SLE Loop Measures

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- Connection with physics: percolation (SLE_6), spin Ising model (SLE_3), FK-Ising model ($SLE_{16/3}$), loop-erased random walk (SLE_2), uniform spanning tree (SLE_8), GFF contour line (SLE_4).

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- Several versions: chordal (boundary to boundary), radial (boundary to interior), whole-plane (interior to interior), etc.

Conformal Markov property

A chordal SLE_κ curve grows in a simply connected domain, say D, from one boundary point a to another boundary point b. If τ is a stopping time before b is reached, then conditional on γ up to τ, the part of γ from τ to its end is a chordal SLE_κ growing in a complement domain.

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A whole-plane SLE_κ curve γ grows in the Riemann sphere $\widehat{\mathbb{C}}$ from one interior point a to another b. It also satisfies CMP, which is slightly different from the above. If τ is a nontrivial stopping time, i.e., does not happen at the initial time, and happens before b is reached, then conditional on γ up to τ, the part of γ from τ to its end is a *radial* SLE_κ growing in a complement domain.

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- **Object**: an SLE_κ loop is a single "random" closed curve, which locally looks like an ordinary SLE_κ curve.
- **Caution**: unlike other SLE_κ curves, the "law" of an SLE_κ loop is not a probability measure, but a σ-finite infinite measure.
- **CMP of SLE loop**: A rooted SLE_κ loop measure is expected to satisfy CMP such that

$$\mathrm{SLE}_\kappa \text{ loop} : \text{chordal } \mathrm{SLE}_\kappa = \text{whole-plane } \mathrm{SLE}_\kappa : \text{radial } \mathrm{SLE}_\kappa.$$

This means that, if γ is an SLE_κ loop in $\widehat{\mathbb{C}}$ rooted at z, and τ is a nontrivial stopping time, then conditional on the part of the curve before τ and the event that τ happens before γ returns to z, the part of γ from τ to its terminal time is a *chordal* SLE_κ curve.

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- Brownian loop measure rooted at 0: $\mu_0^{\mathrm{lp}} := \int_0^\infty \frac{1}{2\pi T} \mu_T^{\mathrm{BB}} dT$. For other roots, $\mu_z^{\mathrm{lp}} = z + \mu_0^{\mathrm{lp}}$.

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- Unrooted Brownian loop masure: $\mu^{\mathrm{lp}} := \int_{\mathbb{C}} \frac{1}{T}\mu_z^{\mathrm{lp}} dA(z)$.
- Möbius invariance: $W(\mu_z^{\mathrm{lp}}) = \mu_{W(z)}^{\mathrm{lp}}$ and $W(\mu^{\mathrm{lp}}) = \mu^{\mathrm{lp}}$.

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- Kemppainen-Werner constructed unrooted SLE_κ loop measures in $\widehat{\mathbb{C}}$ for $\kappa \in (8/3, 4]$ as the intensity measure of a *nested* CLE, which is used to prove the Möbius invariance of nested CLE on $\widehat{\mathbb{C}}$.

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- Field-Lawler and Benoist-Dubédat have been working on the construction of SLE loops using different approaches.

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- Mckean's dimension theorem for SLE (with factor d).

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- The simple SLE_κ loops for $\kappa \in (0, 4]$ give examples of MKS loop measures with $\mathrm{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} \in (-\infty, 1]$.

The first attempt

A whole-plane SLE_κ curve from 0 to ∞ may be constructed by the following procedure. Run a radial SLE_κ curve in the simply connected domain $\widehat{\mathbb{C}} \setminus \{|z| \leq \varepsilon\}$ from ε to ∞, and take the limit as $\varepsilon \to 0$.

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This inspires us to define SLE_κ loop rooted at 0 by the following approach: run a chordal SLE_κ curve in $\widehat{\mathbb{C}} \setminus \{|z| \leq \varepsilon\}$ from ε to $-\varepsilon$, and then let $\varepsilon \to 0$. This procedure does not work because of the following reason. As SLE_κ for $\kappa \in (0, 8)$ is not space-filing, almost surely the curve avoids ∞, i.e., the curve is bounded. By scaling property, we end up with a single point by taking the limit $\varepsilon \to 0$.

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This observation also gives an evidence that the law of an SLE_κ loop can not be a probability measure.

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- Given a σ-finite measure μ on U and a σ-finite kernel from U to V, we may define a measure $\mu \otimes \nu$ on $U \times V$ such that

$$\mu \otimes \nu(E \times F) = \int_E \nu(u, F) d\mu(u), \quad E \in \mathcal{U}, \quad \mathcal{F} \in \mathcal{V}.$$

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- Use $\nu \overleftarrow{\otimes} \mu$ if we want to switch the order of U and V.

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Caution: After the second step, the marginal measure of X is changed unless ν is μ-a.s. a probability kernel, i.e., $\nu(u, V) = 1$ for μ-a.s. $u \in U$. In fact, if ν is finite, then the new marginal measure of X is absolutely continuous w.r.t. the old law: μ, and the RN derivative is $\nu(\cdot, V)$.

CMP and kernel

The rigorous statement of the CMP for a chordal SLE_κ measure $\mu^{\#}_{D;a\to b}$ in D from a to b is as follows. Let T_b be the time that the curve ends at b. If τ is a stopping time, then

$$\mathcal{K}_\tau(\mu^{\#}_{D;a\to b}|_{\{\tau<T_b\}})(d\gamma_\tau) \oplus \mu^{\#}_{D(\gamma_\tau;b);(\gamma_\tau)_{\mathrm{tip}}\to b}(d\gamma^\tau) = \mu^{\#}_{D;a\to b}|_{\{\tau<T_b\}},$$

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where

- $\mathcal{K}_\tau(\gamma)$ is the truncation of γ at time τ.
- $\mu \oplus \nu$ is the pushforward of $\mu \otimes \nu$ under the concatenation map $(\beta, \gamma) \mapsto \beta \oplus \gamma$.
- $D(\gamma_\tau; b)$ is the connected component of $D \setminus \gamma_\tau$ whose boundary contains b and $(\gamma_\tau)_{\mathrm{tip}}$, the tip of γ_τ.
- $\mu^{\#}_{D(\gamma_\tau;b);(\gamma_\tau)_{\mathrm{tip}}\to b}$ is a chordal SLE_κ measure in $D(\gamma_\tau; b)$.

CMP AND KERNEL

Using the same spirit, we may rigorously define the CMP for an SLE_κ loop measure μ_z^1 in $\widehat{\mathbb{C}}$ rooted at z as follows. Let T_z be the time that the loop returns to z. If τ is a *nontrivial* stopping time, then

$$\mathcal{K}_\tau(\mu_z^1|_{\{\tau<T_z\}})(d\gamma_\tau) \oplus \mu^{\#}_{\widehat{\mathbb{C}}(\gamma_\tau;z);(\gamma_\tau)_{\mathrm{tip}}\to z}(d\gamma^\tau) = \mu_z^1|_{\{\tau<T_z\}},$$

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where

- $\mathcal{K}_\tau(\gamma)$ and $\mu \oplus \nu$ have the same meaning as before.
- $\widehat{\mathbb{C}}(\gamma_\tau; z)$ is the connected component of $\widehat{\mathbb{C}} \setminus \gamma_\tau$ whose boundary contains z and $(\gamma_\tau)_{\mathrm{tip}}$, the tip of γ_τ.
- $\mu^{\#}_{\widehat{\mathbb{C}}(\gamma_\tau;z);(\gamma_\tau)_{\mathrm{tip}}\to z}$ is a chordal SLE_κ measure in $\widehat{\mathbb{C}}(\gamma_\tau; z)$.

Natural parametrization

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Lawler and Rezaei proved that NP of SLE agrees with the d-dimensional Minkowski content of SLE, where $d = 1 + \frac{\kappa}{8}$ is the Hausdorff dimension of SLE ([Beffara]). So NP of SLE is determined by the curve itself, and independent of the domain or equation.

Minkowski content measure

Now we recall the Minkowski content and introduce the Minkowski content measure. We fix $d \in (1, 2)$. Let $S \subset \mathbb{C}$ be a closed set. The (d-dimensional) Minkowski content of S is defined to be

$$\mathrm{Cont}(S) = \lim_{\varepsilon \downarrow 0} \varepsilon^{d-2} A(S^{\varepsilon}),$$

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Definition

Let $S \subset \mathbb{C}$. Suppose $\mathcal{M}$ is a measure supported by S such that for every compact set $K \subset \mathbb{C}$, $\mathrm{Cont}(K \cap S) = \mathcal{M}(K) < \infty$. Then we say that $\mathcal{M}$ is the Minkowski content measure on S, or S possesses Minkowski content measure. We will use $\mathcal{M}_S$ to denote this measure.

Space-time homogeneity

Lawler conjectured that an SLE_κ loop measure should satisfy space-time homogeneity: Suppose γ follows the SLE_κ loop measure μ_z^1 rooted at z, and is parameterized periodically by its Minkowski content measure. Then for any deterministic number $a \in \mathbb{R}$, if we reroot the loop at $\gamma(a)$, i.e., we define a new loop: $\mathcal{T}_a(\gamma)(t) := z + \gamma(a+t) - \gamma(a)$, then the "law" of the new loop $\mathcal{T}_a(\gamma)$ is still μ_z^1.

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Here we say that a loop γ is parameterized periodically by its Minkowski content measure if γ is defined on $\mathbb{R}$ with period $T = \mathrm{Cont}(\gamma)$ such that for any $a \le b \le a+T$, $\mathrm{Cont}(\gamma([a,b]) = b - a$.

Minkowski content measure

The work by Lawler and Rezaei showed that a chordal SLE_κ curve in $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ a.s. possesses Minkowski content measure, which is the pushforward measure of NP under the curve function. Moreover, the measure is supported by $\mathbb{H}$, and is parameterizable for the curve.

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Minkowski content measure satisfies conformal covariance with factor d. This means that, if S possesses Minkowski content measure $\mathcal{M}_S$, and if f is a conformal map defined on a domain $D \supset S$, then $f(S)$ also possesses Minkowski content measure, which is absolutely continuous w.r.t. $f_*(\mathcal{M}_S)$, and the RN derivative is $|f'(f^{-1}(\cdot))|^d$.

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In particular, we see that SLE_κ curve in any simply connected domain possesses Minkowski content measure.

Two-sided radial SLE and Green's function

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- The Green's function for a chordal SLE_κ curve is the limit as $\varepsilon \to 0$ of the probability of the event that a chordal SLE_κ curve visits a disc of radius ε divided by the scaling factor ε^{2-d}.
- We use $\mu^{\#}_{D;a\to b}$, $\nu^{\#}_{D;a\to z\to b}$ and $G_{D;a\to b}$ to denote chordal SLE, two-sided radial SLE and chordal SLE Green's function.

Decomposition of SLE

Field proved that, for $\kappa \in (0,4]$, a bounded domain D with analytic boundary, and distinct points $a, b \in \partial D$,

$$\int_D \nu^{\#}_{D;a\to z\to b} G_{D;a\to b}(z) A(dz) = \text{Cont} \cdot \mu^{\#}_{D;a\to b}.$$

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This means, if one integrates the laws of two-sided radial SLE_κ curves in D from a to b passing through different interior points against the Green's function for the chordal SLE_κ curve in D from a to b, then one gets the law of a chordal SLE_κ curve in D from a to b biased by the Minkowski content of the whole curve.

Decomposition of SLE

Field's result was later extended to all $\kappa \in (0,8)$ in a more general form.

Theorem (Z, 2016)

Let $\kappa \in (0,8)$. Let D be a simply connected domain with two distinct prime ends a and b. Then

$$\mu^{\#}_{D;a\to b}(d\gamma) \otimes \mathcal{M}_{\gamma;D}(dz) = \nu^{\#}_{D;a\to z\to b}(d\gamma) \overleftarrow{\otimes} (G_{D;a\to b} \cdot A)(dz).$$

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By looking at the first marginal measure, we recover Field's result.

The above results show that the law of a chordal SLE_κ curve in D from a to b may be constructed by integrating the laws of two-sided radial SLE_κ curves in D from a to b passing through different points z against the Green's function for the chordal SLE_κ, and then unweighting the integrated measure by the Minkowski content of the curve.

Construction of rooted SLE loop

The construction of rooted SLE_κ loops is inspired by the above observation. Since an SLE_κ loop rooted at z may be viewed as a degenerate chordal SLE_κ in $\widehat{\mathbb{C}}$ from z to z, we expect that its law can be constructed by integrating the laws of degenerate two-sided radial SLE_κ curves in $\widehat{\mathbb{C}}$ from z to z passing through different points w against some suitable function, and then unweighting the integrated measure by the Minkowski content.

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A degenerate two-sided radial SLE_κ curve in $\widehat{\mathbb{C}}$ from z to z passing through w is a two-sided whole-plane SLE_κ curve, which is composed of two arms connecting two points in $\widehat{\mathbb{C}}$. A two-sided whole-plane SLE_κ satisfies some CMP such that

$$\text{two-sided whole-plane} : \text{two-sided radial} = \text{whole-plane} : \text{radial}.$$

Rooted loops I

Below is the main theorem on rooted SLE loop measure μ_z^1: the superscript 1 denotes the number of root; the subscript z is the root.

Theorem

Let $G_{\mathbb{C}}(w) = |w|^{-2(2-d)}$, $w \in \mathbb{C} \setminus \{0\}$. Let $\nu^{\#}_{z \rightleftharpoons w}$ denote the law of the two-sided whole-plane SLE$_\kappa$ curve from z to z passing through w (modulo a time change). Define

$$\mu_z^1 = \mathrm{Cont}(\cdot)^{-1} \cdot \int_{\mathbb{C}\setminus\{z\}} \nu^{\#}_{z \rightleftharpoons w} G_{\mathbb{C}}(w - z) A(dw), \quad z \in \mathbb{C}.$$

Then we have the following facts:

Rooted loops II

Theorem

(I) *Each* μ_z^1 *is supported by non-degenerate loops in* $\widehat{\mathbb{C}}$ *rooted at* z *which possess Minkowski content measure that is parameterizable. Moreover, we have the decomposition formula*

$$\mu_z^1(d\gamma) \otimes \mathcal{M}_\gamma(dw) = \nu^{\#}_{z \rightleftharpoons w}(d\gamma) \overleftarrow{\otimes} G_{\mathbb{C}}(w - z) \cdot A(dw), \quad z \in \mathbb{C}.$$

(II) *Each* μ_z^1 *satisfies CMP.*

(III) *Each* μ_z^1 *satisfies the space-time homogeneity.*

(IV) *Möbius covariance:* $W(\mu_z^1) = |W'(z)|^{2-d} \mu^1_{W(z)}$.

(V) *For each* $r > 0$, *(a)* $\mu_z^1(\{\gamma : \mathrm{diam}(\gamma) > r\}) < \infty$; *(b)* $\mu_z^1(\{\gamma : \mathrm{Cont}(\gamma) > r\}) < \infty$.

(VI) *If a measure* μ' *supported by non-degenerate loops rooted at* z *satisfies (II) and (V.a), then* $\mu' = c\mu_z^1$ *for some* $c \geq 0$.

Self similarity and stationary increments

There is an interseting byproduct of the SLE loop measures: Hölder continuity and dimension property of SLE with NP.

Self similarity and stationary increments

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Below is a list of previous works.

- Rohde and Schramm proved that SLE in CP is Hölder continuous for $\kappa \neq 8$.
- Lind improved the Hölder exponents of SLE in CP, which was later proved to be optimal by Lawler and Viklund.
- Werness proved that, for $\kappa \leq 4$, for any $\alpha < 1/d$, an SLE_κ curve may be reparametrized to be α-Hölder continuous.
- Lawler and Rezaei proved that, if SLE_κ curve γ is parameterized by CP, and if Θ_t is such that $\gamma \circ \Theta^{-1}$ is γ parameterized by NP, then Θ is Hölder continuous.
- No result on the Hölder continuity of SLE with NP was known.

If we apply the map $z \mapsto 1/z$ to the formula:

$$\mu_0^1(d\gamma) \otimes \mathcal{M}_\gamma(dw) = \nu^{\#}_{0\rightleftharpoons w}(d\gamma) \overleftarrow{\otimes} (|w|^{-2(2-d)} \cdot A)(dw),$$

then we get

$$\mu_\infty^1(d\gamma) \otimes \mathcal{M}_\gamma(dw) = \nu^{\#}_{\infty\rightleftharpoons w}(d\gamma) \overleftarrow{\otimes} A(dw).$$

Since $\nu^{\#}_{\infty\rightleftharpoons w} = w + \nu^{\#}_{\infty\rightleftharpoons 0}$, using the above formula, we can prove that, if a two-sided whole-plane SLE_κ curve γ with law $\nu^{\#}_{\infty\rightleftharpoons 0}$ is parametrized by its Minkowski content measure such that $\gamma(0) = 0$, then it is a $1/d$-self similar process defined on $\mathbb{R}$ with stationary increments, i.e.,

$$(\gamma(at)) \sim (a^{1/d}\gamma(t)), \quad \forall a > 0;$$

$$(\gamma(a+t) - \gamma(a)) \sim (\gamma(t)), \quad \forall a \in \mathbb{R}.$$

Hölder continuity and dimension theorem

We want to study the Hölder continuity and dimension properties of γ. The problem boils down to the finiteness of momentums of $|\gamma(1)|$:

Lemma

For any $c \in (-d, \infty)$, $\mathbb{E}[|\gamma(1)|^c] < \infty$.

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Theorem (Hölder Continuity)

γ *is locally* α*-Hölder continuous for any* $\alpha < 1/d$.

Theorem (Mckean's Dimension Theorem)

For any deterministic closed set $A \subset \mathbb{R}$, *a.s.* $\dim(\gamma(A)) = d \cdot \dim(A)$.

Unrooted loop

We use rooted SLE_κ loop measures to construct unrooted SLE_κ loop measure in $\widehat{\mathbb{C}}$. It is a σ-finite measure on unrooted loops. An unrooted loop is a continuous function defined on the circle S^1, modulo an orientation-preserving auto-homeomorphism of S^1.

We may view the two-sided whole-plane SLE_κ measure $\nu^{\#}_{z\rightleftharpoons w}$ as a measure on unrooted loops. By the work of Miller and Sheffield, a two-sided whole-plane SLE_κ satisfies reversibility, i.e., we have $\nu^{\#}_{z\rightleftharpoons w} = \nu^{\#}_{w\rightleftharpoons z}$ as measures on unrooted loops.

Unrooted loop

Theorem

Define the measure μ^0 on unrooted loops by

$$\mu^0 = \mathrm{Cont}(\cdot)^{-2} \cdot \int_{\mathbb{C}} \int_{\mathbb{C}} \nu^{\#}_{z \rightleftharpoons w} |w - z|^{-2(2-d)} A(dw) A(dz).$$

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Then μ^0 is a σ-finite measure that satisfies:

(I) *the decomposition formulas:*

$$\mu^0(d\gamma) \otimes \mathcal{M}_\gamma(dz) = \mu^1_z(d\gamma) \overleftarrow{\otimes} A(dz);$$

$$\mu^0(d\gamma) \otimes (\mathcal{M}_\gamma)^2(dz \otimes dw) = \nu^{\#}_{z \rightleftharpoons w}(d\gamma) \overleftarrow{\otimes} |w-z|^{-2(2-d)} \cdot (A)^2(dz \otimes dw).$$

(II) *Möbius invariance: $W(\mu^0) = \mu^0$.*

SLE loops in subdomains of $\widehat{\mathbb{C}}$

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- Let $\mathcal{L}_D(V_1, V_2) = \{\text{loops in } D \text{ that intersect both } V_1 \text{ and } V_2\}$. Let $\mathrm{c} = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$. Recall μ^{lp} is the Brownian loop measure.

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- Let U be a multiply connected domain with two boundary points a, b on the same boundary component. We may find a simply connected domain $D \supset U$ such that ∂D is the component of ∂U containing a, b. Lawler defined the SLE_κ in U from a to b as

$$\mu^D_{U;a\to b} = \mathbf{1}_{\{\cdot\subset U\}} e^{\mathrm{c}\,\mu^{\mathrm{lp}}(\mathcal{L}_D(\cdot,U^c))} \cdot \mu^{\#}_{D;a\to b}.$$

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- Conformal covariance: if $U_j \subset D_j$, $a_j, b_j \in \partial U_j$, $j = 1, 2$, and $f : (U_1; a_1, b_1) \overset{\mathrm{Conf}}{\twoheadrightarrow} (U_2; a_2, b_2)$, then

$$f(\mu^D_{U_1;a_1\to b_1}) = |f'(a_1)|^{\frac{6-\kappa}{2\kappa}} |f'(b_1)|^{\frac{6-\kappa}{2\kappa}} \mu^D_{U_2;a_2\to b_2}.$$

- If $\mu^D_{U;a\to b}$ is finite, we may normalize it to get a probability measure with conformal invariance.

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$$\mu^1_{D;z} = \mathbf{1}_{\{\cdot \subset D\}} e^{\mathrm{c}\,\mu^{\mathrm{lp}}(\mathcal{L}_{\widehat{\mathbb{C}}}(\cdot, D^c))} \cdot \mu^1_z, \quad \mu^0_D = \mathbf{1}_{\{\cdot \subset D\}} e^{\mathrm{c}\,\mu^{\mathrm{lp}}(\mathcal{L}_{\widehat{\mathbb{C}}}(\cdot, D^c))} \cdot \mu^0.$$

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- However, $\mu^{\mathrm{lp}}(\mathcal{L}(\gamma, D^c))$ is not finite for any curve γ in D. The correct alternative is the normalized Brownian loop measure introduced in [Field-Lawler], i.e.,

$$\Lambda^*(V_1, V_2) := \lim_{r \downarrow 0} [\mu^{\mathrm{lp}}_{\{|z-z_0|>r\}}(\mathcal{L}(V_1, V_2)) - \log\log(1/r)],$$

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- The limit converges if V_1 and V_2 are disjoint compact subsets of $\widehat{\mathbb{C}}$; and the value does not depend on the $z_0 \in \widehat{\mathbb{C}}$.
- The correct way to define SLE_κ loop measures in $D \subset \widehat{\mathbb{C}}$ is using $\Lambda^*(\cdot, D^c)$ in place of $\mu^{\mathrm{lp}}(\mathcal{L}_{\widehat{\mathbb{C}}}(\cdot, D^c))$.

SLE loops in subdomains of $\widehat{\mathbb{C}}$

Theorem

The $\mu^1_{D;z}$ and μ^0_D defined using normalized Brownian loop measure satisfy conformal covariance and conformal invariance, respectively: if $W : U \overset{\text{Conf}}{\twoheadrightarrow} V$, and $z \in U$, then

$$W(\mu^1_{U;z}) = |W'(z)|^{2-d} \mu^1_{V;W(z)};$$

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- The proof uses CMP of rooted SLE_κ loop in $\widehat{\mathbb{C}}$ and the generalized restriction property of chordal SLE_κ.
- SLE loop satisfies generalized restriction property:

$$\mu^0_{U_1} = \mathbf{1}_{\{\cdot \subset U_1\}} e^{c\,\mu^{\text{lp}}(\mathcal{L}_{U_2}(\cdot, U_1^c))} \cdot \mu^0_{U_2}, \quad U_1 \subset U_2 \subset \widehat{\mathbb{C}}.$$

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$$W(\mu^1_{U;z}) = |W'(z)|^{2-d} \mu^1_{V;W(z)};$$

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- The proof uses CMP of rooted SLE_κ loop in $\widehat{\mathbb{C}}$ and the generalized restriction property of chordal SLE_κ.
- SLE loop satisfies generalized restriction property:

$$\mu^0_{U_1} = \mathbf{1}_{\{\cdot \subset U_1\}} e^{\mathrm{c}\, \mu^{\mathrm{lp}}(\mathcal{L}_{U_2}(\cdot, U_1^c))} \cdot \mu^0_{U_2}, \quad U_1 \subset U_2 \subset \widehat{\mathbb{C}}.$$

So this is an MKS loop measure with central charge c.

SLE loops in Riemann surfaces

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- The generalized restriction property of SLE_κ loop measures allows to define SLE loops in Riemann surfaces.
- The definition uses regular or normalized Brownian loop measure.
- The generalized restriction property implies a consistency formula: we may define SLE loops on charts, and glue them together.
- There exist other ways of defining SLE loop measures in Riemann surfaces (e.g., using $\text{SLE}_{8/3}$ loops instead of Brownian loops).

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- An SLE_κ bubble is a "random" loop in a simply connected domain rooted at a boundary point.
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- The construction is also similar, except that we now use the degenerate two-sided radial SLE_κ curve in place of two-sided whole-plane SLE_κ curve.
- A degenerate two-sided radial SLE_κ curve grows in a simply connected domain starting and ending at the same boundary point, and passing through a marked interior point. It can be constructed from two-sided radial SLE_κ by merging the two end points. We use the symbol $\nu^{\#}_{D;a\rightleftharpoons w}$, where $a \in \partial D$, $b \in D$.

Theorem

Let $G_{\mathbb{H}}(w) = |w|^{\frac{2}{\kappa}(\kappa-8)}(\Im w)^{\frac{(\kappa-8)^2}{8\kappa}}$. *Then the following are true.*

(I) *There is a unique* σ*-finite measure* $\mu^1_{\mathbb{H};a}$ *(*SLE_κ *bubble in* $\mathbb{H}$*), which is supported by non-degenerate loops in* $\overline{\mathbb{H}}$ *rooted at* a *which possess Minkowski content measure in* $\mathbb{C}\setminus\{a\}$*, and satisfies*

$$\mu^1_{\mathbb{H};a}(d\gamma)\otimes \mathcal{M}_{\gamma;\mathbb{C}\setminus\{0\}}(dw) = \nu^{\#}_{\mathbb{H};a\rightleftharpoons w}(d\gamma)\overleftarrow{\otimes} G_{\mathbb{H}}(w-a)\cdot A(dw).$$

(II) *Every* $\mu^1_{\mathbb{H};a}$ *satisfies CMP.*

(III) *If* $W:\mathbb{H}\overset{\text{Conf}}{\twoheadrightarrow}\mathbb{H}$*, then* $W(\mu^1_{\mathbb{H};a}) = |W'(a)|^{\frac{8}{\kappa}-1}\mu^1_{\mathbb{H};W(a)}$.

(IV) *For any* $r>0$, $\mu^1_{\mathbb{H};a}(\{\gamma:\operatorname{diam}(\gamma)>r\})<\infty$.

(V) *If a measure* μ' *supported by non-degenerate loops in* $\overline{\mathbb{H}}$ *rooted at* a *satisfies (II) and (IV), then* $\mu' = c\mu^1_{\mathbb{H};a}$ *for some* $c\ge 0$.

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We use a result by Lalwer that, for $\kappa \in (4,8)$, the intersection of a chordal SLE_κ curve with the boundary of the domain has $(2-\frac{8}{\kappa})$-dimensional Minkowski content.

SLE bubbles

The construction uses degenerate two-sided chordal SLE_κ measure supported by loops rooted at two boundary points, which is defined by $\nu^{\#}_{D;a\rightleftharpoons b} = \lim_{D\ni w\to b} \nu^{\#}_{D;a\rightleftharpoons w}$.

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To define SLE_κ bubble in $\mathbb{H}$ rooted at $x \in \mathbb{R}$, we integrate $\nu^{\#}_{\mathbb{H};x\rightleftharpoons y}$ against the function $G_{\mathbb{H}}(y-x) := |x-y|^{-2(\frac{8}{\kappa}-1)}$, and then unweight the integrated measure by the $(2-\frac{8}{\kappa})$-dimensional Minkowski content of the intersection of the loop with $\mathbb{R}$. This agrees with the previous SLE_κ bubble measure up to a multiplicative constant.

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If we further integrate the laws of SLE_κ bubble in $\mathbb{H}$ rooted at x against the Lebesgue measure, and then unweight the integrated measure by the $(2-\frac{8}{\kappa})$-dimensional Minkowski content of the intersection of the loop with $\mathbb{R}$, we then get the unrooted SLE_κ bubble measure in $\mathbb{H}$.

Happy Birthday,
Peter!

Thank you!