Brownian Motion and PDEs (and Fourier series)

Stefan Steinerberger

Peter W. Jones Birthday Conference, KIAS

Yale University

Undergraduate Seminar (Paul Müller, JKU Linz, 2009)

Universal Local Parametrizations via Heat Kernels and Eigenfunctions of the Laplacian

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MAT 280: Laplacian Eigenfunctions: Theory, Applications, and Computations
Lecture 1: Overture: Motivations, scope and structure of the course

Lecturer: Naoki Saito

The picture on the Yale homepage (still!)

$$\Delta u = f$$

$\Delta u = f$

PDEs

jones
/jōnz/

US *informal*

verb
gerund or present participle: **jonesing**

have a fixation on; be addicted to.
"Palmer was jonesing for some coke again"

Brownian Motion

Philosophical Overview

- parabolic PDEs make things nice and smooth (easy)
- elliptic PDEs minimize some energy functional (hard)

Alternatively: any solution of

$$-\text{div}(a(x)\nabla u) + \nabla V \nabla u + cu = 0$$

gives rise to a solution of a heat/diffusion equation

$$u_t + (-\text{div}(a(x)\nabla u) + \nabla V \nabla u + cu) = 0.$$

Use Brownian motion to study parabolic (=elliptic) problems!

Quantilized Donsker-Varadhan estimates

M. Donsker, S. Varadhan, On a variational formula for the principal eigenvalue for operators with maximum principle, PNAS 1975

Donsker-Varadhan Inequality

$\Omega \subset \mathbb{R}^n$ (but also works on graphs) and

$$Lu = -\mathrm{div}(a(x)\nabla u) + \nabla V(x) \nabla u.$$

Question. What is the smallest $\lambda > 0$ for which

$$Lu = \lambda u \qquad \text{has a solution with } u\big|_{\partial\Omega} = 0?$$

Example:

$$L = -\Delta + \nabla\left(\frac{1}{2}x^2\right) \qquad \text{on } [0,1].$$

$$\langle Lu, u\rangle \geq \boxed{?} \cdot \|u\|^2$$

Donsker-Varadhan Inequality

$$Lu = -\mathrm{div}(a(x)\nabla u) + \nabla V(x)\nabla u.$$

Question. What is the smallest $\lambda > 0$ for which

$$Lu = \lambda u \qquad \text{has a solution with } u\big|_{\partial\Omega} = 0?$$

Donsker-Varadhan: associate a drift diffusion process (wiggle with $a(x)$, drift towards ∇V) and maximize the expected exit time.

Donsker-Varadhan Inequality

$$\lambda_1 \geq \frac{1}{\sup_{x\in\Omega} \mathbb{E}_x \tau_{\Omega^c}}.$$

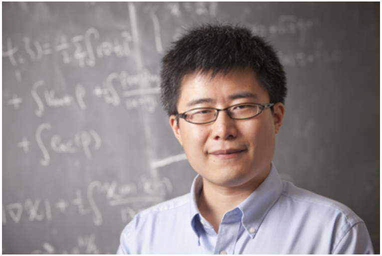


Figure: Jianfeng Lu (Duke)

Instead of looking at the mean of the first exist time, we study quantiles: let $d_{p,\partial\Omega} : \Omega \to \mathbb{R}_{\geq 0}$ be the smallest time t such that the likelihood of exiting within that time is p.

J. Lu and S., 2016

$$\lambda_1 \geq \frac{\log(1/p)}{\sup_{x\in\Omega} d_{p,\partial\Omega}(x)}.$$

Moreover, as $p \to 0$, the lower bound converges to λ_1.

Proof.

Start drift-diffusion in the point, where the solution assumes its maximum. have (Feynman-Kac)

$$\|u\|_{L^\infty} = u(x) = e^{\lambda t}\mathbb{E}_\omega\left(u(\omega(t))\right)$$

with the convention that $u(\omega(t))$ is 0 if the drift-diffusion processes leaves Ω at some point in the interval $[0, t]$. Let now $t = d_{p,\partial\Omega}(x)$, in which case we see that

$$\mathbb{E}_\omega\left(u(\omega(t))\right) \leq p\|u\|_{L^\infty} + (1-p)0.$$

Altogether, we obtain

$$\|u\|_{L^\infty} = e^{\lambda d_{p,\partial\Omega}(x)}\mathbb{E}_\omega\left(u(\omega(t))\right) \leq e^{\lambda d_{\partial\Omega}(x)} p\|u\|_{L^\infty}$$

from which the statement follows. □

Example 1

Let us consider

$$L = -\Delta \qquad \text{on } [0,1].$$

Then $\lambda_1 = \pi^2$.

p	$1/2$	$1/4$	10^{-1}	10^{-2}	10^{-8}	Donsker-Varadhan
lower bound	7.28	8.40	8.92	9.39	9.74	8

Example 2

Let us consider

$$L = -\Delta + \nabla\left(\frac{1}{2}x^2\right) \qquad \text{on } [0,1].$$

Then $\lambda_1 = 2$.

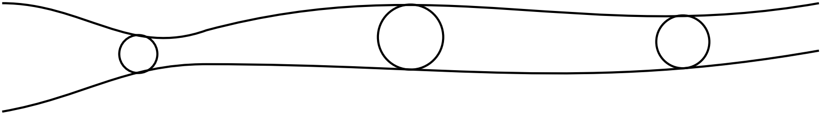
p	0.5	0.3	0.2	0.1	0.05	Donsker-Varadhan
lower bound	1.52	1.67	1.74	1.79	1.83	1.678

Lieb's inradius result and the Polya-Szegő conjecture

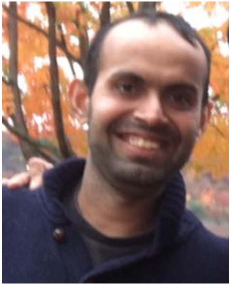
Polya

In two dimensions, we have (Osserman, Makai, Hayman, Polya-Szegő, ...)

$$\lambda_1(\Omega) = \inf_{f \neq 0} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx} \sim \frac{1}{\text{inradius}^2}$$



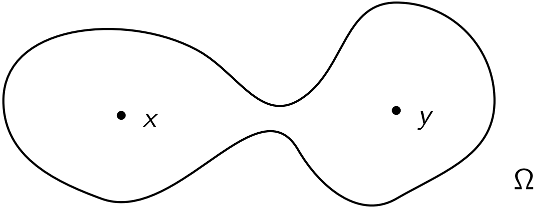
One direction ($\lesssim$) is trivial. The other direction ($\gtrsim$) was posed as a conjecture by Polya & Szegő in 1951 (proven by Makai (1965) and, independently, Hayman (1978)).



Theorem (M. Rachh and S, CPAM 2017)

Let $\Omega \subset \mathbb{R}^2$ be simply connected and $u : \Omega \to \mathbb{R}^2$ vanish on $\partial\Omega$. If u assumes a global extremum in $x_0 \in \Omega$, then

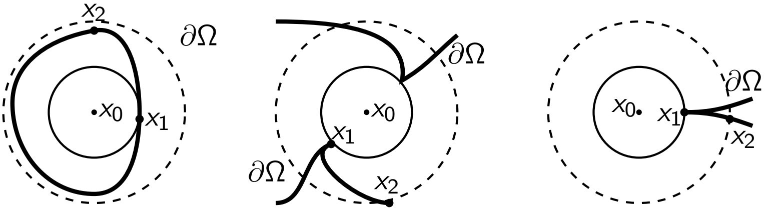
$$\inf_{y \in \partial\Omega} \|x_0 - y\| \geq c \left\| \frac{\Delta u}{u} \right\|_{L^\infty(\Omega)}^{-1/2}.$$



Proof.

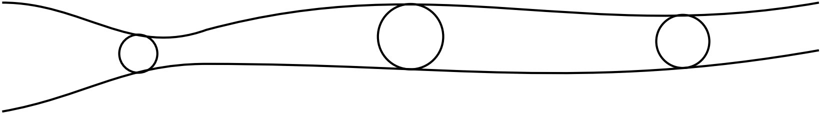
$$\begin{aligned}\|u\|_{L^\infty} = u(x_0) = \mathbb{E}_{x_0}\left(u(\omega(t))e^{\int_0^t V(\omega(z))dz}\right) \\ \leq (1 - p_{x_0}(t))\|u\|_{L^\infty(\Omega)}\mathbb{E}_{x_0}\left(e^{\int_0^t V(\omega(z))dz}\right) \\ \leq (1 - p_{x_0}(t))\|u\|_{L^\infty} e^{t\|V\|_{L^\infty}},\end{aligned}$$

Therefore $(1 - p_{x_0}(t))e^{t\|V\|_{L^\infty}} \geq 1$. □



Lieb's theorem

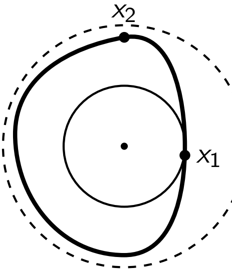
Such results are impossible in dimensions ≥ 3: one can take a ball and remove one-dimensional lines without affecting the PDE.



Theorem (Elliott Lieb, 1984, Inventiones)

Ω contains a $(1-\varepsilon)$-fraction of a ball with radius

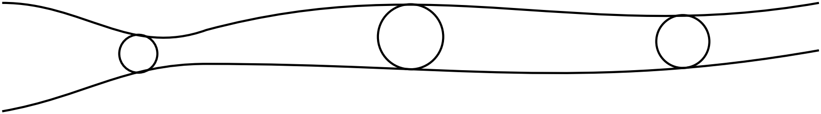
$$r \sim \frac{c_\varepsilon}{\sqrt{\lambda_1(\Omega)}}$$



Lemma (S, 2014, Comm. PDE)

If you start Brownian motion in the maximum of the eigenfunction $-\Delta u = \lambda u$, then the likelihood of it impacting the nodal set within time $t = \lambda^{-1}$ is less than 64%.

This means that not 'much' boundary can be close to the maximum.



Theorem (Rachh and S, 2017, CPAM)

If, with Dirichlet conditions,

$$-\Delta u = Vu \qquad \text{in } \Omega$$

then Ω contains a $(1-\varepsilon)$-fraction of a ball with radius

$$r \sim \frac{c_\varepsilon}{\sqrt{\|V\|_{L^\infty}}}$$

centered around the maximum of u.

A 'Real Life' Application(?)

Anomaly detection

Fiddling with results of this type suggest that for $-\Delta \phi_\lambda = \lambda \phi_\lambda$, the quantity

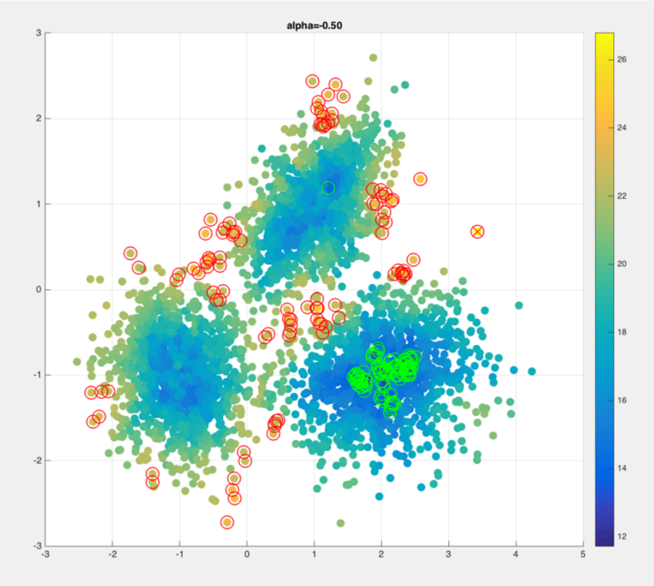
$$\frac{1}{\sqrt{\lambda}} \frac{|\phi_\lambda(x)|}{\|\phi_\lambda\|_{L^\infty}}$$

is a decent proxy for the distance to the nearest nodal set.

How about summing over distances to nodal lines

$$\sum_{\lambda \leq N} \frac{1}{\sqrt{\lambda}} \frac{|\phi_\lambda(x)|}{\|\phi_\lambda\|_{L^\infty}} \qquad ?$$

Anomaly detection



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Anomaly detection

On the torus $\mathbb{T}$, the quantity

$$\sum_{\lambda \leq N} \frac{1}{\sqrt{\lambda}} \frac{|\phi_\lambda(x)|}{\|\phi_\lambda\|_{L^\infty}}$$

simplifies to

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

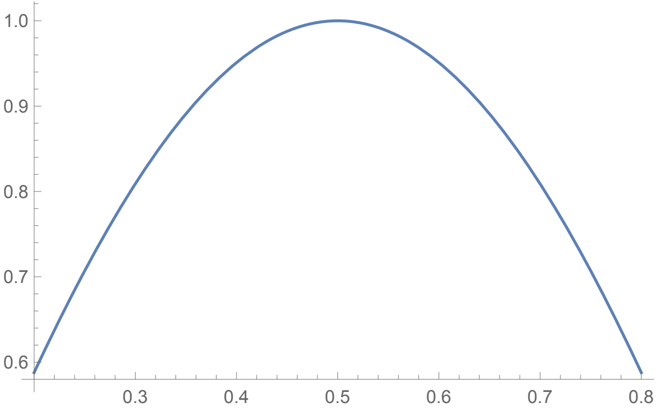

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Figure: $n = 1$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

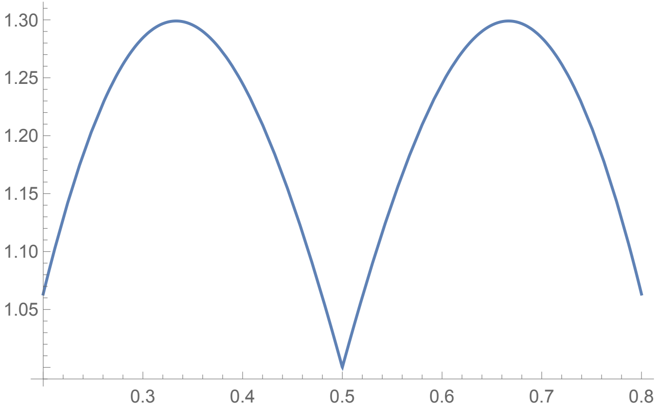

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Figure: $n = 2$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

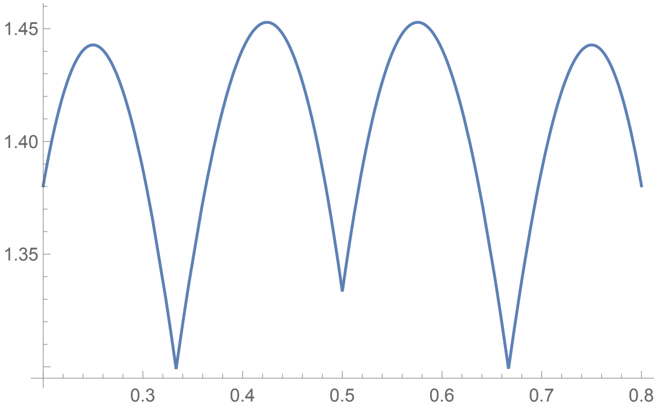

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Figure: $n = 3$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

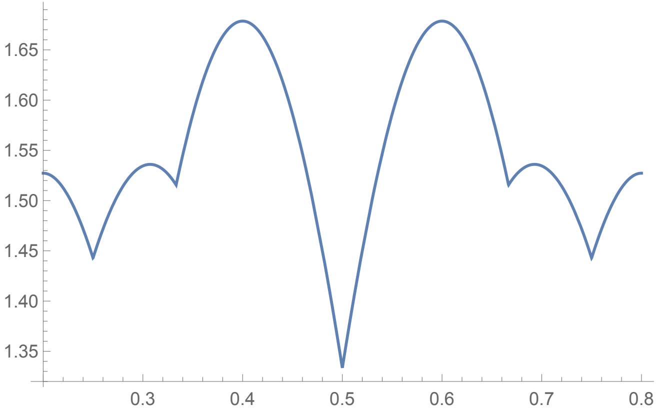

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Figure: $n = 4$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

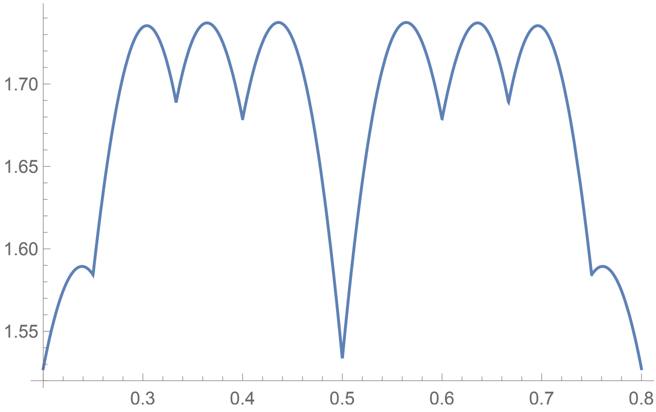

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Figure: $n = 5$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

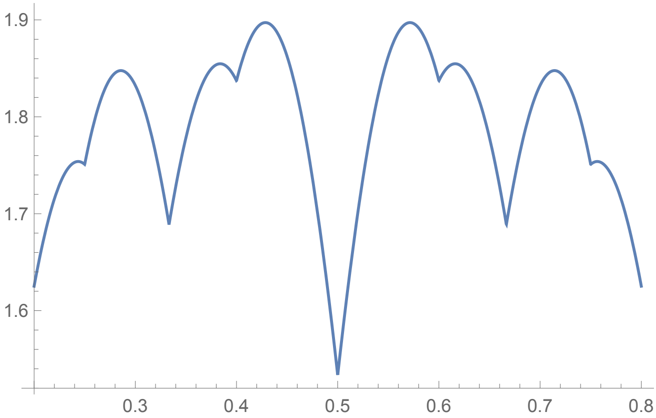

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Figure: $n = 6$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

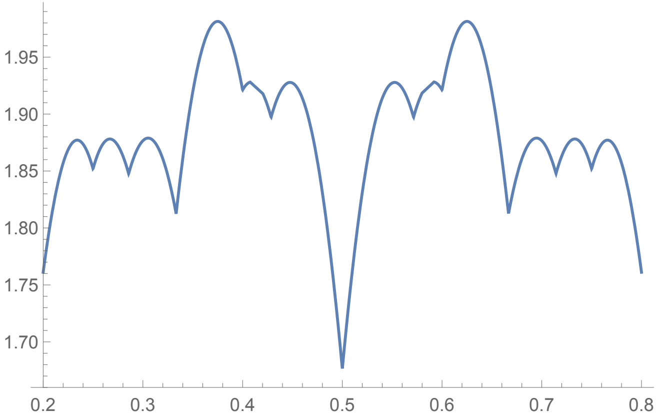

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Figure: $n = 7$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

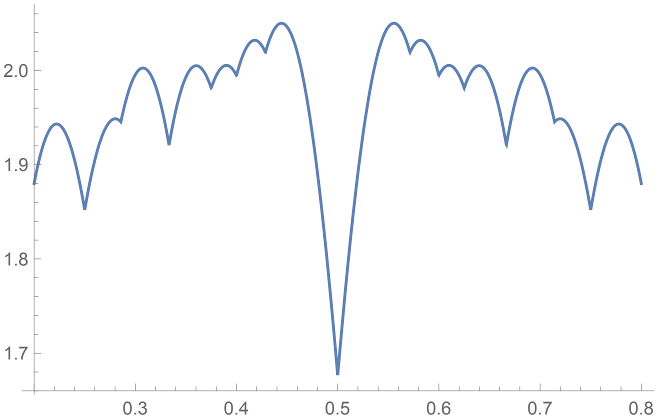

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Figure: $n = 8$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

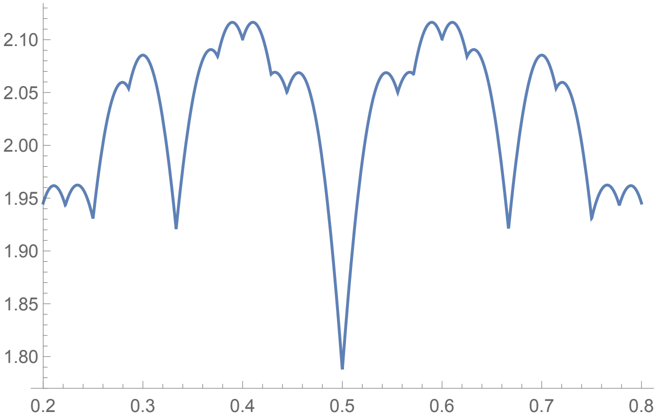

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Figure: $n = 9$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

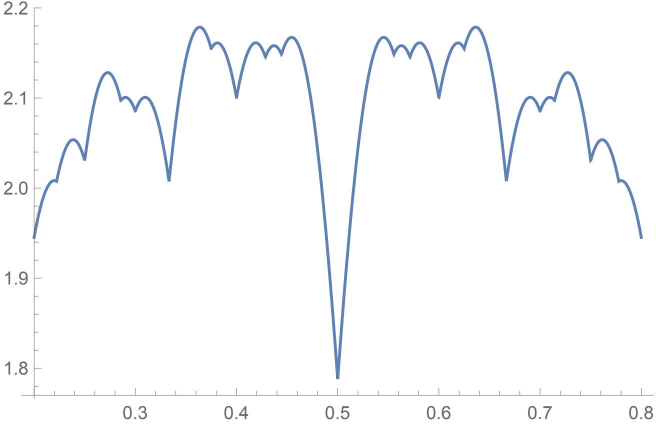

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Figure: $n = 10$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

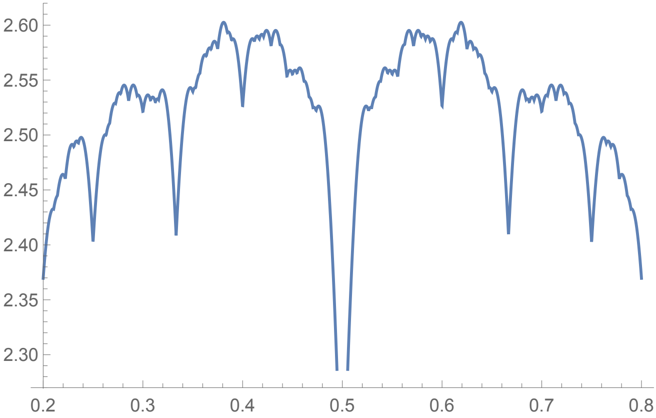

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Figure: $n = 20$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

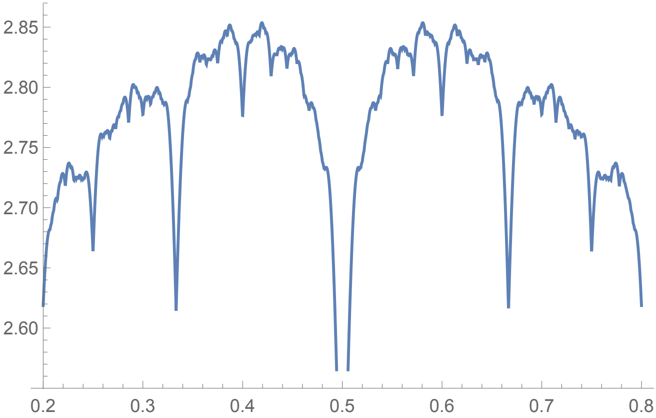

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Figure: $n = 30$ on $[0.2, 0.8]$

Anomaly detection

$$\sum_{k=1}^{n} \frac{|\sin k\pi x|}{k}.$$

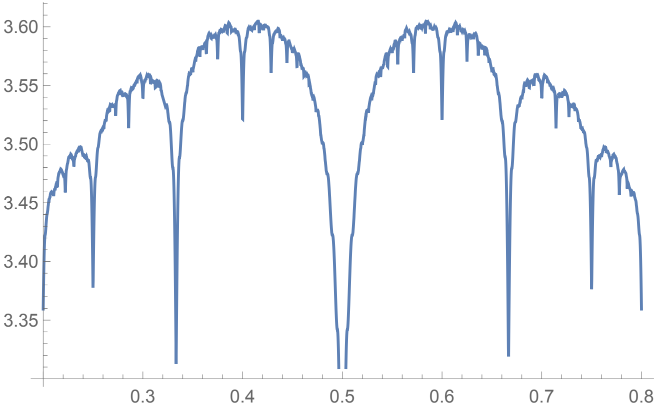

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Figure: $n = 100$ on $[0.2, 0.8]$

$$f_n(x) = \sum_{k=1}^{n} \frac{|\sin(k\pi x)|}{k}.$$

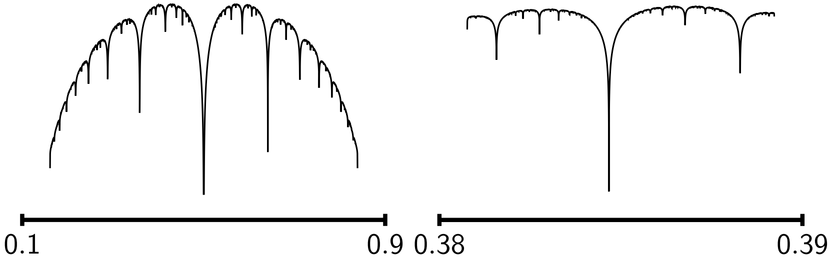
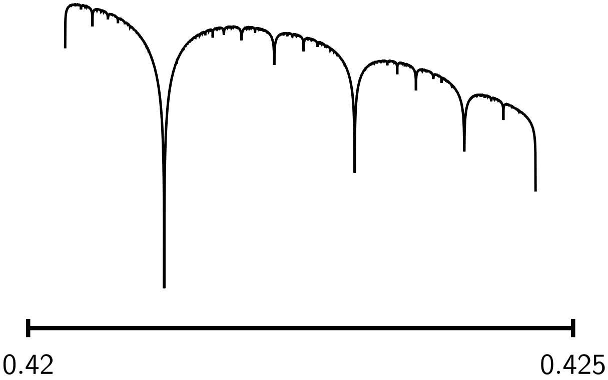


Figure: Right: the big cusp in the right picture is located at $x = 5/13$, the two smaller cusps are at $x = 8/21$ and $x = 7/18$.

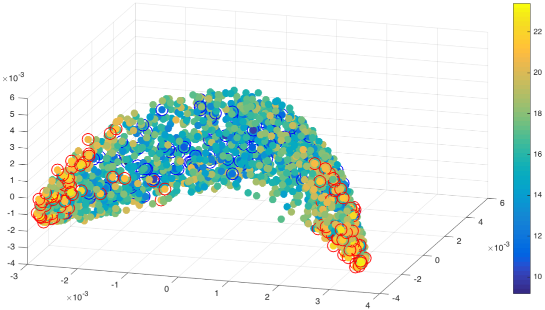


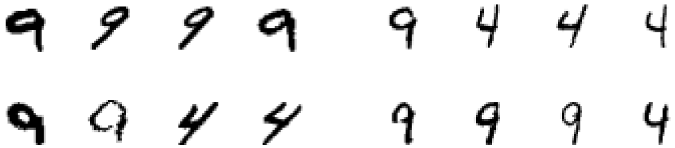
Theorem (S. 2016)

f_n has a strict local minimum in $x = p/q \in \mathbb{Q}$ as soon as

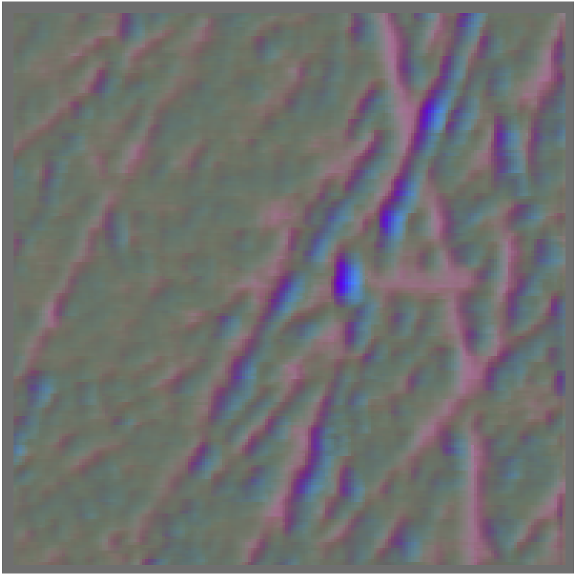
$$n \geq (1 + o(1))\frac{q^2}{\pi}.$$

Handwritten digits (ongoing w/ X. Cheng/Gal Mishne)

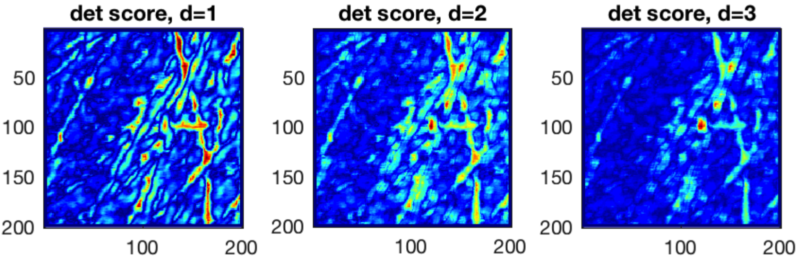




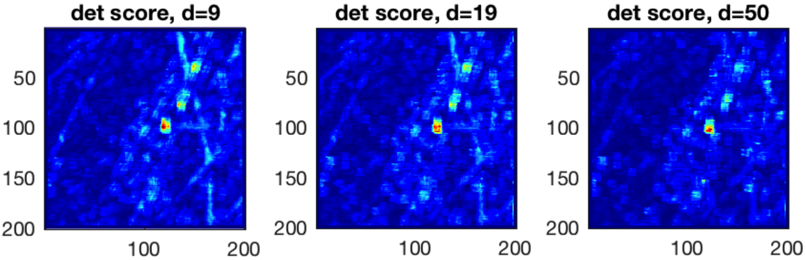
Seamine (ongoing w/ X. Cheng/Gal Mishne)



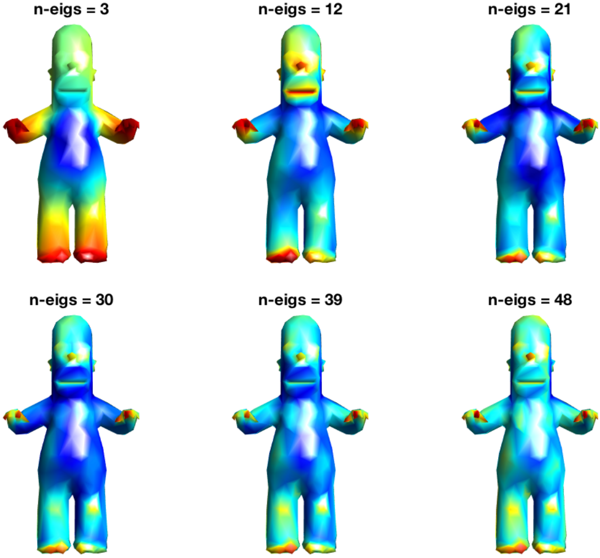
Seamines (ongoing w/ X. Cheng/Gal Mishne)



Seamines (ongoing w/ X. Cheng/Gal Mishne)

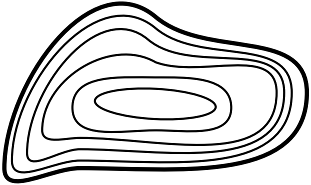


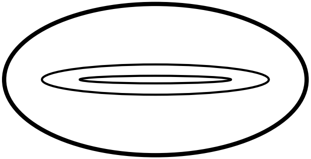
Homer (ongoing w/ X. Cheng/Gal Mishne)



Strict local maxima, elliptic PDEs, lifetime of Brownian motion and topological bounds on Fourier coefficients

Level sets of elliptic PDEs

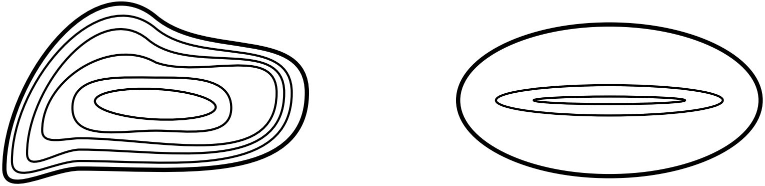




Generally tricky.

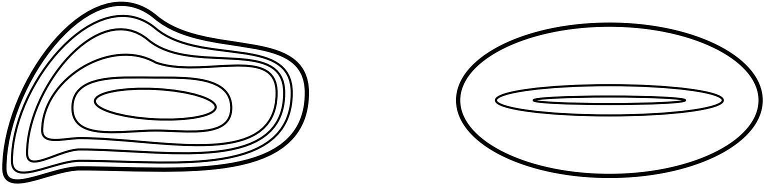
Maybe (P.-L. Lions) convex Ω and $-\Delta u = f(u)$ implies convex level sets?

Level sets of elliptic PDEs



Maybe (P.-L. Lions) convex Ω and $-\Delta u = f(u)$ implies convex level sets?
Yes for $-\Delta u = 1$ (Makar-Limanov, 70s)
Yes for $-\Delta u = \lambda_1 u$ (Brascamp-Lieb, 70s).
Yes, for some other f (various).
No: Hamel, Nadirashvili & Sire (2016).

Level sets of elliptic PDEs

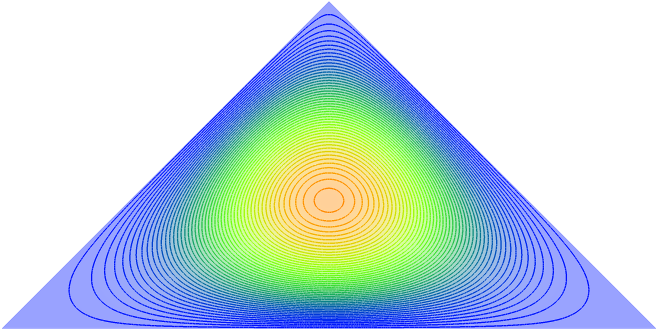


Can level sets ever be fundamentally more eccentric than the domain?

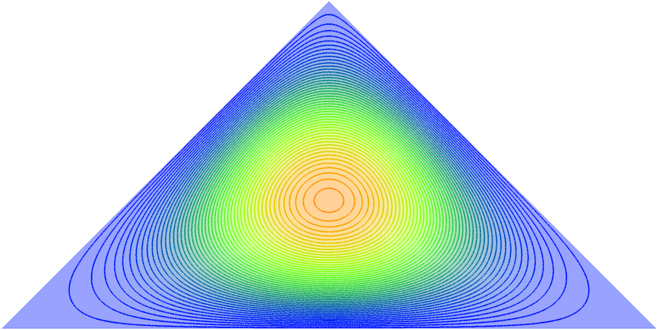
Let $\Omega \subset \mathbb{R}^2$ be convex and consider

$$-\Delta u = 1 \qquad \text{with Dirichlet boundary conditions.}$$

This is the expected lifetime of Brownian motion. It also has some meaning in mechanics (St. Venant torsion).



The three basic questions in hiking



1. How big is the mountain? $(\|u(x_0)\|_{L^\infty} \sim \text{inrad}(\Omega)^2)$
2. Where is the maximum? (x_0, maximal lifetime)
3. What's the view from the top? $(D^2 u(x_0)?)$

The three basic questions in hiking

Let $\Omega \subset \mathbb{R}^2$ be convex and consider

$$-\Delta u = 1 \qquad \text{with Dirichlet boundary conditions.}$$

Some facts.

- $\|u\|_{L^\infty} \sim \text{inrad}(\Omega)^2$
- Maximum is in unique $x_0 \in \Omega$ (Makar-Limanov, 1971)
- Eccentricity of level sets close to the maximum is determined by the Hessian $D^2u(x_0)$ in the maximum.
- $D^2u(x_0)$ is negative semi-definite. $\text{tr} D^2u(x_0) = \Delta u(x_0) = -1$.
- How close can the eigenvalues of $D^2u(x_0)$ be to 0?

Let $\Omega \subset \mathbb{R}^2$ be convex and consider

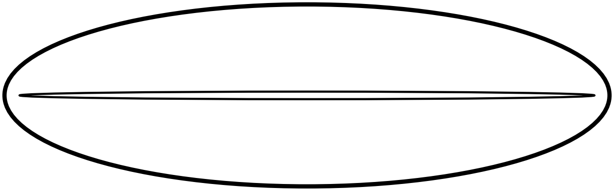
$$-\Delta u = 1 \qquad \text{with Dirichlet boundary conditions.}$$

Theorem (Spectral gap in the maximum, S, 2017)

There are universal constants $c_1, c_2 > 0$ such that

$$\lambda_{\max}\left(D^2 u(x_0)\right) \leq -c_1 \exp\left(-c_2 \frac{\text{diam}(\Omega)}{\text{inrad}(\Omega)}\right).$$

This is the sharp scaling.



Theorem (S, 2017)

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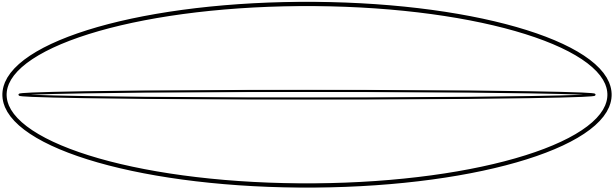
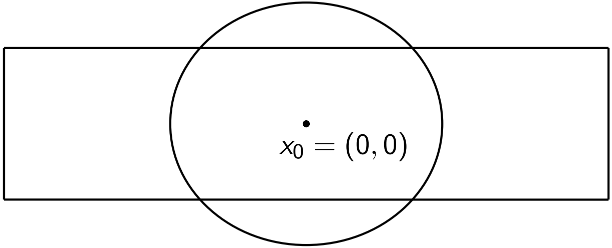


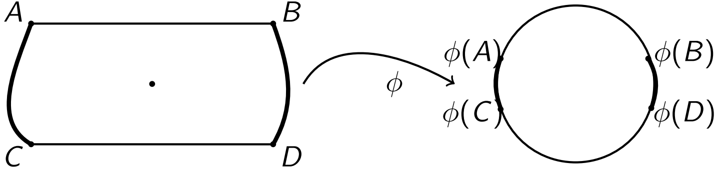
On domains Ω where $\partial\Omega$ has strictly positive curvature

$$\lambda_{\max}\left(D^2 u(x_0)\right) \leq -\frac{c}{\mathrm{inrad}(\Omega)^2} \frac{\min_{\partial\Omega} \kappa}{\max_{\partial\Omega} \kappa^3}.$$

Build a suitable rectangle and solve $\Delta v = -1$ on the rectangle. Imitate the local structure around the maximum up to and including the Hessian.



$\Delta(u - v) = 0$ on the intersection. Riemann mapping to the disk gives a harmonic function on the disk that is flat around the origin.



A Fourier series surprise

Harmonic function

$$\Delta u = 0 \qquad \text{on the unit disk } \mathbb{D}.$$

We know that

- $u(0,0) = 0$
- $\nabla u = 0$
- u is continuous on $\partial\mathbb{D}$ and has exactly 4 roots.

Does this force the $D^2 u(0,0)$ to have a large eigenvalue?

Yes (in all the ways that it could possibly be true) **.**

A Fourier series surprise

Proposition (New?)

If $f : \mathbb{T} \to \mathbb{R}$ is continuous, orthogonal to $1, \sin x, \cos x$ and has 4 roots, then f cannot be orthogonal to both $\sin 2x$ and $\cos 2x$.

Complex Analysis.

Consider the harmonic conjugate and perform a Poisson extension. The multiplicity of the root in the origin is at least 3. The argument principle implies that

$$f(t) + \tilde{f}(t) \qquad \text{winds around the origin at least 3-times,}$$

which creates at least 6 roots. $\square$

A Fourier series surprise

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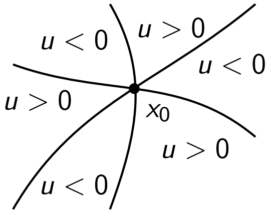
PDEs.

This means that the solution of the Dirichlet problem vanishes at least to third order in the origin.



This lines can never meet (maximum principle), therefore at least 6 roots on the boundary. □

Topological bounds on the Fourier coefficients

Theorem (S. 2017)

Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function that changes sign n times. Then

$$\sum_{k=0}^{n/2} |\langle f, \sin kx\rangle| + |\langle f, \cos kx\rangle| \gtrsim_n \frac{|f\|_{L^1(\mathbb{T})}^{n+1}}{\|f\|_{L^\infty(\mathbb{T})}^{n}}.$$

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Theorem (Harmonic function formulation)

Let $u : \mathbb{D} \to \mathbb{R}$ be harmonic, let $u\big|_{\partial\mathbb{D}}$ be continuous and assume it changes sign n times. Then

$$\sum_{k=0}^{n/2} \left| \frac{\partial^k u}{\partial x^k} \right| + \left| \frac{\partial^k u}{\partial y^k} \right| \geq c_n \frac{\|u\big|_{\partial\mathbb{D}}\|_{L^1(\mathbb{T})}^{n+1}}{\|u\big|_{\partial\mathbb{D}}\|_{L^\infty(\mathbb{T})}^{n}}.$$

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Theorem (Heat equation formulation)

Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function that changes sign n times. Then

$$\|e^{t\Delta} f\|_{L^1(\mathbb{T})} \gtrsim_{n,t} \frac{\|f\|_{L^1(\mathbb{T})}^{n+1}}{\|f\|_{L^\infty(\mathbb{T})}^{n}}.$$

Sketch of proof

Show

$$\|e^{t\Delta}f\|_{L^1(\mathbb{T})} \gtrsim_{n,t} \frac{\|f\|_{L^1(\mathbb{T})}^{n+1}}{\|f\|_{L^\infty(\mathbb{T})}^{n}}$$

and then recover all the other statements. Proof is based on using multiple interpretations:

- semigroup (to get many related families of estimates)
- Fourier multiplier (keeps Fourier eigenspaces separated)
- convolution with the Jacobi theta function θ_t
- diffusion process (does not increase the number of roots).

Happy Birthday!