Harmonic measure, Riesz transforms, and uniform rectifiability

Xavier Tolsa

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Rectifiability

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E is n-AD-regular if

$$\mathcal{H}^n(B(x,r) \cap E) \approx r^n \quad \text{ for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

E is uniformly n-rectifiable if it is n-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes (originating from P. Jones TST).

The Riesz and Cauchy transforms

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In $\mathbb{R}^d$, the n-dimensional Riesz transform of $f \in L^1_{loc}(\mu)$ is $\mathcal{R}_\mu f(x) = \lim_{\varepsilon \searrow 0} \mathcal{R}_{\mu,\varepsilon} f(x)$, where

$$\mathcal{R}_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mu(y).$$

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In $\mathbb{C}$, the Cauchy transform of $f \in L^1_{loc}(\mu)$ is $\mathcal{C}_\mu f(z) = \lim_{\varepsilon \searrow 0} \mathcal{C}_{\mu,\varepsilon} f(z)$, where

$$\mathcal{C}_{\mu,\varepsilon} f(z) = \int_{|z-\xi|>\varepsilon} \frac{f(\xi)}{z-\xi} \, d\mu(\xi).$$

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The existence of principal values is not guarantied, in general.

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We also denote

$$\mathcal{R}\mu = \mathcal{R}_\mu 1, \qquad \mathcal{C}\mu = \mathcal{C}_\mu 1.$$

Harmonic measure

$\Omega \subset \mathbb{R}^{n+1}$ open and connected.
For $p \in \Omega$, ω^p is the harmonic measure in Ω with pole in p.

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Questions about the metric properties of harmonic measure:
When $\mathcal{H}^n \approx \omega^p$?
Which is the connection with rectifiability?
Dimension of harmonic measure?

Metric properties of harmonic measure

- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^1 \approx \omega^p$. (F.& M. Riesz)
- Many results in $\mathbb{C}$ using complex analysis (Makarov, Jones, Bishop, Wolff,...).
- The analogue of Riesz theorem fails in higher dimensions (counterexamples by Wu and Ziemer).
- In higher dimension, need real analysis techniques.
  Connection with uniform rectifiability studied recently by Hofmann, Martell, Uriarte-Tuero, Mayboroda, Azzam, Badger, Bortz, Toro, Akman, etc.

Connection between harmonic measure and Riesz transform

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Therefore, for $x \in \Omega$:

$$c \, \nabla_x G(x,p) = K(x-p) - \int K(x-y) \, d\omega^p(y).$$

That is, $$\mathcal{R}\omega^p(x) = K(x-p) - c \, \nabla_x G(x,p).$$

Riesz transforms and rectifiability

Theorem (Nazarov, T., Volberg, 2012)

Let $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}^n_E$. If $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded, then E n-rectifiable.
If, additionally, E is n-AD-regular, then E is uniformly n-rectifiable.

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- David-Semmes problem.

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- The case $n = 1$ proved previously by Mattila-Melnikov-Verdera (AD-regular case) and David and Léger, using curvature.

A converse to Riesz theorem

Theorem

Let $\Omega \subset \mathbb{R}^{n+1}$ be open. Let $E \subset \partial\Omega$ with $0 < \mathcal{H}^n(E) < \infty$ such that $\omega|_E \approx \mathcal{H}^n|_E$. Then E is n-rectifiable.

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- Proof by
  [Azzam, Mourgoglou, T.]
  + [Hofmann, Martell, Mayboroda, T., Volberg].
- A special case for $n = 1$ by Pommerenke.
- It solves a question posed by Bishop in the 1990's.
- Previous related results by Martell, Hofmann and Uriarte-Tuero with stronger quantitative assumptions and consequences.

Idea of the proof of the theorem

Let $x \in E$ and $x' \in \Omega$ such that $\text{dist}(x', \partial\Omega) \approx \varepsilon$. Then use the identity

$$\mathcal{R}\omega^p(x') = K(x' - p) - c\,\nabla_{x'} G(x', p).$$

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$$|\nabla_{x'} G(x', p)| \lesssim \frac{G(x', p)}{\varepsilon} \lesssim \frac{\omega^p(B(x, 4\varepsilon))}{\mathcal{H}^n_\infty(B(x, \varepsilon) \cap \partial\Omega)} \lesssim 1.$$

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Then

$$|\mathcal{R}_\varepsilon\omega^p(x)| \leq |\mathcal{R}\omega^p(x')| + C \sup_{r>0} \frac{\omega^p(B(x, r))}{r^n} < \infty.$$

Thus $\mathcal{R}_*\omega^p(x) < \infty$ for all $\mathcal{H}^n$-a.e. $x \in E$.

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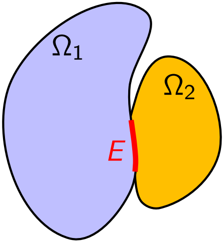
Thus $\mathcal{R}_* \omega^p(x) < \infty$ for all $\mathcal{H}^n$-a.e. $x \in E$.
This implies there exists $F \subset E$ with $\mathcal{H}^n(F) > 0$ such that $\mathcal{R}_{\mathcal{H}^n|F}$ is bounded in $L^2(\mathcal{H}^n|_F)$.
So F is n-rectifiable by the Nazarov-T.-Volberg theorem.

A two-phase problem



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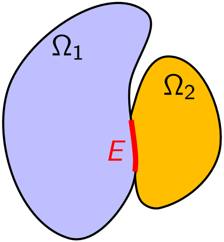

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Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$, with $\Omega_1 \cap \Omega_2 = \varnothing$, $\partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$, with harmonic measures ω^1, ω^2. Let $E \subset \partial\Omega_1 \cap \partial\Omega_2$ be such that $\omega^1 \ll \omega^2 \ll \omega^1$ on E.

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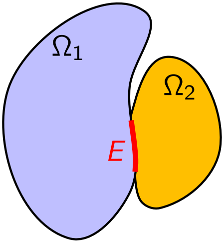

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- Other contributions: ACF, Badger, Engelstein,...

Harmonic functions and uniform rectifiability

Theorem

Let $E \subset \mathbb{R}^{n+1}$ be n-AD-regular and $\Omega = \mathbb{R}^{n+1} \setminus E$.
The following are equivalent:

(a) *E is uniformly n-rectifiable.*

(b) *There is $C > 0$ such that if u is a bounded harmonic function on Ω and B is a ball centered at E,*

$$\int_B |\nabla u(x)|^2 \operatorname{dist}(x, \partial\Omega)\, dx \leq C\, \|u\|^2_{L^\infty(\Omega)}\, r(B)^n. \tag{1}$$

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- (a)⇒(b) by Hofmann, Martell, Mayboroda.
- (b)⇒(a) by Garnett, Mourgoglou and T.

To prove (b)⇒(a) we show:

Theorem

Let $E, \Omega \subset \mathbb{R}^{n+1}$ *be as above. Let* $\mu = \mathcal{H}_{|E}$ *and let* $\mathcal{D}_\mu$ *be a dyadic lattice of cubes associated to* μ*.*

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Let $E, \Omega \subset \mathbb{R}^{n+1}$ be as above. Let $\mu = \mathcal{H}_{|E}$ and let $\mathcal{D}_\mu$ be a dyadic lattice of cubes associated to μ. Then E is uniformly n-rectifiable if and only if there exists a partition of $\mathcal{D}_\mu$ into **trees** *$\mathcal{T} \in I$ satisfying:*

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(a) *The family of roots of $\mathcal{T} \in I$ fulfils the packing condition*

$$\sum_{\mathcal{T} \in I : \mathrm{Root}(\mathcal{T}) \subset S} \mu(\mathrm{Root}(\mathcal{T})) \leq C\, \mu(S) \quad \textit{for all } S \in \mathcal{D}_\mu.$$

(b) *For each $\mathcal{T} \in I$ with $R = \mathrm{Root}(\mathcal{T})$, there exists a point $p_{\mathcal{T}} \in \Omega$ with*

$$c^{-1}\ell(R) \leq \mathrm{dist}(p_{\mathcal{T}}, R) \leq \mathrm{dist}(p_{\mathcal{T}}, \partial\Omega) \leq c\, \ell(R)$$

such that, for all $Q \in \mathcal{T}$, $\omega^{p_{\mathcal{T}}}(5Q) \approx \dfrac{\mu(Q)}{\mu(R)}$.

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- This is a characterization of uniform rectifiability in terms of harmonic measure.
- ω may be mutually singular with $\mathcal{H}^n|_E$ (Bishop - Jones).

More remarks

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- An extension to operators $Lu = \mathrm{div} A \nabla u$, with A elliptic, perturbation of constant coefficient matrix, by Azzam, Garnett, Mourgoglou, T.

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- Solve the David-Semmes problem in $\mathbb{R}^d$ for $n = 2, \ldots, d-2$.

Another open problem and a conjecture

About the dimension of harmonic measure

Jones and Wolff showed that for all $\Omega \subset \mathbb{R}^2$, there exists $E \subset \partial\Omega$ with $\dim_{\mathcal{H}} E \leq 1$ such that ω is concentrated on E, i.e. $\omega(\partial\Omega \setminus E) = 0$.

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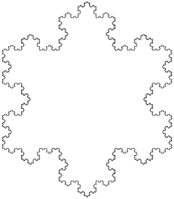

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Conjecture: $\varepsilon_n = \frac{1}{n}$.

Thank you.

Happy birthday Peter!