{1}{a\sigma_T n_e}$, $\dot{\tau} = \dfrac{1}{\eta_c}$

& neglect metric terms

$$\boxed{\dot{\Delta}_I + ik\mu\Delta_I + \frac{1}{\eta_c}\Delta_I = \frac{1}{\eta_c}\left[\Delta_{I_0} - 4\mu\Theta_B\right]} \quad \text{B.E.}$$

$$\xi = \Delta_{I_0} - 4\mu\Theta_B$$

Solve iteratively in powers of η_c

$$\Delta_I = \xi - \eta_c\dot{\Delta}_I - \eta_c ik\mu\Delta_I$$

replace to get next order

$$\Delta_I = \xi - \eta_c(\dot{\xi} + ik\mu\xi) + \eta_c^2(\ddot{\xi} + ik\mu\dot{\xi} - k^2\mu^2\xi)$$

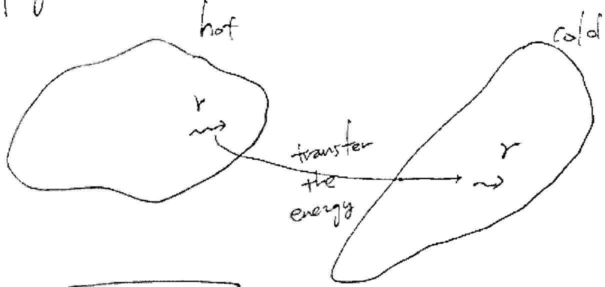
We can rewrite the equation ;

$$\begin{cases} \Delta_{I_0} = \cdots \\ \Theta_B = \cdots \end{cases}$$

Solve : we want the dispersion equation (relation) $(e^{i\omega})$

$$\omega = \underbrace{\pm \frac{k}{\sqrt{3(1+R)}}}_{c_s k} + \underbrace{\frac{i\, k^2 \eta_c}{6}\left(1 - \frac{6}{5(1+R)} + \frac{1}{(1+R)^2}\right)}_{\text{damping term}}$$

Silk damping



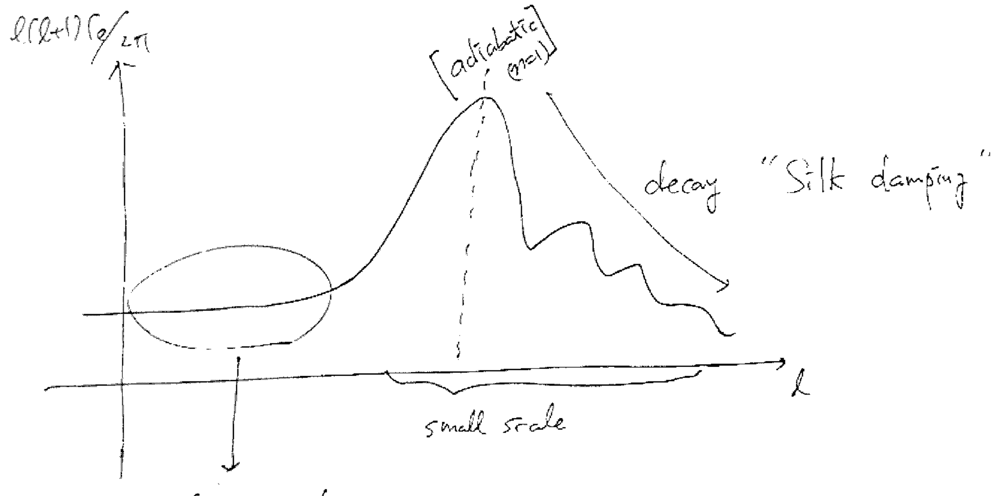
$$\boxed{\lambda_D \sim \frac{1}{k}} \ ?$$

$\eta_c = \dfrac{1}{a \sigma_T n_e}$ (MFP)

$\lambda_D = \sqrt{N}\, \eta_c$

$N = \dfrac{\eta}{\eta_c} \Rightarrow \lambda_D = (\eta \eta_c)^{1/2}$

N: # of scattering



large scale

(Sachs-wolfe) → gravitational potential at LS.

Putting it all together

Solve for Δ $\left(= \frac{\delta T}{T} \longleftrightarrow a_{lm} \right)$

$$\langle a^{I*}_{lm} \, a^{I}_{l'm'} \rangle = C^{I}_{l} \underbrace{\delta_{ll'} \delta_{mm'}}$$

↳ statistical isotropy

$$\langle \delta^*(\vec{k}') \, \delta(\vec{k}) \rangle = P(k) \, \delta^3(\vec{k} - \vec{k}')$$

$$\boxed{C^{I}_{l} = \frac{1}{8\pi} \int k^2 dk \, \left| \Delta_{Il}(k, \eta_0) \right|^2}$$

relation btw. what we solved & what we will plot.

Polarization

· polarization's only generated at weak coupling

a) $\Delta_{I_2} = -\frac{1}{3} \Delta_{Q_0} = -\frac{2}{15} ik \eta_c \Delta_I$ finite

· Polarization is generated at recombination through the quadrupole

b) scalar perturbations (δ, Φ, Ψ) will only generate E type polarization. (even) in parity

c) Tensor perturbation $\Rightarrow$ both E+B types (gravity waves)

d) typically, E mode polarization is $\sim 10\% \, \frac{\delta T}{T}$

What can we learn from the C_ℓ

I) test the initial conditions

adiabatic : $\cos^2(k r_s) \sim C_\ell$

isocurvature : $\sin^2(k r_s) \sim C_\ell$

$$r_s k = n\pi \quad (\text{for adiabatic})$$

$$r_s k = (n+\tfrac{1}{2})\pi \quad (\text{for iso.})$$

adiabatic : first peak at $\ell \sim 220$ ⟵ data

isocurvature : first peak at $\ell \sim 330$

II) The spectral index

$$\frac{\delta T}{T}(\hat{n}) = \frac{1}{3}\Phi(d_* \hat{n})$$

$$k^3 \Phi \propto k^{n-1}$$

The C_ℓ depend on n

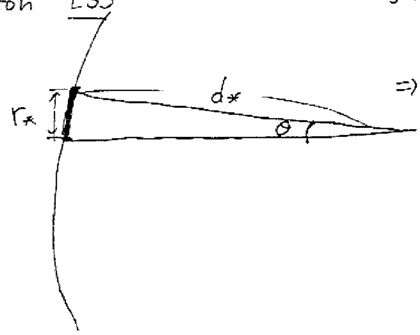
$$C_\ell \propto \frac{\Gamma[\ell + \frac{n-1}{2}]\,\Gamma[\frac{9-n}{2}]}{\Gamma[\ell + \frac{5-n}{2}]\,\Gamma[\frac{3+n}{2}]} \quad \left[\Gamma \text{ is the gamma function}\right]$$

$n=1$ is scale invariant $\quad C_\ell \propto \frac{1}{\ell^2}$

III) Geometry

$r_* = r_s(\eta_*)$: sound horizon LSS

$d_* = \eta_0 - \eta_*$

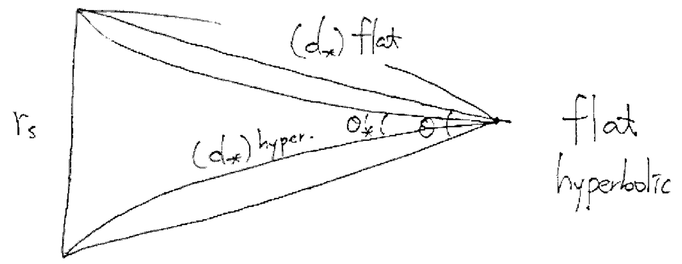


Flat universe

$$\sin\frac{\theta}{2} = \frac{r_*}{2d_*}$$

$$\Rightarrow \theta = 2\arcsin\left(\frac{r_*}{2d_*}\right)$$

Suppose the universe is hyperbolic



$$\theta'_* = 2 \arcsin \left[\frac{\sqrt{\frac{\cosh\left(\frac{r_*}{R_c}\right) - 1}{2}}}{\sinh\left(\frac{d_*}{R_c}\right)} \right]$$

R_c is the curvature of space.