Long term dynamics for nonlinear dispersive equations

W. Schlag (University of Chicago)

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Overview and Summary

Lecture describes advances on asymptotic behavior of solutions to nonlinear evolution equations.

- For linear equations with time-independent coefficients description based on spectral resolution, functional calculus. Classical asymptotic completeness, Agmon-Kato-Kuroda (60's, 70's) for potentials, ongoing studies on variable metrics (trapping, nontrapping, hyperbolic trapped trajectories).
- Two types of nonlinear Hamiltonian equations: those that admit "solitons" (focusing), and those that do not (defocusing). For the latter much better understanding, ultimately want to show that all excess energy radiates off to spatial infinity (scattering). Focusing equations typically exhibit finite-time blowup for large data (small data expect global existence and scattering).
- Concentration compactness: Analogue of elliptic technique (Lions, Struwe, Lieb 80s), developed by Bahouri-Gérard (1998), Kenig-Merle (2006 etc.)
- Invariant manifolds in infinite dimensions (center-stable mfld).

Linear asymptotic completeness

Linear Schrödinger equation in $\mathbb{R}^n$ with suitable decaying potential

$$i\partial_t\psi - \Delta\psi + V\psi = 0, \quad \psi(0) \in L^2(\mathbb{R}^d)$$

exhibits long-term dynamics

$$\psi(t) = \sum_j e^{itE_j}\psi_j + e^{-it\Delta}\phi_0 + o_{L^2}(1), \qquad t \to \infty$$

where $(-\Delta + V)\psi_j = E_j\psi_j$, $E_j \le 0$ are bound states, $\phi_0 \in L^2$.

Asymptotic completeness of the wave operators

Analogue for nonlinear equation? Soliton resolution problem.

Linear Klein-Gordon equation

Solve Cauchy problem

$$\Box u + u = F \ \text{ in } \ \mathbb{R}^{1+d}_{t,x}, \quad u(0) = f,\ u_t(0) = g$$

by explicit Duhamel formula ($\langle a \rangle = (1 + |a|^2)^{\frac{1}{2}}$)

$$u(t) = \cos(t\langle\nabla\rangle)f + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}g + \int_0^t \frac{\sin((t-s)\langle\nabla\rangle)}{\langle\nabla\rangle}F(s)\, ds$$

Energy estimate for $\vec{u} := (u, u_t)$, $\mathcal{H} = H^1 \times L^2(\mathbb{R}^d)$

$$\|\vec{u}(t)\|_{\mathcal{H}} \lesssim \|(f,g)\|_{\mathcal{H}} + \int_0^t \|F(s)\|_2\, ds$$

No time decay. Long-term analysis of nonlinear equations requires decay properties.

Stationary phase gives that

$$e^{\pm it\langle\nabla\rangle} f(x) = \int_{\mathbb{R}^d} e^{\pm it\langle\xi\rangle} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi$$
$$= \int_{\mathbb{R}^{2d}} e^{\pm it\langle\xi\rangle} e^{i(x-y)\cdot\xi}\, d\xi\, f(y)\, dy \quad \text{formal}$$

decays like $t^{-\frac{d}{2}}$. Critical points: $\pm t\xi\langle\xi\rangle^{-1} + x = 0$, Hessian nondegenerate, but as $\xi \to \infty$ one principal curvature vanishes. Stein-Tomas theorem for extension of Fourier transform:

$$(e^{\pm it\langle\nabla\rangle} f)(x) = \int_{\mathbb{R}^{d+1}} e^{i(x\cdot\xi+t\tau)} \delta(\tau \mp \langle\xi\rangle) \hat{f}(\xi)\, d\xi\, d\tau = (G\,\sigma)^\vee$$

with σ the lift of $d\xi$ to hyperboloid, satisfies (with $|\xi| \simeq \lambda$) the bound $\|u\|_{L^p_t L^q_x} \lesssim \langle\lambda\rangle^\beta \|f\|_2$, where $2 < p \le \infty$, $2 \le q \le \infty$, $\frac{1}{p} + \frac{d}{2q} = \frac{d}{4}$, $\beta = \frac{1}{2}(d/2+1)(1/q' - 1/q)$

Cubic nonlinear Klein-Gordon

Energy subcritical model equation:

$$\Box u + u = u^3 \quad \text{in} \quad \mathbb{R}^{1+3}_{t,x}$$

$\forall\ \vec{u}(0) \in \mathcal{H} := H^1 \times L^2$, there $\exists!$ strong solution (Duhamel sense)

$$u \in C^0([0,T); H^1),\ \dot{u} \in C^0([0,T); L^2)$$

for some $T \geq T_0(\|\vec{u}[0]\|_{\mathcal{H}}) > 0$.

Properties: continuous dependence on data; persistence of regularity; energy conservation:

$$E(u,\dot{u}) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\dot{u}|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4\Big)\, dx$$

If $\|\vec{u}(0)\|_{\mathcal{H}} \ll 1$, then global existence; let $T^* > 0$ be maximal forward time of existence:

$$T^* < \infty \Longrightarrow \|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} = \infty$$

If $T^* = \infty$ and $\|u\|_{L^3([0,T^*),L^6(\mathbb{R}^3))} < \infty$, then u scatters: $\exists\,(\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}$ s.t. for $v(t) = S_0(t)(\tilde{u}_0, \tilde{u}_1)$ one has

$$(u(t), \dot{u}(t)) = (v(t), \dot{v}(t)) + o_{\mathcal{H}}(1) \quad t \to \infty$$

where $S_0(t)$ is the free KG evolution. If u scatters, then $\|u\|_{L^3([0,\infty),L^6(\mathbb{R}^3))} < \infty$.

Finite propagation speed: if $\vec{u}(0) = 0$ on $\{|x - x_0| < R\}$, then $u(t,x) = 0$ on $\{|x - x_0| < R - t, 0 < t < \min(T^*, R)\}$.

$T > 0$, exact solution to cubic NLKG

$$\varphi_T(t) \sim \sqrt{2}(T - t)^{-1} \qquad \text{as} \;\; t \to T_+,$$

Use finite prop-speed to cut off smoothly to neighborhood of cone $|x| < T - t$. Gives smooth solution to NLKG, blows up at $t = T$ or before.

Payne-Sattinger theorem 1975

Small data: global existence and scattering.
Large data: can have finite time blowup.
Is there a criterion to decide finite time blowup/global existence?
YES if energy is smaller than the energy of the **ground state** Q
unique positive, radial solution (Coffman) of :

$$-\Delta\varphi + \varphi = \varphi^3, \quad \varphi \in H^1(\mathbb{R}^3) \tag{1}$$

Minimization problem

$$\inf\left\{ \|\varphi\|_{H^1}^2 \mid \varphi \in H^1,\ \|\varphi\|_4 = 1 \right\}$$

has radial solution $\varphi_\infty > 0$, decays exponentially,
$Q = \lambda\varphi_\infty,\ \lambda > 0$. Minimizes the stationary energy (or action)

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}|\varphi|^4 \right) dx$$

amongst all nonzero solutions of (1). Dilation functional:

$$K_0(\varphi) = \langle J'(\varphi)|\varphi\rangle = \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + \varphi^2 - |\varphi|^4)(x)\, dx$$

Payne-Sattinger theorem

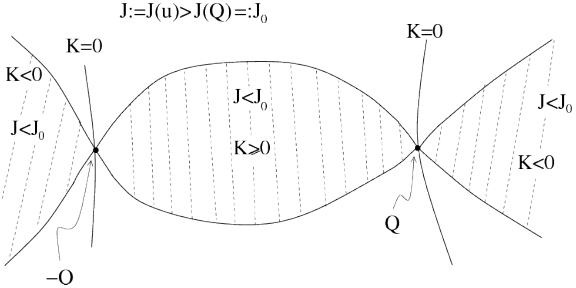


Figure: The splitting of $J(u) < J(Q)$ by the sign of $K = K_0$

Theorem (PS 1975)

If $E(u_0, u_1) < E(Q, 0)$, the dichotomy: $K(u_0) \geq 0$ global existence, $K(u_0) < 0$ finite time blowup

Ibrahim-Masmoudi-Nakanishi (2010): **Scattering** in addition to global existence. Why wait 35 years? See next slides...

Concentration Compactness by Bahouri-Gérard

Let $\{u_n\}_{n=1}^\infty$ free Klein-Gordon solutions in $\mathbb{R}^3$ s.t.

$$\sup_n \|\vec{u}_n\|_{L_t^\infty \mathcal{H}} < \infty$$

$\exists$ free solutions v^j bounded in $\mathcal{H}$, and $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^3$ s.t.

$$u_n(t,x) = \sum_{1\le j<J} v^j(t + t_n^j, x + x_n^j) + w_n^J(t,x)$$

satisfies $\forall\ j < J$, $\vec{w}_n^J(-t_n^j, -x_n^j) \rightharpoonup 0$ in $\mathcal{H}$ as $n \to \infty$, and

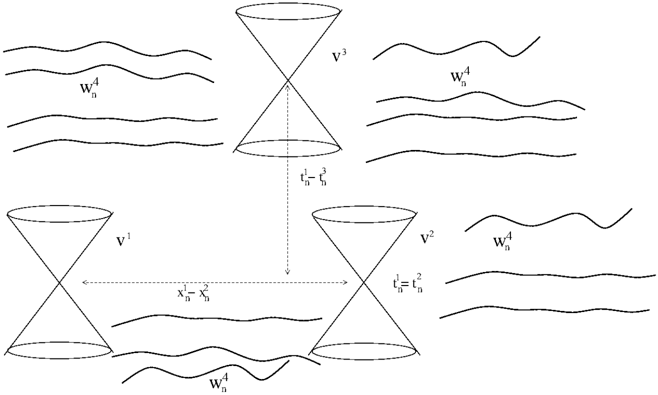
- $\lim_{n\to\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = \infty \ \forall\, j \neq k$
- dispersive errors w_n^J vanish asymptotically:

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0 \quad \forall\, 2 < p < 6$$

- orthogonality of the energy:

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{1\le j<J} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{w}_n^J\|_{\mathcal{H}}^2 + o(1) \quad n \to \infty$$

Profiles and Strichartz sea



We can extract further profiles from the Strichartz sea if w_n^4 does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n\to\infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$.

Critical wave equation: Kenig-Merle

Payne-Sattinger regime for the energy critical focusing NLW in $\mathbb{R}^3$:

$$u_{tt} - \Delta u - u^5 = 0$$

Stationary solution $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$, unique radial solution. Aubin-Talenti solution, **extremizer** for the critical embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Theorem (KM2007)

Assume $(u_0, u_1) \in \dot{H}^1 \times L^2$, $E(u_0, u_1) < E(W, 0)$.

- *If* $\|\nabla u_0\|_2 < \|\nabla W\|_2$ *then global existence and scattering (both time directions)*
- *If* $\|\nabla u_0\|_2 > \|\nabla W\|_2$ *then finite time blowup (both time directions). type I blowup, based on later work*

Kenig-Merle blueprint for scattering

- Small data scattering. Perturbative, based on Strichartz estimates.
- Induction on energy (Bourgain). Suppose result fails at some energy $0 < E_* < E(W, 0)$. Use Bahouri-Gérard decomposition to find special solution u_* of energy E_*, with infinite scattering norm $\|u_*\|_{L^8_{0<t<T^*,x}} = \infty$. It follows that trajectory (up to time of existence T^*) is precompact, modulo scaling symmetry. Main point in concentration-compactness: there can be only one profile, and dispersive error vanishes in energy norm.
- Rigidity step: Show there can be no precompact solution of energy below ground state energy other than zero. Key role played by monotone quantities such as virial or Morawetz which express asymptotic outgoing property of waves. $\langle u_t, x \cdot \nabla u \rangle$. Spatial cutoffs needed. Alternative tool: Exterior energy estimates.

Acta 2008 Kenig-Merle paper more complicated, exclusion of self-similar blowup, self-similar coordinates.

Beyond Payne Sattinger in unstable case (subcritical)

Theorem (Nakanishi-S. 2010)

Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2$, $(u_0, u_1) \in \mathcal{H}_{\mathrm{rad}}$. *In* $t \geq 0$ *for NLKG:*

1. *finite time blowup*
2. *global existence and scattering to* 0
3. *global existence and scattering to* Q*:*
   $u(t) = Q + v(t) + o_{H^1}(1)$ *as* $t \to \infty$, *and*
   $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ *as* $t \to \infty$, $\Box v + v = 0$, $(v, \dot{v}) \in \mathcal{H}$.

All 9 *combinations of this trichotomy allowed as* $t \to \pm\infty$.

- Applies to all dimensions, subcritical equations for which small data scattering is known.
- Linearized operator $L_+ = -\Delta + 1 - 3Q^2$ has unique negative eigenvalue.
- third alternative is center-stable manifold of codimension 1. Uniqueness of center-stable manifold.

The invariant manifolds

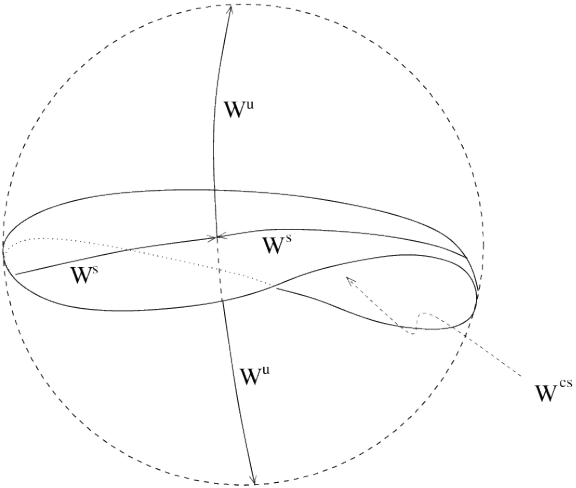
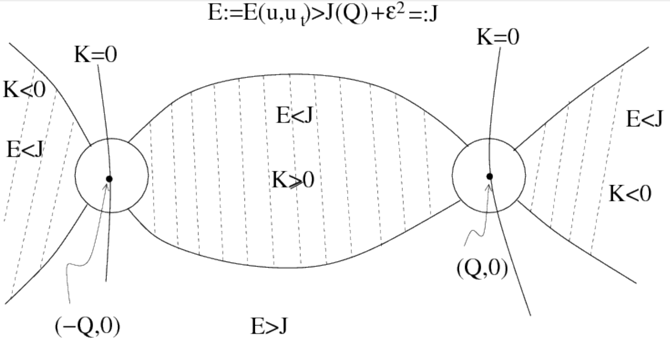


Figure: Stable, unstable, center-stable manifolds

Variational structure above $E(Q, 0)$



- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.
- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.
- Point: Stabilization of the sign of $K(u(t)) = \int |\nabla u|^2 + u^2 - u^4 \, dx$.

Numerical 2-dim section through $\partial\mathcal{S}_+$ (with R. Donninger)

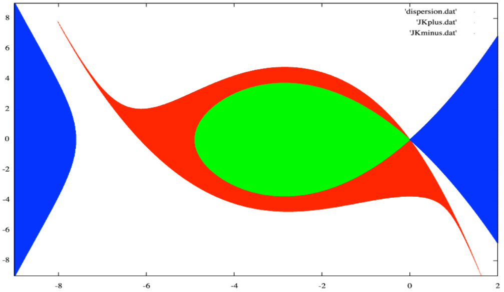
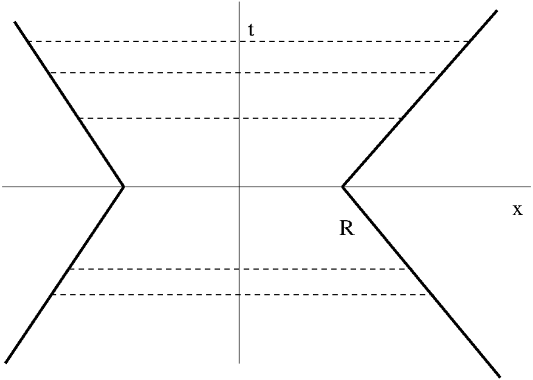


Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, (A, B) vary in $[-9, 2] \times [-9, 9]$
- RED: global existence, WHITE: finite time blowup, GREEN: $\mathcal{PS}_+$, BLUE: $\mathcal{PS}_-$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)



$\mathbb{R}^3$ radial data, free wave $\Box u = 0$. Then ($R = 0$ case!) for one sign $\pm$

$$\lim_{t\to\pm\infty} \int_{|x|\geq|t|} (|\nabla u|^2 + u_t^2)(t,x)\,dx \geq c \int_{\mathbb{R}^3} (|\nabla u|^2 + u_t^2)(0,x)\,dx$$

Exterior Energy Estimates

Extends to all odd dimensions, nonradial data. Fails in even dimensions, but holds for data $(u_0, 0)$, $d = 4, 8, \ldots$, or $(0, u_1)$, $d = 6, 10, \ldots$ (**Côte, Kenig. S.**)

Obstruction for the case $R > 0$: Newton potential $u(x) = |x|^{-1}$ solves $\Box u = 0$ in $|x| > |t|$, has finite energy on $|x| \geq R > 0$ but infinite energy on $\mathbb{R}^3$.

If $u_0 \perp |x|^{-1}$ in $\dot{H}^1(|x| \geq R)$ radial, then

$$\lim_{t\to\pm\infty} \int_{|x|\geq|t|+R} (|\nabla u|^2 + u_t^2)(t,x)\,dx \geq c \int_{|x|\geq R} (|\nabla u|^2 + u_t^2)(0,x)\,dx$$
$$= c \int_R^\infty ((ru)_r^2 + (ru)_t^2)(0,r)\,dr$$

Analogue in higher odd dimensions but with more obstructions (**Lawrie, Liu, Kenig, S.**).

Exterior Energy Estimates, nonlinear context

Critical equation, $W_\lambda(x) = \sqrt{\lambda} W(\lambda x)$.

Theorem (DKM2012)

Let (u, u_t), radial finite energy solution of $\Box u - u^5 = 0$, $0 \le t < T^*$. If $u \notin \{0, \pm W_\lambda \,|\, \forall\, \lambda > 0\}$, then $\exists\, R > 0, \eta > 0$

$$\int_{|x| \ge |t| + R} (|\nabla u|^2 + u_t^2)(t,x)\, dx \ge \eta, \qquad 0 \le \pm t < T^*$$

- In particular, nonstationary global solutions radiate off a positive amount of energy.
- Find sequence $t_n \to \infty$ so that $\vec{u}(t_n)$ bounded in $\dot{H}^1 \times L^2$. Apply concentration compactness to $\vec{u}(t_n) - \vec{u}_L(t_n)$ where u_L is a free wave which carries all energy of $\vec{u}$ in $|x| \ge t - A$.
- Use theorem to identify all nonzero profiles as W_λ, and radiative error vanishes.

DKM soliton resolution

Theorem (DKM2012)

Let (u, u_t), radial finite energy solution of $\Box u - u^5 = 0$, $0 \le t < T^*$.

- *Type I finite time blowup ($\dot{H}^1 \times L^2$ norm becomes infinite).*
- *Type II finite time blowup, multi-bubble representation via W_λ plus a function constant in time.*
- *Global bounded solutions, multi-bubble representation via W_λ plus free radiation.*

Multi-bubble in infinite time: exists free wave $\vec{v}$ s.t.

$$\vec{u}(t) = \sum_{j=1}^{J} (\pm W_{\lambda_j(t)}(t), 0) + (v(t), v_t(t)) + o(1)$$

$$\lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll t, \qquad t \to \infty$$

In finite time, replace $\vec{v}$ by a constant. Absence of self-similar solutions.

DKM soliton resolution in other contexts

Existence of such solutions known for one bubble:
Krieger-S-Tataru for finite time, **Donninger-Krieger** in infinite time. One expects multi-bubble solutions to be unstable.
DKM method applied to other scenarios:

- Exterior equivariant wave maps $u : \mathbb{R}^3 \setminus B(0,1) \to \mathbb{S}^3$ with Dirichlet condition on ∂B and arbitrary data of finite energy. Scatter to the unique harmonic map in the same equivariance and degree class as the data. **Lawrie-S 11** for zero degree and 1-equivariance, **Kenig-Lawrie-S 13** for nonzero degree, **Kenig-Lawrie-Liu-S 14** for all equivariance classes, degrees. Observed numerically by **Bizon-Chmaj-Maliborski**.
- Defocusing (and thus stable) radial u^5 NLW in $\mathbb{R}^3$ with a potential well. Combines exterior energy estimates with center-stable manifolds, one-pass theorem **(Jia, Liu, S, Xu).**
- **Method appears not to apply in the subcritical case (propagation speed of Klein-Gordon).**

Defocusing u^5 NLW with potential

Consider

$$\Box u + Vu + u^5 = 0$$

radial, decaying V, deep enough to trap bound states $-\Delta\varphi + V\varphi + \varphi^5 = 0$. For generic V finitely many bound states, and linearized operator $H_\varphi := -\Delta + V + 5\varphi^4$ has no anomalies (zero energy resonance or eigenvalues). $\dot{H}^1 \times L^2(\mathbb{R}^3)$ data lead to global solutions (standard). *Long term dynamics?*

Theorem (Jia, Liu, S, Xu '14, '15)

All radial finite energy solutions scatter (asymptotically free) to one of the stationary solutions φ. Data scattering to φ are (i) open if H_φ has no negative eigenvalues (ii) form a C^1 path-connected manifold $\mathcal{M}$ in $\dot{H}^1 \times L^2(\mathbb{R}^3)$ of co-dimension equal to number of negative eigenvalues of H_φ.

The manifold $\mathcal{M}_\varphi$ is a global, unbounded, center-stable manifold associated with stationary solution φ. Is it closed?

Defocusing u^5 NLW with potential

- Scattering result is an adaptation of DKM technique. One profile in Bahouri-Gérard decomposition sees potential V (no scaling) the others do not (scaling).
- Potential V perturbative error in $|x| \geq t - A$, so exterior energy methods still apply.
- Local construction of $\mathcal{M}_\varphi$ near any solution scattering to φ. Delicate, radial endpoint for Strichartz. Note difference from standard center-stable manifold constructions: not near stationary solution but near a given scattering solution.
- The local manifold has repulsive property: If solution remains near it for all times $t \geq 0$, then it lies on it. Perturbative.
- Solution leaves, comes back eventually? Nonperturbative.
- **No-return** or **one-pass theorem**: if the solution exits small neighborhood of $\mathcal{M}_\varphi$ then it must emit a fixed quantum of energy which pushes it away from $\mathcal{M}_\varphi$, precluding a near return. So near but off of $\mathcal{M}_\varphi$ solution cannot scatter to φ.

Dispersive equations with dissipation

Consider in $\mathbb{R}^d$, $d \le 6$

$$\partial_{tt} u - \Delta u + 2\alpha \partial_t u + u - f(u) = 0$$

data $(u(0), \partial_t u(0)) \in H^1 \times L^2(\mathbb{R}^d)$, $\alpha > 0$, $f \in C^{1,\beta}(\mathbb{R})$, odd, $f'(0) = 0$, subcritical. Ambrosetti-Rabinowitz condition: there exists $\gamma > 0$ so that

$$\int_{\mathbb{R}^d} 2(1+\gamma)F(\varphi) - \varphi f(\varphi) \le 0 \quad \forall \varphi \in H^1(\mathbb{R}^d), \quad F' = f \qquad (\star)$$

For example

$$f(u) = \sum_{i=1}^{m_1} a_i |u|^{p_i - 1} u - \sum_{j=1}^{m_2} b_j |u|^{q_j - 1} u \ , \ 1 < q_j < p_i \le \frac{d+2}{d-2}, \forall i, j \qquad (\dagger)$$

$$a_i, b_j \ge 0, a_{m_1} > 0 \ .$$

For this class existence, uniqueness of ground state known, hyperbolicity of linearized operator. We only assume $(\star)$ not $(\dagger)$.

Convergence to equilibria or blowup

Theorem (Burq-Raugel-S '15)

Let $\alpha > 0$. Assume that $1 \leq d \leq 6$ and that nonlinearity satisfies above conditions. Then any solution with radial $H^1 \times L^2$ data

1. *either blows up in finite time,*
2. *or exists globally and converges to an equilibrium point (stationary solution) as $t \to +\infty$.*

Does not use concentration-compactness, but relies heavily of results from dynamical systems in infinite dimensions (invariant manifold theory, **Chen-Hale-Tan, Brunovsky-Polacik** 90s).

Energy is monotone decreasing:

$$E(\vec{u}(T)) - E(\vec{u}(0)) = -2\alpha \int_0^T \|u_t(t)\|_2^2 \, dt$$

Implies: ω-limit set of any global solution consists of equilibria (stationary solutions), or empty.

Convergence to equilibria or blowup: scheme of proof

- Not clear a priori if global solution is bounded in $H^1 \times L^2$.
- Let $K_0(\varphi) = \int_{\mathbb{R}^d} |\nabla\varphi|^2 + \varphi^2 - \varphi f(\varphi)\, dx$. Show $\exists\, t_n \to \infty$ s.t. $K_0(t_n) \to 0$.
- Then show that $\vec{u}(t_n) \to (Q, 0)$, a stationary solution.
- Linearize about $(Q, 0)$. We may or may not have hyperbolicity of the linearized equation, depends on whether $L_Q := -\Delta + 1 - f'(Q)$ has trivial kernel or not; in latter case kernel is 1-dimensional (due to radial assumption).
- Construct stable, unstable, center(un)stable manifolds near $(Q, 0)$ (Chen-Hale-Tan 1997). Latter only present if L_Q has nontrivial kernel. If present, center manifold is a curve.
- Now apply Brunovsky-Polacik (97): if center dynamics is stable, $\vec{u}(t) \not\to (Q, 0)$ as $t \to \infty$ implies $\vec{u}(\tilde{t}_n) \to (\tilde{Q}, 0) \neq (Q, 0)$ which belongs to unstable manifold. But such an equilibrium cannot lie on unstable manifold, so done. Stability of center manifold: it is a curve, and infinitely many equilibria on it. So evolution is trapped between them.

The spectrum of the linearized flow with dissipation

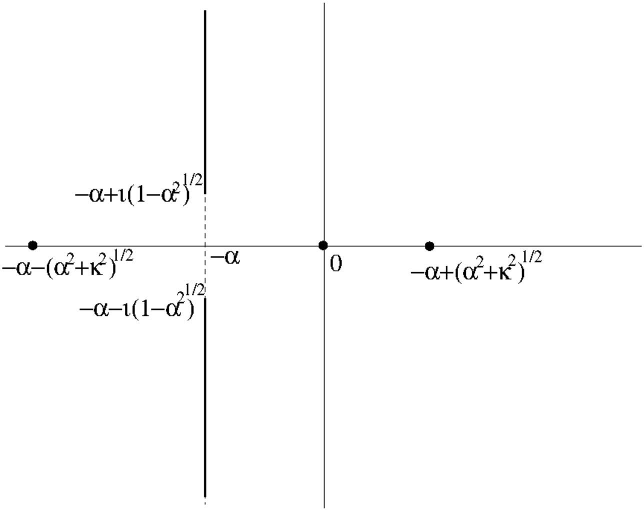


Figure: The spectrum of the damped equation, $0 < \alpha < 1$.

Some details of proof

One has

$$\gamma(\|\phi\|_{H^1}^2 + \|\psi\|_2^2) \leq 2(1+\gamma)E(\phi,\psi) - K_0(\phi)$$

So $K_0(u(t)) \geq -M$ implies solution global. Define

$$y(t) = \frac{1}{2}\|u(t)\|_2^2 + \alpha \int_0^t \|u(s)\|_2^2\, ds$$

Then

$$\ddot{y}(t) = \|\dot{u}(t)\|_2^2 - K_0(u(t)) \qquad (\star)$$

If $K_0(u(t)) \leq -\delta$, then by strict convexity $y(t) \to \infty$ (assume global solution). If $\alpha = 0$ then deduce via energy that

$$\ddot{y}(t) \geq \frac{2+\gamma}{\gamma}\frac{\dot{y}(t)^2}{y(t)} \quad \text{or} \quad \frac{d^2}{dt^2}\left(y^{-\eta}(t)\right) < 0 \qquad \lightning$$

So finite time blowup.

Some more details of proof

Suppose $u(t)$ global solution (and so energy remains positive). Cannot have $K_0(u(t)) \leq -\kappa < 0$ for large times. Also cannot have $K_0(u(t)) \geq \kappa > 0$ for large times: (i) solution is bounded (ii) $(\star)$ implies that

$$\dot{y}(t_2) - \dot{y}(t_1) \leq \int_{t_1}^{t_2} \|\dot{u}(t)\|_2^2 \, dt - (t_2 - t_1)\kappa \qquad \lightning$$

Thus, $K_0(u(t_n)) \to 0$ for some sequence $t_n \to \infty$. Thus, $\|\vec{u}(t_n)\|_{\mathcal{H}}$ uniformly bounded, $\int_{t_n-1}^{t_n+1} \|\partial_t u(t)\|_2^2 \, dt \to 0$ and $\vec{u}_n(s) := \vec{u}(t_n + s)$, $-1 \leq s \leq 1$ converges to $\vec{u}^* = (Q, 0)$ (equilibrium). How to obtain strong convergence in H^1: (i) $u_n(0) \rightharpoonup u^*$ in H^1 (ii) $K_0(u^*) = 0$ by equilibrium (iii) $K_0(u_n(0)) \to 0$ (iv) Thus $\|u_n(0)\|_{H^1} \to \|u^*\|_{H^1}$ (use compact radial Rellich embedding on nonlinear term) (v) strong convergence. More work needed to prove that $\|\partial_t u_n(0)\|_2 \to 0$.

Time dependent asymptotically vanishing damping

Consider in $\mathbb{R}^d$, $d \le 6$

$$\partial_{tt} u - \Delta u + 2\alpha(t)\partial_t u + u - f(u) = 0$$

We assume $\alpha(t) > 0$, $\int_0^\infty \alpha(t)\,dt = \infty$. In fact, $\alpha(t) = (1+t)^{-a}$, $0 < a < \frac{1}{3}$. Let $f(u)$ be as above.

Theorem (Burq-Raugel-S., 17): Any solution with radial $H^1 \times L^2$ data

1. either blows up in finite time,
2. or exists globally and converges to an equilibrium point (stationary solution) as $t \to +\infty$.

Not a dynamical proof, does not use invariant manifold. Rather rely on functional approach, Łojasiewicz-Simon inequality. Nonlinearity $f(u) = |u|^{p-1}u$, case $d = 3$ and $4 < p < 5$ more delicate, requires more PDE techniques.

Functional, Lojasiewicz-Simon inequality

Let $\vec{u}(t)$ be a global trajectory. Then energy remains positive, so

$$\int_0^\infty \alpha(t)\|u_t(t)\|_2^2\,dt < \infty$$

Conclude along some subsequence of integers

$$\int_{n^\gamma}^{(n+1)^\gamma} s^a\|u_t(s)\|_2^2\,ds \to 0, \quad (n+1)^\gamma - n^\gamma \geq n^{a\gamma},\ \gamma = (1-a)^{-1}$$

In analogy with constant damping conclude

$$\max_{I_n} \|\vec{u}(s) - (Q,0)\|_{\mathcal{H}} \to 0, \quad I_n = [n^\gamma, (n+1)^\gamma]$$

Delicate analysis of functional with $\nu = 1+$

$$H_\nu(t) = E(\vec{u}(t)) - E(Q,0) + \frac{\varepsilon_0}{(1+t)^{a\nu}}\langle -\Delta u + u - f(u), u_t\rangle_{H^{-1}}$$

Functional, Łojasiewicz-Simon inequality

Main point is to show that H_ν is non-negative, decreasing (for $\dim = 3$, and $4 < p < 5$ more complicated). Interplay between $H_\nu, \dot{H}_\nu$, and the stationary energy J. Analysis hinges on Łojasiewicz-Simon inequality in the radial setting

$$|J(u) - J(Q)| \leq C\| - \Delta u + u - f(u)\|^2_{H^{-1}}, \quad \|u - Q\|_{H^1} \ll 1$$

Note that $J'(u)$ appears on the right-hand side. Easy if linearization $-\Delta + 1 - f'(Q)$ has trivial kernel. In the radial setting we know that kernel is at most one-dimensional.

$F : \Omega \to \mathbb{R}$ real-analytic on some domain $\Omega \subset \mathbb{R}^n$, $\nabla F(a) = 0$. There exists $1 < \theta \le 2$ so that

$$|F(x) - F(a)| \le C|\nabla F(x)|^\theta, \qquad \forall\, |x - a| < \varepsilon \qquad (\star)$$

Consider ODE

$$\dot{u}(t) + \nabla F(u(t)) = 0, \quad u(0) = u_0 \in \mathbb{R}^n$$

If trajectory global and bounded, then ω limit set is exactly one point. Idea: $u(t_n) \to p$ along some sequence going to ∞. Consider Lyapunov functional with $\theta = 2, a = p$ in $(\star)$

$$G(u) := F(u) - F(p)$$

Then $\frac{d}{dt} G(u(t)) = -|\nabla F(u(t))|^2 = -|\dot{u}(t)|^2 \le -cG(u(t))$. Exponential decrease and convergence.